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New Analytic Solution for Ambartsumian Equation

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Abstract: Analytical solution is obtained for Ambartsumian equation in this paper. This equation is of application in astronomy. The obtained solution has many advantages over the published one in the literature as shown by several comparisons.

Key words: Delay equation; analytic solution.

1. Introduction

The current work is devoted to analyzing the first order delay differential equation given by Ref. [1].

$$y'(t) = -y(t) + \frac{1}{q}\, y\left(\frac{t}{q}\right), \quad q > 1, \tag{1}$$

where q is a constant and the initial condition (IC) is

$$y(0) = \lambda, \tag{2}$$

where λ is a further constant. Eq. (1) is called Ambartsumian equation which describes the surface brightness in Astronomy [1, 2]. Uniqueness and existence of this model has been investigated by Kato and McLeod [3]. Patade and Bhalekar [1] solved Eqs. (1) and (2) by using the Daftardar-Gejji method [4]. Their solution was expressed as a power series in the independent variable t and they have also addressed the issue of convergence of such series. However, obtaining analytical solution for the present model is still of practical interest. In the literature [5-12], several analytical methods were used to solve various problems in different areas of researches, however, a direct approach is to be introduced in this paper. In order to obtain the desired solution, an effective ansatz is to be implemented to treat Eq. (1) under the IC Eq. (2). The proposed approach is based on a suitable series substitution in terms of exponential functions with undetermined coefficients. Details of the proposed method are presented in the next section and the advantages over the previous solution by Daftardar-Gejji method [1] will be discussed in a subsequent section.

2. Analysis

First, we rewrite Eq. (1) as

$$y'(t) = -y(t) + \alpha\, y(\alpha t), \quad \alpha = \frac{1}{q}. \tag{3}$$

Here, the solution of Eq. (3) is assumed as

$$y(t) = \sum_{n=0}^{\infty} a_n(\alpha) e^{-c\,\alpha^n t}, \tag{4}$$

where c is unknown. Accordingly, we have

$$y'(t) = \sum_{n=0}^{\infty} -c\alpha^n a_n(\alpha) e^{-c\,\alpha^n t}, \tag{5}$$

and

$$y(\alpha t) = \sum_{n=0}^{\infty} a_n(\alpha) e^{-c\,\alpha^{n+1} t}. \tag{6}$$

Inserting Eqs. (4)-(6) into Eq. (3), yields

$$\sum_{n=0}^{\infty} -c\,\alpha^n a_n\, e^{-c\alpha^n t} = -\sum_{n=0}^{\infty} a_n\, e^{-c\alpha^n t} + \sum_{n=0}^{\infty} \alpha\, a_n\, e^{-c\alpha^{n+1} t}, \tag{7}$$

or

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$$a_0(1-c)e^{-ct} + \sum_{n=0}^{\infty}\left[\left(1-c\,\alpha^{n+1}\right)a_{n+1}-\alpha\,a_n\right]e^{-c\alpha^{n+1}t}=0, \tag{8}$$

which gives

$$a_0(1-c)=0, \qquad a_0\neq 0, \tag{9}$$

$$\left(1-c\,\alpha^{n+1}\right)a_{n+1}-\alpha\,a_n=0. \tag{10}$$

Therefore, $c=1$ and

$$a_{n+1}=\left(\frac{\alpha}{1-c\,\alpha^{n+1}}\right)a_n$$

$$=\left(\frac{\alpha}{1-\alpha^{n+1}}\right)a_n, \qquad n\geq 0. \tag{11}$$

Accordingly,

$$a_1=\left(\frac{\alpha}{1-\alpha^1}\right)a_0,$$

$$a_2=\left(\frac{\alpha^2}{(1-\alpha^1)(1-\alpha^2)}\right)a_0=a_0\left(\frac{\alpha^2}{\prod_{k=1}^{2}\left(1-\alpha^k\right)}\right),$$

$$a_3=\left(\frac{\alpha^3}{(1-\alpha^1)(1-\alpha^2)(1-\alpha^3)}\right)a_0=a_0\left(\frac{\alpha^3}{\prod_{k=1}^{3}\left(1-\alpha^k\right)}\right),$$

$$a_4=\left(\frac{\alpha^4}{(1-\alpha^1)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)}\right)a_0$$

$$=a_0\left(\frac{\alpha^4}{\prod_{k=1}^{4}\left(1-\alpha^k\right)}\right),$$

$$a_m=\left(\frac{\alpha^m}{(1-\alpha^1)(1-\alpha^2)(1-\alpha^3)\ldots(1-\alpha^m)}\right)a_0$$

$$=a_0\left(\frac{\alpha^m}{\prod_{k=1}^{m}\left(1-\alpha^k\right)}\right), \qquad m\geq 1. \tag{12}$$

Hence

$$y(t)=a_0e^{-t}+\sum_{n=1}^{\infty}a_ne^{-\alpha^n t},$$

i.e,

$$y(t)=a_0e^{-t}+a_0\sum_{n=1}^{\infty}\left(\frac{\alpha^n}{\prod_{k=1}^{n}\left(1-\alpha^k\right)}\right)e^{-\alpha^n t}, \tag{13}$$

On using the initial condition in Eq. (2), a_0 is obtained as

$$a_0=\frac{\lambda}{1+\sum_{n=1}^{\infty}\left(\frac{\alpha^n}{\prod_{k=1}^{n}\left(1-\alpha^k\right)}\right)}. \tag{14}$$

Therefore, the final closed form solution is obtained by inserting the value of a_0 in Eq. (14) into Eq. (13) as

$$y(t)=\lambda\left(\frac{e^{-t}+\sum_{n=1}^{\infty}\left(\frac{\alpha^n}{\prod_{k=1}^{n}\left(1-\alpha^k\right)}\right)e^{-\alpha^n t}}{1+\sum_{n=1}^{\infty}\left(\frac{\alpha^n}{\prod_{k=1}^{n}\left(1-\alpha^k\right)}\right)}\right). \tag{15}$$

This closed form solution can be used to extract an approximate solution by taking m -terms of the series instead of infinity. Consequently, the approximate solution $\Phi_m(t)$ is given by

$$\Phi_m(t)=\lambda\left(\frac{e^{-t}+\sum_{n=1}^{m}\left(\frac{\alpha^n}{\prod_{k=1}^{n}\left(1-\alpha^k\right)}\right)e^{-\alpha^n t}}{1+\sum_{n=1}^{m}\left(\frac{\alpha^n}{\prod_{k=1}^{n}\left(1-\alpha^k\right)}\right)}\right). \tag{16}$$

3. Results and Discussion

In this section, the effectiveness of the present analytic approximate solution $\Phi_m(t)$ will be declared when compared with the previous ones in the literature [1]. The solution obtained by Patade and Bhalekar [1] was in the form:

$$y(t) = \lambda\left[1 + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\left(q^{-k}-1\right)\right)\frac{t^n}{n!}\right], \qquad (17)$$

which can be approximated as

$$\Psi_r(t) = \lambda\left[1 + \sum_{n=1}^{r}\left(\prod_{k=1}^{n}\left(q^{-k}-1\right)\right)\frac{t^n}{n!}\right]. \qquad (18)$$

Our task is now to compare between the present m-term approximate solution $\Phi_m(t)$ in Eq. (16) and the published r-term approximate solution $\Psi_r(t)$ in Eq. (18). At first, we declare that the sequence Eq. (16) is convergent and valid in a wider range than Ref. [1]. In addition, we will use the residual $|RE_m(t)|$ defined by

$$\left|RE_m(t)\right| = \left|\Phi'_m(t) + \Phi_m(t) - \alpha\,\Phi_m(\alpha\, t)\right|, m \ge 1, \qquad (19)$$

to validate our results. Also, the residual $|RE_m(t)|$ in Eq. (19) of the solution Eq. (16) will be compared with the following residual $|RE_r(t)|$ of the solution Eq. (18) in Eq. (20) as

$$\left|RE_r(t)\right| = \left|\Psi'_r(t) + \Psi_r(t) - \alpha\,\Psi_r(\alpha\, t)\right|, \quad r \ge 1. \qquad (20)$$

Let us first demonstrate that $\Phi_n(t)$ is convergent. Fig. 1 shows the convergence of the present approximate solutions $\Phi_{10}(t)$, $\Phi_{12}(t)$, $\Phi_{14}(t)$ and $\Phi_{16}(t)$ at $\lambda = 1$ and $q = 1.4$. Moreover, they are valid for all *t*. While the approximate ones in literature [1]; $\Psi_{40}(t)$, $\Psi_{60}(t)$, $\Psi_{80}(t)$ and $\Psi_{100}(t)$ are valid in sub-intervals as shown from Fig. 2.

Besides, Fig. 3 also demonstrates the convergence for $\lambda = 1$ and at a relatively lower value of q ($q = 1.2$), where the number of terms increases as $q \to 1^+$.

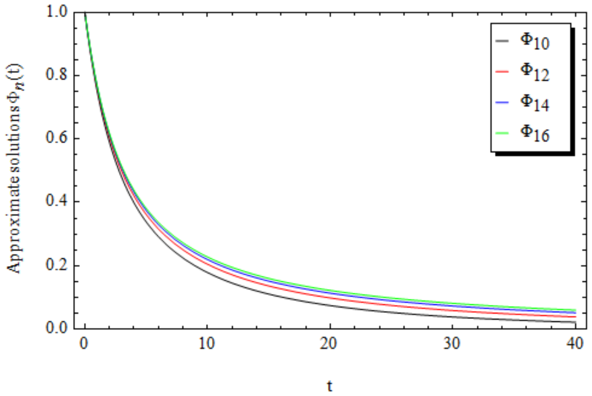

Fig. 1 Convergence of the solution Eq. (16) at $\lambda = 1, q = 1.4$.

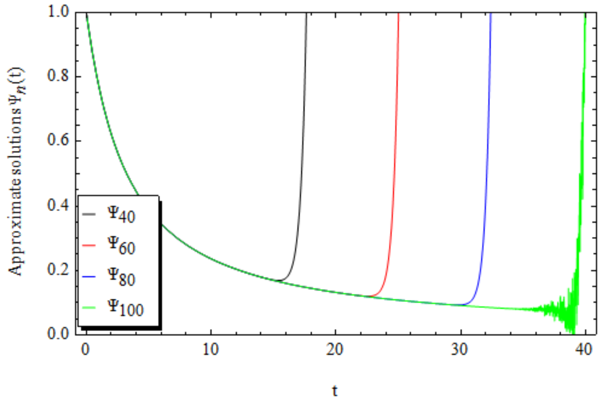

Fig. 2 The approximate solutions Eq. (18) in Ref. [1] at $\lambda = 1, q = 1.4$.

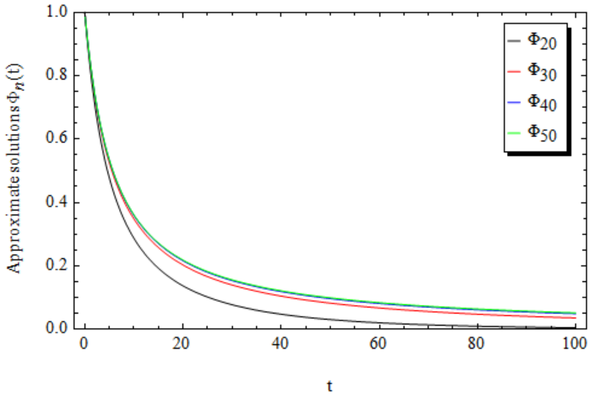

Fig. 3 Convergence of the present approximate solutions Eq. (16) at $\lambda = 1$ **and a lower value** $q = 1.2$.

To confirm the efficiency of our approach over the previous one [1], a comparison is displayed in Fig. 4 between the present approximate solution by using only 20 terms; $\Phi_{20}(t)$ and the previous approximate

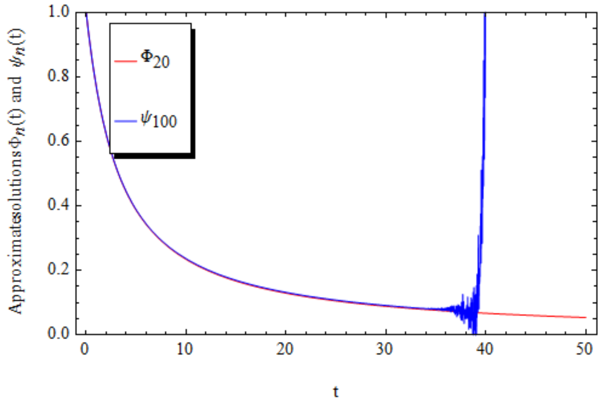

Fig. 4 **Comparison between the present 20 terms and the approximate solution Eq. [1] by using 100 terms at** $\lambda = 1, q = 1.4$.

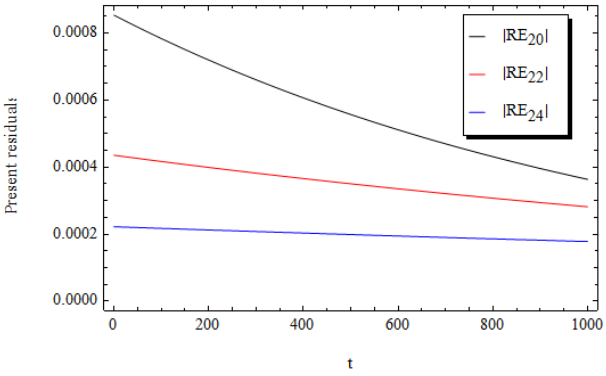

Fig. 5 **The present residuals at** $\lambda = 1, q = 1.4$.

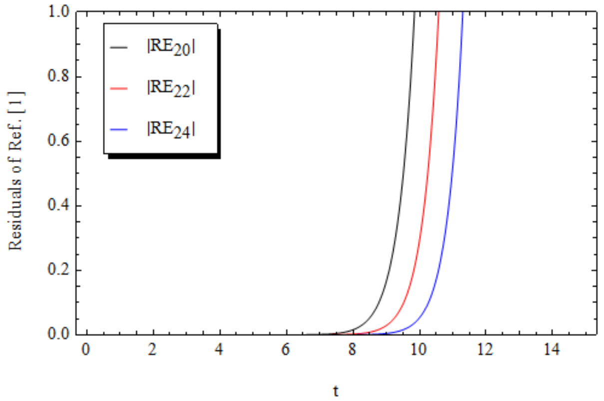

Fig. 6 **The residuals of Ref. [1] at** $\lambda = 1, q = 1.4$.

solution by using 100 terms; $\Psi_{100}(t)$. It is clear from this figure that the developed approach not only is valid in the whole domain but also requires a fewer number of terms. For a further validation, plots for the residuals $|RE_{20}|$, $|RE_{22}|$, and $|RE_{24}|$ versus t at $\lambda = 1, q = 2$ in Figs. 5 (obtained from Eq. (19)) and 6 (obtained from Eq. (20)), respectively. The results reveal that the present residuals (Fig. 5) are much better than those of Ref. [1] (Fig. 6). In addition, the present approximations using only few terms are much accurate than those previously obtained in Ref. [1], even at large q in the whole domain.

4. Conclusions

The Ambartsumian equation was analytically solved. The proposed method was based on an effective ansatz in terms of exponential order of the independent variable. The comparisons between the present solution and the standard power series in the literature reveal that our approach is not only simple but also effective to achieving higher accuracy.

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