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Understanding Expected Utility for Decision Making

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In this paper, we present a brief version of de Finetti-Ramsey's subjective probability theory and provide a rigorous yet intuitively plausible explanation of expected utility using elementary mathematics. In a final section, we take up the case of some "Paradoxes in Expected Utility Theory" and try to reconcile them with the help of subjective probabilities.

Keywords: subjective probability, uncertain prospects, expected utility of monetary gains

Introduction

The purpose of this paper is to define probability so that we have a rigorous yet intuitively plausible explanation (i.e., economic interpretation) of the concept of "expected utility" that goes beyond its current expression as a mere mathematical entity, i.e., numerical representation of preferences over uncertain prospects as in Lahiri (2023). The problem with the definition of probability provided by John Maynard Keynes (1921) in his *Treaties on Probability*, is that it is a "unit free" numerical representation of one's "degree of belief in the likelihood of an event or plausibility of a statement". This may explain those decision-making procedures that are entirely based on the "likelihood of events or plausibility of statements", e.g., maximization or minimization of likelihood. Expected utility is not one of them.

Frank Ramsey (1931; 1980) and Bruno de Finetti (1937; 1974; 2017) independently defined the (subjective) probability of an event assessed by an individual as the price in money units the individual is willing to pay (without being vulnerable to "sure loss"), for a lottery ticket that would yield 1 unit of money to the individual if the event did occur and nothing otherwise, the price of the ticket itself being non-refundable. An alternative definition of the probability of an event as assessed by an individual, that leads to the same result as in Finetti (1937; 1974; 2017) and Ramsey (1931; 1980) is to define it as the price in money units the individual is willing to pay for a lottery ticket that yields 1 unit of money to the individual if the event occurs and nothing otherwise, the price of the ticket itself being non-refundable, such that the expected net gain to the individual from the transaction is zero. This is the approach we follow here. For our approach to go through, we need to assume the existence of a (mathematical) probability measure (in the sense of Kolmogrov) which is used to calculate "mathematical expectations" and then show that such a probability measure assigns the subjective probabilities to the events under consideration. The underlying mathematics in our approach is simpler and considerably more compact than in the work of de Finetti and Ramsey. In fact, the two approaches to definition of probability are

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equivalent. Critiques and "doubting Thomases" of the probability theory due to de Finetti and Ramsey are advised to read and benefit from the paper by Velupillai (2015).

While the seminal work of Leonard Jimmie Savage (1954) entitled *The Foundations of Statistics*, is entirely compatible and even uses this definition of probability, it does not go beyond an "axiomatic foundation for the existence of a numerical representation of preferences over the set of 'acts' that are obtained by applying the operation of mathematical expectation to a real valued function of consequences". The function to which the operation of mathematical expectation is applied is in spirit no different from the ones to which the expectation operator is applied in Lahiri (2023). Whether one considers numerical representations of preferences on the set of "uncertain prospects" or numerical representations of the preferences on the set of "acts", the fact remains that these numerical representations are devoid of any economic meaning, and their best known use in decision-making theory is to facilitate optimization on the basis of preferences from a given set of alternatives. Finding such numerical representations is an extremely worthwhile exercise and there are scientists (most notably Late Professor Peter Fishburn) who have devoted their entire lives to such pursuits.

Our purpose in this paper is not related to the vast literature that investigates necessary and/or sufficient conditions for numerical representation of preferences over a set of alternatives. Our purpose, in this paper is to seek a definition of probability, that permits a "mathematically simple" but "intuitively plausible" exercise to yield exactly one numerical representation of preferences over the set of uncertain prospects, that assigns to each uncertain prospect its "use value" obtained as the mathematical expectation of a given "use value function" of monetary gains and losses. As far as I am aware, there is no precedent in this line of endeavor for whatever reasons.

Probability is a personal assessment based on a "thought experiment" prone to subjectivity. This subjective probability which forms the cornerstone of our discussion here is formally presented beginning with the next section. (See Nau (2001) for an interesting discussion of the concept).

Money by itself does not provide satisfaction, but has "person specific use value", when it is used by individuals as an "instrument" to derive satisfaction from the consumption of goods and services on which they spend this money. We refer to the unit of measurement of this "person specific use value" as "util". Util could be money units or any other that is used to measure something that is desirable to the individual, so long as it is the unit of measurement of the "use value" of monetary gains and losses. The individual specific strictly increasing function that assigns to each amount of gain (negative implying loss), its use value to the individual and assigns zero utils to zero gains, is called the "utility function for monetary gains" of the individual.

The purpose of the calculus of probability presented here is largely pedagogical, and our goal for this is to eventually arrive at an explanation/interpretation of the concept of expected utility for graduate or upper-level undergraduate students in Economics and Management, using no more than the "unitary method of arithmetic" that one usually learns in primary school. The simple economics that we invoke is that if the price paid in anticipation of gaining x units of money is p, then the utility or use value voluntarily foregone is "p times the average utility of gaining x", where the average is per unit of money, i.e., utility of x divided by x, provided x is not equal to zero. It is important to note, that in our context, price is "voluntarily foregone" by the buyer of an uncertain prospect.

Subsequently, we show that for individuals whose preferences over a set of alternatives, each of which are random returns of monetary gains and losses (i.e. an uncertain prospect) can be numerically represented by a "von Neumann Morgenstern utility function" as well as the function that assigns to each alternative expected utility of monetary gains and losses, it must be the case that the Bernoulli utility function of gains and losses whose expectation is the von-Neumann Morgenstern utility function, is a strictly increasing and linear transformation of the "utility function of monetary gains and losses". To the best of my knowledge, the earliest use of the word "prospect" in the theory of decision making under uncertainty occurs in a March 1973 discussion paper of the University of Minnesota (Number 20), by Professor Clifford Hildreth, entitled "Ventures, Bets and Initial Prospects", which is a revised version of a manuscript dated August 1972.

It is important to note that while difference in the nomenclature of the two utility functions is that one of the two explicitly mentions "monetary", whereas the other does not, the expected utility of monetary gains and losses of an uncertain prospect measures the "use value" of an uncertain prospect (i.e., UV of an uncertain prospect), while the von Neumann Morgenstern utility function is simply a mathematical entity, with no specific economic interpretation implied by it. It makes eminent sense to talk about the "use value of an uncertain prospect", since not being a consumable, an uncertain prospect does provide satisfaction, although it can be used as an instrument to derive satisfaction from consumable goods and services.

In the final section of this paper, we take up the case of "Paradoxes in Expected Utility Theory" whose starting point is the critique of "Expected Monetary Value" posed by the St. Petersburg paradox. Our position is that such paradoxes arise because of unwarranted "mathematical liberties" that are taken while describing the behavior of individuals facing uncertain monetary prospects. We do not question the robust validity of expected utility theory as opposed to a theory that argues for expected monetary value and no further. We only insist that the reasons for claiming the superiority of one theory over another should be right and not incorrect. We try to reconcile the "apparent inconsistencies" by using subjective probabilities.

The Framework of Analysis

Let S be a non-empty set, subsets of which are events. An event is a state of nature that is of concern to us, the status of its occurrence (which may be either "true" or "false") being unknown in the current state.

An event E occurs if the status of its occurrence is true. The non-occurrence of E is also a state of nature and is denoted by E^c. In such a situation we say that E^c occurs.

We consider an individual who chooses to measure his/her personal assessment of the price and stakes of lottery tickets in "utils".

A *finite lottery ticket* is a pair $(\{E_1, ..., E_n\}, \pi)$, where $\{E_1, ..., E_n\}$ is a finite partition for some positive integer "n", each member of which is an event and a pay-off function $\pi:\{E_1, ..., E_n\}\rightarrow\mathbb{R}$, where for each $i\in\{1, ..., n\}$, $\pi(E_i)$ is money units to a buyer of the lottery in the current state, if the event E_i occurs. We do not specify the unit of measurement of the pay-offs.

It is easy to see that if n = 2, then $(E_1)^c = E_2$ and $(E_2)^c = E_1$.

Hereafter, we will refer to a finite lottery ticket, as a "lottery ticket".

In what follows, "lottery ticket" and "*uncertain prospect*" are used interchangeably.

The Calculus of Subjective Probability

In this section all prices and returns are measured in money.

A non-empty subset $\mathfrak{B}$ of 2^S (i.e., power set of S) is said to be a *Boolean algebra* (of subsets of S) if and only if it satisfies the following properties:

(a) $S\in \mathfrak{B}$ and $\phi\in\mathfrak{B}$;

(b) If $E\in\mathfrak{B}$, then $E^c\in \mathfrak{B}$;

(c) If E, $F\in\mathfrak{B}$, then $E\cup F\in\mathfrak{B}$.

A member of $\mathfrak{B}$ is said to be an *event*.

Since for all E, $F\in\mathfrak{B}$, $E\cap F = (E^c\cup F^c)^c$, it must be the case that $E\cap F\in \mathfrak{B}$.

In particular, if for some positive integer n, $(E_1, ..., E_n)$ is a partition of S, then the set $\mathfrak{B}(E_1, ..., E_n)$ comprising of all unions in the partition $(E_1, ..., E_n)$ and the null (empty) set, is said to be the *Boolean algebra generated by* $(E_1, ..., E_n)$.

A simple bet on an event E is (i) if E, $E^c\neq\phi$, a lottery ticket $(\{E, E^c\}, \pi)$ with $\pi(E) = 1$ and $\pi(E^c) = 0$; (ii) if $E^c = \phi$, in which case E = S, the lottery ticket $(\{S\}, \pi)$ with $\pi(S) = 1$; (iii) if $E = \phi$, in which case $E^c = S$, the lottery ticket $(\{S\}, \pi)$ with $\pi(S) = 1$.

Let P(E) a real number denote the *price* of a simple bet on an event E.

If an agent *buys* a simple bet on an event E at a price P(E) then the agent gains 1-P(E) if the state of nature E occurs and gains -P(E) (i.e., loses P(E)) if the event E does not occur.

A function $P:\mathfrak{B}\rightarrow\mathbb{R}$ which associates with each simple bet on an event E the price P(E) is called a *price function*.

A function $\psi: \mathfrak{B}\rightarrow\mathbb{R}$ is said to be a (*finitely additive*) *probability measure* if the following conditions are satisfied:

(i) $\psi(E)\in[0, 1]$ for all $E\in\mathfrak{B}$;

(ii) $\psi(S) = 1$, $\psi(\phi) = 0$;

(iii) For all E, $F\in \mathfrak{B}$ with $E\cap F = \phi$, it is the case that $\psi(E\cup F) = \psi(E) + \psi(F)$.

From (i) and (iii) it follows that for E, $F\in\mathfrak{B}$, $E \subset F$ implies $\psi(F) = \psi(E) + \psi(F\backslash E)$.

It follows from (iii) of the definition of a probability measure that for all E, $F\in \mathfrak{B}$ it is the case that $\psi(E\cup F) = \psi(E) + \psi(F)-\psi(E\cap F)$.

The reasoning is as follows: $E\cup F = E\cup(F\backslash E)$ where $E\cap(F\backslash E) = \phi$ and $F = (E\cap F)\cup(F\backslash E)$ where $(E\cap F)\cap(F\backslash E) = \phi$. Thus, $\psi(E\cup F) = \psi(E) + \psi(F\backslash E)$ and $\psi(F) = \psi(E\cap F) + \psi(F\backslash E)$. Substituting for $\psi(F\backslash E)$ from the second equation to the first we get, $\psi(E\cup F) = \psi(E) + \psi(F)-\psi(E\cap F)$.

A function $X:S\rightarrow\mathbb{R}$ is said to be a *random variable*.

For $s\in S$, X(s) is the *realization of the random variable* X at *s*.

A random variable X is said to be *finitely generated* if there exists a finite partition $\{E_1, \ldots, E_n\}$ if for all $j\in\{1, \ldots, n\}$ and s, $s'\in E_j$: $X(s) = X(s')$ (= x_k say).

It is very easy to establish the following result.

If X is a finitely generated random variable, then its mathematical expectation (in the sense of Kolmogorov) with respect to the probability measure ψ is given by $\mathcal{E}_{\psi}(X) = \sum_{j=1}^{n} x_j\psi(E_j)$.

A price function P is said to be a *de Finetti price function* if there exists a probability measure such that for every event E the mathematical expectation of the gains from the simple bet on E is zero.

Theorem 1: Every de Finetti price function is a finitely additive probability measure.

Proof: Let P be a de Finetti price function. Then there exists a probability measure $\psi: \mathfrak{B}\rightarrow\mathbb{R}$, such that for every event E if P(E) is the price of the simple bet on E, the expected gain in utils to the buyer of this simple bet is 0.

Let X be the random variable such that X(s) = 1-P(E) if $s \in E$ and X(s) = -P(E) if $s \in E^c$.

Thus, $\mathcal{E}_{\psi}(X) = \psi(E)(1-P(E)) + (1-\psi(E)(-P(E)) = 0$, i.e. $P(E) = \psi(E)$.

This proves the theorem. Q.E.D.

Thus, if P is a de Finetti price function, then P is said to be a *subjective probability measure* and for each event E, P(E) is said to be *the subjective probability* of E.

Note that for each event E, P(E) is the "expectation with respect to P" of the simple bet on E interpreted as a random variable measured in money units.

If $(\{E_1, ..., E_n\}, \pi)$ is a lottery ticket, then $\mathcal{E}_P(\pi)$ is simply the expected return from the lottery ticket and may thus be interpreted as the *price (according to de Finetti)* of the lottery ticket $(\{E_1, ... ,E_n\}, \pi)$ corresponding to the pricing function P. This interpretation requires no more than an application of the unitary method, since if for $i \in \{1, \dots, n\}$, $P(E_i)$ is the price of a lottery ticket that yields 1 unit of money if E_i occurs and nothing otherwise, then by the "linearity condition" $\pi(E_i)P(E_i)$ is the price of a lottery ticket that yields $\pi(E_i)$ units of money if E_i occurs and nothing otherwise, and the lottery ticket $(\{E_1, ..., E_n\}, \pi)$ is simply a collection of "n" such lottery tickets.

Given a subjective probability measure P and events G and E, since G is a disjoint union of $G \cap E$ and $G \cap E^c$, we know that $P(G) = P(G \cap E) + P(G \cap E^c)$.

Thus, if P(G) > 0, then $1 = \frac{P(E \cap G)}{P(G)} + \frac{P(E^c \cap G)}{P(G)}$.

Given a subjective probability measure P and an event G with P(G) > 0, a function $P(\cdot|G): \mathfrak{B} \rightarrow \mathbb{R}$ is said to be a *price function conditional on* G *and consistent with* P if, for each event E, $P(E|G) = \mathcal{E}_{\psi}(X(E|G))$ with respect to P, where $X(E|G)(s) = 1$ if $s \in E \cap G$, $X(E|G) = 0$ if $s \in E^c \cap G$ and $X(E|G) = P(E|G)$ if $s \in G^c$.

Since, $\mathcal{E}_{\psi}(X(E|G)) = P(E \cap G) + P(E|G)(1-P(G))$, $E(X(E|G)) = P(E|G)$ implies $P(E|G) = \frac{P(E \cap G)}{P(G)}$.

Relationship Between "Utility Function" for Money and "Bernoulli Utility Function"

Recall that the "utility function of monetary gains and losses" is a strictly increasing function of gains that assigns to each amount of gain of money its "use value" measured in utils and satisfies the property that the use value of zero monetary gains is zero utils. In this sense, as mentioned in the Introduction, utility is simply "use value".

Thus, consider an individual with initial monetary wealth w > 0 and suppose that the maximum gain that is possible for the individual is M > 0. Hence the set of possible gains for the individual is the closed interval [-w, M], where a negative gain denotes loss.

Let $\mu: [-w, M] \rightarrow \mathbb{R}$ be a strictly increasing function satisfying $\mu(0) = 0$ that denotes the "utility function of monetary gains and losses" the values of which are measured in "utils".

Consider the uncertain prospect $(\{E, E^c\}, \pi)$, where $\pi(E)$ = (a monetary gain of) x units, $\pi(E^c)$ = (a monetary gain of) y.

Given a subjective probability measure P the price of a simple bet on E is P(E). Thus, the price of the uncertain prospect that returns x units of money if E occurs and nothing otherwise, is xP(E). For x ($\neq$) units of monetary gain, the average utility or use value per unit of money is $\frac{\mu(x)}{x}$.

Thus, the utility (or use value) paid/foregone voluntarily in anticipation of gaining x units of money from the uncertain prospect "that returns x units of money if E occurs and nothing otherwise" is $\frac{\mu(x)}{x}xP(E) = \mu(x)P(E)$.

Note: Although the money forgone for the uncertain prospect is xP(E), the average utility per unit of money that is used to calculate the "foregone utility" is not the average utility of foregone money, i.e., $\frac{\mu(xP(E))}{xP()}$. The average utility per unit of money that is used to calculate the "foregone utility" is the average utility of "anticipated gain", i.e., $\frac{\mu(x)}{x}$.

Similarly, the utility (or use value) paid/foregone voluntarily in anticipation of gaining $y \neq 0$ units of money from the uncertain prospect "that returns y units of money if E^c occurs and nothing otherwise" is $\frac{\mu(y)}{y}yP(E^c) = \mu(y)P(E^c) = \mu(y)(1\text{-}P(E))$.

Since, the uncertain prospect $(\{E, E^c\}, \pi)$ is nothing but a collection of two uncertain prospects one of which is "gaining x units of money if E occurs and nothing otherwise" and "gaining y units of money if E^c occurs and nothing otherwise", the utility (or use value) paid/foregone voluntarily in anticipation of the gains from the uncertain prospect $(\{E, E^c\}, \pi)$ is $\mu(x)P(E) + \mu(y)(1\text{-}P(E))$, with the understanding that a zero monetary gain does not require foregoing any utility.

Given a subjective probability measure P, *the use value (UV)* of the uncertain prospect $(\{E, E^c\}, \pi)$ is simply $\mu(x)P(E) + \mu(y)(1\text{-}P(E))$.

Thus, expected utility of monetary gains and losses of an uncertain prospect $(\{E, E^c\}, \pi)$, measures its use value.

Note: Unless, μ is a linear function, there is no reason to assume $\mu(x)P(E) + \mu(y)(1\text{-}P(E)) = \mu(xP(E) + y(1\text{-}P(E))$, the latter being the utility or use value foregone in buying the uncertain prospect $(\{E, E^c\}, \pi)$.

Given preferences over the set of alternatives, each of which is uncertain prospects, in Lahiri (2023) we provide a theorem, using a small number of very reasonable assumptions which are sufficient for the existence of a "von Neumann Morgenstern utility function" over the set of alternatives, that numerically represents the preferences. A von Neumann Morgenstern utility function over the set of all random returns of monetary gains and losses, assigns to each alternative the expectation—determined by it—of a "Bernoulli utility function of money", the latter being a strictly increasing function of monetary gains and losses that assigns 0 to "zero gains".

There is the extremely realistic example related to portfolio diversification discussed in Section 1.1 of the book by Eeckhoudt, Gollier, and Schlesinger (2005) which compels us to conclude that a "Bernoulli utility function of money", may not always be a linear function of money with positive slope. In fact, the only interpretation of "expected monetary value" that is unconditionally justifiable in our framework, is that of the expectation of a "Bernoulli utility function of money" of a "hypothetical individual" whose utility function for money is linear and has a positive slope.

In this section we establish a relationship between "the utility function of monetary gains and losses" and a "Bernoulli utility function of money" for individuals whose preferences over a set of alternatives, each of which is random returns of monetary gains and losses can be numerically represented by a function that assigns the "use value" for each alternative in the set as well as a "von Neumann Morgenstern utility function". We call the former "UV representation" and the latter "von Neumann Morgenstern utility representation".

Let u: [-w, M] $\to\mathbb{R}$ be a strictly increasing function satisfying u(0) = 0 be a "Bernoulli utility function of money". Thus, if p$\in$[0, 1] is the (subjective) probability of an event E occurring, then the expected Bernoulli utility (i.e., the "von Neumann Morgenstern utility") of gaining x$\in$[-w, M] units of money if event E occurs and y$\in$[-w, M] otherwise, is pu(x) + (1-p)u(y). Similarly, if q$\in$[0, 1] is the probability of an event F occurring, then the "von Neumann Morgenstern utility" of gaining z$\in$[-w, M] units of money if event F occurs and r$\in$[-w, M] units of money if it does not, is qu(z) + (1-q)u(r).

Given two such alternatives, according to the "von-Neumann Morgenstern utility representation" the first alternative is "at least as good" as the second if and only if pu(x) + (1-p)u(y) $\geq$ qu(z) + (1-q)u(r). The first is "no different from" the second if and only if pu(x) + (1-p)u(y) = qu(z) + (1-q)u(r) and the first is "strictly preferred to" the second if and only if pu(x) + (1-p)u(y) > qu(z) + (1-q)u(r).

On the other hand, according to the "UV representation", the first alternative is "at least as good" as the second if and only if pμ(x) + (1-p)μ(y) $\geq$ qμ(z) + (1-q)μ(r). The first is "no different from" the second if and only if pμ(x) + (1-p)μ(y) = qμ(z) + (1-q)μ(r) and the first is "strictly preferred to" the second if and only if pμ(x) + (1-p)μ(y) > qμ(z) + (1-q)μ(r).

Suppose the preferences have a "von-Neumann Morgenstern utility representation" with Bernoulli utility function u, as well as a "UV representation".

Let x$\in$(0, M].

Then since μ is a strictly increasing function satisfying μ(0) = 0, there exists λ(x)$\in$(0, 1] such that μ (x) = λ(x)μ(M) + (1-λ(x))μ(0) = λ(x)μ(M).

Thus $\lambda(x) = \frac{\mu(x)}{\mu(M)}$.

Thus, the UV of the prospect of gaining x units of money for sure is the same as the UV of the uncertain prospect of gaining M with probability λ(x) and nothing otherwise. Hence the prospect of gaining x units of money for sure "is no different from" the uncertain prospect of gaining M with probability λ(x) and nothing otherwise.

Since we have assumed that the preferences have a "von Neumann Morgenstern utility representation", it must be the case that $u(x) = \lambda(x)u(M) = \mu(x)\,\frac{u(M)}{\mu(M)}$.

Let $\alpha = \frac{u(M)}{\mu\,(M)} > 0$.

Thus, u(x) = $\alpha\mu$(x) for all x$\in$[0, M].

Now let x$\in$[-w, 0).

By a reasoning like the one above we get $u(x) = \mu(x)\,\frac{u(-w)}{util\,(-w)}$.

Let $\beta = \frac{u(-w)}{\mu\,(-w)} > 0$.

Thus, u(x) = $\beta\mu$(x) for all x$\in$[-w, 0].

Towards a contradiction suppose $\alpha \neq \beta$.

Let x$\in$(0, M) and y$\in$[-w, 0). Thus, x > 0 > y and hence μ(x) > 0 > μ(y).

Let $p = -\frac{\mu(y)}{\mu(x)-\mu(y)}$. Clearly p$\mu$(x) + (1-p) μ(y) = 0.

Thus, according to the “UV representation”, the uncertain prospect which yields a gain of x if an event whose probability of occurrence is p, occurs and a loss of -y if the event does not occur is “no different from” the prospect that yields “zero gains” with probability 1.

Thus, by our assumption the von Neumann Morgenstern utility of the uncertain prospect should also be “0” (= u(0)), i.e., $pu(x) + (1\text{-}p)u(y) = 0$.

However, $pu(x) + (1\text{-}p)u(y) = p\alpha\mu(x) + (1\text{-}p)\beta\mu(y)$ which combined with $p\mu(x) + (1\text{-}p)\mu(y) = 0$ implies:

(1) $pu(x) + (1\text{-}p)u(y) = p\alpha\mu(x) + (1\text{-}p)\beta\mu(y) > 0$ if $\alpha > \beta$;

(2) $pu(x) + (1\text{-}p)u(y) = p\alpha\mu(x) + (1\text{-}p)\beta\mu(y) < 0$ if $\alpha < \beta$.

If $\alpha > \beta$, then the uncertain prospect which yields a gain of x if an event whose probability of occurrence is p, occurs and a loss of -y if the event does not occur is “strictly preferred to” the prospect that yields “zero gains” with probability 1.

If $\alpha < \beta$, the prospect that yields “zero gains” with probability 1 is “strictly preferred to” the uncertain prospect which yields a gain of x if an event whose probability of occurrence is p, occurs and a loss of -y if the event does not occur.

In either case our earlier conclusion that the uncertain prospect which yields a gain of x if an event whose probability of occurrence is p, occurs and a loss of -y if the event does not occur is “no different from” the prospect that yields “zero gains” with probability 1 is violated.

Hence, it must be the case that $\alpha = \beta$, i.e., the “Bernoulli utility function” is a linear and strictly increasing transformation of the utility function of monetary gains and losses.

Explanation of Paradoxical Behavior

In this section, we will provide scenarios which have either been misunderstood or misrepresented, to justify the use of expected (monetary or Bernoulli) utility of gains and losses, instead of expected monetary values, in decision analysis. However, the very realistic example (considerably more realistic than the St. Petersburg Paradox) about a hypothetical individual by the name of Sempronius in section 1.1 of Eeckhoudt et al. (2005), shows that more non-linearity of the Bernouli utility function (and consequently the utility function for monetary gains and losses), is required for explaining the preference for a more diversified portfolio of assets over a less diversified one, than what is required simply for the purpose of explaining “loss aversion”.

A major concern is about the frequent observation that a potential buyer of a fair bet may refuse to buy it, in the sense that the potential buyer may attach a negative price to such a lottery ticket. Why?

The reason for this appears to us to be an asymmetry in the roles of the buyers and sellers. True to his word, the seller of a fair bet foresees the real possibility of an equal number of gains and losses on the utilitarian lottery tickets that he sells, whereas this is not the case with the buyer who may buy just one such ticket. A simple example may help to illustrate what we are trying to suggest.

Consider a seller of lottery tickets, each of which yields its buyer a “gain” of $1/- if the toss of an unbiased coin shows up heads and the buyer incurs a “loss” of $1/- if the toss of the same coin shows up tails. As far as the seller is concerned the coin is an unbiased one and not loaded in favor of any outcome. Hence it would not be unreasonable for him to assume that he would gain $1 almost as many times as he would lose $1 and thus his net gain from selling such lottery tickets is zero. But how about a buyer of the lottery ticket who gets the opportunity to lose or gain from a toss of the unbiased coin exactly once? What the outcome of “that particular toss" is going to be is neither implied nor does it have any implications for the kind of observations that one would expect from multiple repetitions of the toss. The perspective of the buyer is completely different from that of the seller, since the question that confronts the buyer is: will the outcome of this toss be among the approximately 50% times "heads" show up or will the outcome of this toss be among the approximately 50% times "tails" show up? The price the buyer would be willing to pay for the lottery ticket would very likely depend on whether the buyer is an optimist or a pessimist, or what his mood is at the time of buying the ticket. While there may be definite results in physics which could determine the outcome of a coin toss on the basis of force, spin etc. of the toss, rarely are such considerations invoked when one has to decide on buying a lottery ticket or not—even if the potential buyer is a professional physicist. It is quite possible that the buyer may be feeling pessimistic at the moment, and seek a non-refundable compensation from the seller of the lottery ticket in case he did agree to participate in the lottery. Of course, it is important to note, that while the seller of the lottery ticket considers the two possible outcomes of the toss of the coin as heads and tails, the same is not true for the potential buyer. The potential buyer would very likely be viewing the two possible outcomes as lucky and unlucky or as a good day and a bad day or elements in the set {\$1, \$(-1)}. More formally, the potential buyer views the two outcomes as elements in the set {this particular toss is among the approximately 50% times "heads" show up, this particular toss is among the approximately 50% times "tails" show up}. It is precisely for this reason, that the subjective price or the de Finetti price of the event "less than or equal to zero" may be greater than $\frac{1}{2}$, the latter being the de Finetti price of the event to the seller.

A well-known but not well-recognized red-herring in decision-making theory is the so-called St. Petersburg paradox which led to Bernoulli utility functions and expected utility maximization. The experiment consists of repeated trials of an unbiased coin till the first head shows up. If the first head shows up on the nth toss, then the participant in this game gets $\$2^n$. How much should a person be willing to pay to play this game? While no reasonable person would be willing to pay more than a couple of dollars for it, apparently an expected monetary value maximiser should be willing to pay an unbounded sum of money to play the game, provided one believes the coin is fair and not loaded in favor of showing either heads or tails. However, the conclusion that a person would be willing to stake any amount of money that is conceivable, to participate in such a game, rests crucially on the assumption that simply because the person initially started off by believing that the coin under consideration is fair and unbiased continues to do so after no head has appeared till "n" tosses, however large "n" may be. Hence after observing a string of one million consecutive tails showing up—if that was humanly possible to endure—one would continue to abide by one's initial belief that heads and tails would show up almost equally often. That clearly requires a "Giant Leap of Faith"—and absolutely nothing less. In fact, that such a game is on offer would make one suspect, whether the coin is "fair", i.e., the two sides are indeed head and tail and/or the engineering behind the coin has not favored any one side to show up more often than another.

The approach using Kolmogorov (mathematical) probability on which the St. Petersburg paradox is based, is that there is no justification required for assuming an infinite sequence of independent and identically distributed (IID) random variables. It is perfectly consistent with mathematical probability to assume a sequence of randomizations to be IID even if such an assumption is inconsistent with science and/or empirical observations and it is the assumption of IID random outcomes of an unbiased coin in the St. Petersburg paradox which leads to the conclusion that the probability of the first head appearing after "n" tails is $(\frac{1}{2})^{n+1}$ for all positive integers "n". It is precisely this assumption that leads to expected monetary value being $+\infty$ of this so-called paradox proposed by Nicolas Bernoulli, who first stated it in a letter to Pierre Raymond de Montmort on September 9, 1713. However, it was not he, but his cousin Daniel Bernoulli who is considered to be the pioneer of Expected Utility Theory with his arguments in the *Commentaries of the Imperial Academy of Science of Saint Petersburg* (1738) in favor of strictly increasing and bounded utility functions of money whose expectation rather than the expected monetary value he suggested as a criterion for choosing uncertain prospects. It is perfectly fine for us if the Bernoulli brothers or whoever else subscribes to the kind of probability theory that finds nothing unreasonable about the probability of the first head appearing after "n" tails to be $(\frac{1}{2})^{n+1}$ for all positive integers denoted generically by "n". The approach of subjective probability theory we subscribe to, unlike the approach adopted above, would allow the conditional probability of a head on the $(n+1)^{th}$ toss given tails on the previous "n" toss, to be equal to $\frac{1}{2}$ for n= 0, 1, 2, equal to $\frac{1}{4}$ for n = 3, $\frac{1}{8}$ for n = 4, and the same to be equal to 0 for n ≥ 5. While expected utility maximization with an increasing and bounded utility function and IID Bernoulli random variables would theoretically accommodate the "mathematical madness" of the type that confused Nicolas Bernoulli, the subjective probability of the kind that leads to our more sober (gu-)es(s)timates, would recommend updating and evaluating probabilities and then using expected monetary value to evaluate lotteries/uncertain monetary prospects. The conditional probabilities that we have suggested to explain the so-called St. Petersburg paradox, is one among innumerable possibilities, that not only contests the assumption that human understanding of the uncertainty inherent in a "thought experiment" or "demonstration" involving an infinite sequence of coin tosses that are claimed to be fair and independent of one another, is programmed to process the information provided as it is, but goes a step further and challenges the next best assumption, that it may not mimic the behavior of robot that is programmed to interpret the inherent or underlying uncertainty as a Markov process. In fact, we would be inclined to view the St. Petersburg paradox as an example against thoughtless invocation in social science of an infinite sequence of IID random variables with a well-behaved probability distribution or the next best possibility of a Markovian stochastic process, rather than a reason for expected utility theory. By this, we do not wish to cast aspersions on or deny "the possibly immense significance" of expected utility theory. Not at all. We simply do not view the St. Petersburg paradox as a valid argument against using expected monetary value as a reasonable evaluator of uncertain prospects, wherever it is applicable.

For the uncertain prospect in the St. Petersburg paradox, the rational decision would be to offer no more than a dollar or two and that too for the sake of some entertainment, which is what most people would do.

On the other hand, a very realistic example (considerably more realistic than the St. Petersburg Paradox) about a hypothetical individual by the name of Sempronius in section 1.1 of Eeckhoudt et al. (2005), shows the need for using expected utility instead of expected monetary value to evaluate the worth of uncertain monetary prospects.

An apparent violation of expected utility that may challenge the confidence of mathematicians in "mathematical probabilities" but can be easily explained using subjective probabilities is available in Lahiri and Sikdar (2020), where we consider the example of a uniformly distributed random variable on the closed interval [0, 1]. The first alternative on offer is a "huge" monetary loss if the random variable realizes a rational number and a comparatively small monetary gain otherwise. For the same random, there is another alternative available which yields a modest monetary gain less than "comparatively small monetary gain" mentioned earlier if the random variable realizes an irrational number and nothing otherwise. How such a random variable could be operationalized is also discussed in Lahiri and Sikdar (2020). In fact, operationalizing a random variable which is uniformly distributed on the closed interval [0, 1], may turn out to be easier than operationalizing an unending sequence of IID tosses of an "unbiased coin".

Most individuals—including mathematicians—rejected the first alternative considering it to be too risky, although the mathematical expectation associated with the first alternative was a positive gain greater than the mathematical expectation associated with the second alternative. The comparatively less mathematically trained individuals confessed that they were associating a probability of 1/2 to each of the two outcomes, thereby ending up with a huge expected loss for the first alternative as opposed to a positive expected gain for the second. Clearly, such probabilities disagree with those prescribed by the uniform probability distribution on [0, 1], according to which the probability of realizing a rational number is "0". Even mathematicians preferred to use subjective probabilities that were different from the mathematically "objective" probabilities, in the context of the experiment discussed in Lahiri and Sikdar (2020).

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