Journal of Mathematics and System Science 9 (2019) 115-118
doi: 10.17265/2159-5291/2019.04.003

On the Pair of Equations $x + y = z + w$, $y + z = (x - w)^2$

A. Vijayasankar[1], Sharadha Kumar[1] and M. A. Gopalan[2]

1. Department of Mathematics, National College, Trichy 620 001, Tamil Nadu, India

2. Department of Mathematics, Shrimati Indira Gandhi College, Trichy 620 002, Tamil Nadu, India

Abstract: The system of double equations given by $x + y = z + w$, $y + z = (x - w)^2$, is studied for obtaining its non-zero distinct solutions in integers.

Key words: Double equations, integer solutions, pair of equations with 5 unknowns.

1. Introduction

Systems of indeterminate quadratic equations of the form $ax + c = u^2$, $bx + d = v^2$ where a, b, c, d are non-zero distinct constants, have been investigated for solutions by several authors [1, 2] and with a few possible exceptions, most of them were primarily concerned with rational solutions. Even those existing works wherein integral solutions have been attempted, deal essentially with specific cases only and do not exhibit methods of finding integral solutions is a general form. In Ref. [3], a general form of the integral solutions to the system of equations $ax + c = u^2$, $bx + d = v^2$ where a, b, c, d are non-zero distinct constants is presented when the product ab is a square free integer whereas the product cd may or may not be a square integer. For other forms of system of double diophantine equations, one may refer to Refs. [4-25]. This communication concerns with yet another interesting system of double Diophantine equations namely $x + y = z + w$, $y + z = (x – w)^2$, for its infinitely many non-zero distinct integer solutions.

2. Method of Analysis

Consider the system of double equations:

$$x + y = z + w \tag{1}$$

$$y + z = (x - w)^2 \tag{2}$$

Corresponding author: Mayilrangam A. Gopalan, Ph.D., research fields: diophantine equations and number patterns.

Four different methods of solving Eqs. (1) and (2) are illustrated below.

2.1 Method 1

The introduction of the transformations:

$$x = u + v,\ w = u - v \tag{3}$$

in Eqs. (1) and (2) leads to $z – y = 2v$, $z + y = 4v^2$ from which, on solving, we get:

$$z = 2v^2 + v,\ y = 2v^2 - v \tag{4}$$

Note that Eqs. (3) and (4) satisfy Eqs. (1) and (2).

Properties:

- Each of the following expressions represents a perfect square:

$4xw – (z + y)$

$4xw + (z – y)^2$

- $z^2 - y^2$ is a Cubical integer
- Each of the following expressions represents a Bi-quadratic integer:

– $4zy + z + y$

– $4(zy + z + y) – 3(x – w)^2$

2.2 Method 2

Assume

$$z = u + v,\ y = u - v,\ w = s \tag{5}$$

Substituting Eq. (5) in Eqs. (1) and (2) and simplifying, note that:

$$u = 2v^2 \tag{6}$$

The substitution of Eq. (6) in Eq. (5) leads to:

$$z = 2v^2 + v,\ y = 2v^2 - v,\ w = s \tag{7}$$

Also, from Eq. (1),

$$x = 2v + s \tag{8}$$

Observe that Eqs. (7) and (8) satisfy Eqs. (1) and (2)

Properties:

$6\left((x-w)^4 - 4zy\right)$ is a nasty number

$2\left(z^2 + y^2\right) - (x-w)^2$ is a bi-quadratic integer

$\dfrac{2\left(z^4 - y^4\right)}{(x-w)^2 + 1}$ is a quintic integer

$(2v^2 + v, 2v^2 - v, 8v^2)$ is the diophantine triple with the property $D(v^2)$ as the product of any two members of the set added with v^2 is a perfect square.

$(2v^2 + v, 2v^2 - v, 8v^2 + 3)$ is the special dio-triple with the property $D(v^2 + 1)$ as the product of any two members of the set added with the same members and increased by $v^2 + 1$ is a perfect square.

2.3 Method 3

Assume w is chosen arbitrarily and take:

$$w = s \ (\neq 0) \tag{9}$$

Eliminating x between Eqs. (1) and (2), we have

$z^2 - (2y + 1)z + y^2 - y = 0.$

Treating the above equation as a quadratic in z and solving for z, one gets:

$$z = \frac{1}{2}\left(2y + 1 \pm \sqrt{8y+1}\right) \tag{10}$$

The square-root on the R.H.S of Eq. (10) is eliminated when:

$$y = \frac{n(n+1)}{2} \tag{11}$$

and

$$z = \frac{1}{2}(n+1)(n+2)\ , \ \frac{1}{2}n(n-1) \tag{12}$$

Substituting Eqs. (11), (12) and (9) in Eq. (1), we have:

$$x = s + z - y = \begin{cases} s + n + 1 \\ s - n \end{cases}$$

Thus, there are two sets of solutions to Eqs. (1) and (2) represented as below:

Set: 1

$x = s + n + 1$, $y = t_{3,n}$, $z = t_{3,n+1}$, $w = s$

Set: 2

$x = s - n$, $y = t_{3,n}$, $z = t_{3,n-1}$, $w = s$

where $t_{3,a}$ is the triangular number of rank a.

2.4 Method 4

Consider the transformations:

$$x = p + q,\ y = p - q,\ z = p + r,\ w = p - r \tag{13}$$

where p, q, r are non-zero distinct integers.

Note that Eq. (1) is automatically satisfied. The substitution of Eq. (13) in Eq. (2) leads to

$q^2 + (2r + 1)q + r^2 - r - 2p = 0.$

The above equation is quadratic in q and solving for q, we have:

$$q = \frac{1}{2}\left(-2r - 1 \pm \sqrt{8r + 1 + 8p}\right) \tag{14}$$

The square root on the R.H.S of Eq. (14) is removed by choosing suitably the values of r and p and the corresponding values of q are obtained from Eq. (14). Substituting these values of p, q, r in Eq. (13), the values of x, y, z, w satisfying Eqs. (1) and (2) are obtained. A few examples are given below:

Example: 1

Take

$$r = \frac{s(s+1)}{2}\ ,\ p = \frac{k^2 + (2s+1)k}{2}$$

$$\therefore \quad q = \frac{1}{2}\left(-s^2 + s + 2k\right),\ \frac{1}{2}\left(-s^2 - 3s - 2 - 2k\right)$$

In view of Eq. (13), the corresponding 2 sets of solutions to Eqs. (1) and (2) are as follows:

Set: 1

$$x = \frac{1}{2}\left(k^2 + (2s+1)k - s^2 + s + 2k\right)$$

$$y = \frac{1}{2}\left(k^2 + (2s+1)k + s^2 - s - 2k\right)$$

$$z = \frac{1}{2}\left(k^2 + (2s+1)k + s(s+1)\right)$$

$$w = \frac{1}{2}\left(k^2 + (2s+1)k - s(s+1)\right)$$

Set: 2

$$x = \frac{1}{2}\left(k^2 + (2s+1)k - s^2 - 3s - 2 - 2k\right)$$

$$y = \frac{1}{2}\left(k^2 + (2s+1)k + s^2 + 3s + 2 + 2k\right)$$

$$z = \frac{1}{2}\left(k^2 + (2s+1)k + s(s+1)\right)$$

$$w = \frac{1}{2}\left(k^2 + (2s+1)k - s(s+1)\right)$$

Example: 2

Consider:

$$r = \frac{1}{2}(s^2 + s + 2),\ p = \frac{1}{2}(k^2 + (2s+1)k - 2)$$

$$\therefore \quad q = \frac{1}{2}\left(-s^2 + s + 2k - 2\right),\ \frac{1}{2}\left(-s^2 - 3s - 2k - 4\right)$$

Employing Eq. (13), the corresponding 2 sets of solutions to Eqs. (1) and (2) are as follows:

Set: 3

$$x = \frac{1}{2}\left(k^2 + (2s+1)k - 2 - s^2 + s + 2k - 2\right)$$

$$y = \frac{1}{2}\left(k^2 + (2s+1)k - 2 + s^2 - s - 2k + 2\right)$$

$$z = \frac{1}{2}\left(k^2 + (2s+1)k - 2 + s^2 + s + 2\right)$$

$$w = \frac{1}{2}\left(k^2 + (2s+1)k - 2 - s^2 - s - 2\right)$$

Set: 4

$$x = \frac{1}{2}\left(k^2 + (2s+1)k - 2 - s^2 - 3s - 2k - 4\right)$$

$$y = \frac{1}{2}\left(k^2 + (2s+1)k - 2 + s^2 + 3s + 2k + 4\right)$$

$$z = \frac{1}{2}\left(k^2 + (2s+1)k - 2 + s^2 + s + 2\right)$$

$$w = \frac{1}{2}\left(k^2 + (2s+1)k - 2 - s^2 - s - 2\right)$$

3. Conclusions

In this paper, an attempt has been made to obtain many integer solutions to the pair of equations $x + y = z + w$, $y + z = (x - w)^2$. The authors wish that the researchers of diophantine equations may be motivated in solving other choices of double diophantine equations.

References

[1] Dickson, L. E. 1952. *History of the Theory of Numbers*, Vol. II. New York: Chelsea Publishing Company.

[2] Batta, B., and Singh, A. N. 1938. *History of Hindu Mathematics*, Asia Publishing House.

[3] Mignotte, M., and Petho, A. 1995. "On the System of Diophantine Equations." *Mathematica Scandinavica* 76 (1): 50-60.

[4] Cohn, J. H. E. 1998. "The Diophantine System $x^2 - 6y^2 = -5$, $x = 2z^2 - 1$." *Mathematica Scandinavica* 82 (2): 161-4.

[5] Le, M. H. 2004. "On the Diophantine System $x^2 - Dy^2 = 1 - D$, $x = 2z^2 - 1$." *Mathematica Scandinavica* 95 (2): 171-80.

[6] Anglin, W. S. 1996. "Simultaneous Pell Equations." *Maths. Comp*. 65: 355-9.

[7] Baker, A., and Davenport, H. 1969. "The Equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$." *Quart. Math. Oxford* 20 (2): 129-37.

[8] Walsh, P. G. 1997. "On Integer Solutions to $x^2 - dy^2 = 1$ and $z^2 - 2dy^2 = 1$." *Acta Arith*. 82: 69-76.

[9] Mihai, C. 2007. "Pairs of Pell Equations Having at Most One Common Solution in Positive Integers." *An. St. Univ. Ovidius Constanta* 15 (1): 55-66.

[10] Abu Muriefah, F. S., and Al Rashed, A. 2006. "The Simultaneous Diophantine Equations $y^2 - 5x^2 = 4$ and $z^2 - 442x^2 = 441$." *The Arabian Journal for Science and Engineering* 31 (2A): 207-11.

[11] Gopalan, M. A., and Devibala, S. 2004. "Integral Solutions of the Double Equations $x(y - k) = v^2$, $y(x - h) = u^2$." *IJSAC* 1 (1): 53-7.

[12] Gopalan, M. A., and Devibala, S. 2004. "On the System of Double Equations $x^2 - y^2 + N = u^2$, $x^2 - y^2 - N = v^2$." *Bulletin of Pure and Applied Sciences* 23E (2): 279-80.

[13] Gopalan, M. A., and Devibala, S. 2005. "Integral Solutions of the System $a(x^2 - y^2) + N_1^2 = u^2$, $b(x^2 - y^2) + N_2^2 = v^2$." *Acta Ciencia Indica* XXXIM (2): 325-6.

[14] Gopalan, M. A., and Devibala, S. 2005. "Integral Solutions of the System $x^2 - y^2 + b = u^2$, $a(x^2 - y^2) + c = v^2$." *Acta Ciencia Indica* XXXIM (2): 607.

[15] Gopalan, M. A., and Devibala, S. 2006. "On the System of Binary Quadratic Diophantine Equations $a(x^2 - y^2) + N = u^2$, $b(x^2 - y^2) + N = v^2$." *Pure and Applied Mathematika Sciences* LXIII (1-2): 59-63.

[16] Gopalan, M. A., Vidhyalakshmi S., and Lakshmi, K.

2014. "On the System of Double Equations $4x^2 - y^2 = z^2$, $x^2 + 2y^2 = w^2$." *Scholars Journal of Engineering and Technology (SJET)* 2 (2A): 103-4.

[17] Gopalan, M. A., Vidhyalakshmi, S., and Janani, R. 2016. "On the System of Double Diophantine Equations $a_0 + a_1 = q^2$, $a_0a_1 \pm 2(a_0 + a_1) = p^2 - 4$." *Transactions on Mathematics^{TM}* 2 (1): 22-6.

[18] Gopalan, M. A., Vidhyalakshmi, S., and Nivetha, A. 2016. "On the System of Double Diophantine Equations $a_0 + a_1 = q^2$, $a_0a_1 \pm 6(a_0 + a_1) = p^2 - 36$." *Transactions on Mathematics^{TM}* 2 (1): 41-5.

[19] Gopalan, M. A., Vidhyalakshmi S., and Bhuvaneswari, E. 2016. "On the System of Double Diophantine Equations $a_0 + a_1 = q^2$, $a_0a_1 \pm 4(a_0 + a_1) = p^2 - 16$." *Jamal Academic Research Journal* 279-82.

[20] Meena, K., Vidhyalakshmi S., and Priyadharsini, C. 2016. "On the System of Double Diophantine Equations $a_0 + a_1 = q^2$, $a_0a_1 \pm 5(a_0 + a_1) = p^2 - 25$." *Open Journal of Applied & Theoretical Mathematics (OJATM)* 2 (1): 8-12.

[21] Gopalan, M. A., Vidhyalakshmi S., and Rukmani, A. 2016. "On the System of double Diophantine Equations $a_0 - a_1 = q^2$, $a_0a_1 \pm (a_0 - a_1) = p^2 + 1$." *Transactions on Mathematics^{TM}* 2 (3): 28-32.

[22] Devibala, S., Vidhyalakshmi, S., and Dhanalakshmi, G. 2017. "On the System of Double Equations $N_1 - N_2 = 4k + 2$ ($k > 0$), $N_1N_2 = (2k + 1)a^2$." *International Journal of Engineering and Applied Sciences (IJEAS)* 4 (6): 44-5.

[23] Vidhyalakshmi, S., Gopalan, M. A., and Aarthy Thangam, S. 2017. "Three Special Systems of Double Diophantine Equations." *IJRSR* 8 (12): 22292-6.

[24] Vidhyalakshmi, S., Gopalan, M. A., and Aarthy Thangam, S. 2017. "On the Pair of Diophantine Equations." *IJSIMR* 5 (8): 27-34.

[25] Gopalan, M. A., Vidhyalakshmi, S., and Aarthy Thangam, S. 2018. *Systems of Double Diophantine Equations*. Guntur, AP: KY Publication.