

# On The Cauchy Problem For Some Parabolic Fractional Partial Differential Equations With Time Delays

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**Abstract:** The Cauchy problem for some parabolic fractional partial differential equation of higher orders and with time delays is considered. The existence and unique solution of this problem is studied. Some smoothness properties with respect to the parameters of these delay fractional differential equations are considered.

**Keywords:** Cauchy problem- fractional partial differential equations with time delays- successive approximations.

## 1. Introduction

We will adhere the following notations,  $R^n$  is  $n$ -dimensional Euclidean space,  $x, y$  are elements of this space,  $q = (q_1, q_2, \dots, q_n)$  is a multi-index,

$$D_x = \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}$$

$$D_x^q = \frac{\partial^{|q|}}{\partial x_1^{q_1} \partial x_2^{q_2} \dots \partial x_n^{q_n}}$$

$$|q| = q_1 + q_2 + \dots + q_n$$

$$D_t = \frac{\partial}{\partial t}, |x|^2 = x_1^2 + x_2^2 + \dots + x_n^2$$

Consider the parabolic fractional partial differential equations with time delay of the form

$$D_t^\alpha u(x, t, p) + \sum_{|q| \leq 2m} a_q(x, t) D_x^q u(x, t, p) = \sum_{j=1}^k \sum_{|q| < 2m} b_{qj}(x, t) D_x^q u(x, t - p_j, p) + h(x, t) \quad (1.1)$$

$$0 < \alpha \leq 1, 0 < t \leq T, p = (p_1, \dots, p_k), 0 < p_j < \nu < T, j = 1, 2, \dots, k,$$

where  $\nu$  is a given positive number

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Let us suppose that  $u$  satisfies the Cauchy condition,

$$u(x, 0, p) = f(x) \quad (1.2)$$

Where  $f$  is a given function

Let  $C^r(Q)$  denote the set of all functions defined in a neighborhood of each point of a domain  $Q \subset R^l$ , and having bounded continuous partial derivatives of all orders less than or equal  $r$  in  $Q$  ( $r$  denotes a nonnegative integer and  $l$  a positive integer).

We assume that  $f \in C^{2m-1}(R^n)$ . Let  $u$  satisfy

$$u(x, t, p) = F(x, t), (x, t) \in Q_1 \quad (1.3)$$

$$Q_1 = \{(x, t): x \in R^n, -\nu < t < 0, 0 < p_j < \nu\}$$

Where  $F$  is a given bounded continuous function on  $Q_1$ .

Assume that  $F \in C^{2m-1}(R^n)$  for every  $t \in (-\nu, 0)$ , and

$$F(x, 0) = f(x), x \in R^n \quad (1.4)$$

The linear partial differential operator  $\sum_{|q|=2m} a_q(x, t) D_x^q$  is supposed to be uniform elliptic on  $R^n$ , i.e. there exists a positive number  $\lambda$  such that for every

$$(x, t) \in Q_2 = \{(x, t): x \in R^n, 0 \leq t \leq T\},$$

and for every  $y \in R^n$ ,

$$(-1)^m \sum_{|q|=2m} a_q(x, t) y^q \geq \lambda |y|^{2m} \quad (1.5)$$

Where the number  $\lambda$  is independent of  $x, t$  and  $y$ .

It is suitable to rewrite the problem (1.1), (1.2) in the form

$$\begin{aligned}
 u(x, t, p) &= f(x) + \\
 \frac{1}{\Gamma(\alpha)} \int_0^t (t - \eta)^{\alpha-1} \sum_{|q| \leq 2m} a_q(x, t) D_x^q u(x, \eta, p) d\eta &= \\
 \frac{1}{\Gamma(\alpha)} \int_0^t (t - \eta)^{\alpha-1} \sum_{j=1}^k \sum_{|q| < 2m} b_{qj}(x, \eta) D_x^q u(x, \eta, p_j, p) d\eta &
 \end{aligned}
 \tag{1.6}$$

Where  $\Gamma$  is Gamma function

Following Friedman [11], we assume the following conditions.

(C<sub>1</sub>) All the coefficients  $a_q, |q| \leq 2m$ , are continuous and bounded functions on  $Q_2$ . (C<sub>2</sub>) All the coefficients  $a_q, |q| \leq 2m$ , satisfy a Holder condition with respect to uniformly in  $Q_2$ , namely there exists a positive constant  $K$  and a constant  $\gamma \in (0, 1)$  such that

$$|a_q(x, t) - a_q(y, t)| \leq K|x - y|^\gamma, \text{ for all } x, y \in R^n, t \in [0, T]$$

(C<sub>3</sub>) The coefficients  $a_q, |q| = 2m$ , of the principle term are uniformly continuous as functions of  $t$  relative to  $(x, t)$  in  $Q_2$ .

(C<sub>4</sub>)  $D_x^\beta a_q \in C(Q_2)$ ,  $\beta = (\beta_1, \dots, \beta_n)$  is a multi-index  $|\beta| \leq 2m, |q| \leq 2m$ , and

$C(Q_2)$  is the set of all continuous bounded functions on  $Q_2$ .

We assume also that coefficients  $b_{qj}, |q| \leq 2m$ , relative to iple term are uniformly continuous as functions of  $j = 1, 2, \dots, k$ , satisfy a Holder condition with respect to  $x$ , (similar to cond. (C<sub>2</sub>)) and  $D_x^\beta b_{qj} \in C(Q_2), |\beta| < 2m$ .

In section 2, we study the existence and uniqueness of the solution of the Cauchy problem (1.1), (1.2). In section 3, we consider the dependence of the parameters  $p_1, \dots, p_k$  of the solution of the considered

Cauchy problem, compare [1], [10], [13], and [14].

## 2. Uniqueness and Existence Theorems

Consider the Cauchy problem

$$D_t^\alpha w(x, t) + \sum_{|q| \leq 2m} a_q(x, t) D_x^q w(x, t) = 0
 \tag{2.1}$$

$$w(x, 0) = f(x)
 \tag{2.2}$$

It is well known under the conditions (C<sub>1</sub>) – (C<sub>4</sub>) that the solution of the Cauchy problem (2.1), (2.2) represented in the form [7]

$$w(x, t) = \int_0^\infty \int_{-\infty}^\infty G(x - y, t^\alpha \theta) f(y) \zeta_\alpha(\theta) dy d\theta
 \tag{2.3}$$

$\zeta_\alpha(\theta)$  is a probability density function define on  $(0, \infty)$

$G$  is the fundamental solution of equation (2.1), when  $\alpha = 1$ , which has the following conditions [11].

$$(C_5) D_x^q G, D_y^q G, D_t G \in (Q_2 \times Q_2), |q| \leq 2m$$

$$(C_6) |D_x^q D_y^B G(x, t, y, \theta)| \leq \frac{Kz(x - y, t - \theta)}{(t - \theta)^{v_1}}$$

Where  $z(x - y, t - \theta) = \exp[-v_2(|x - y|^{2m}/t - \theta)^{\frac{1}{2m-1}}]$ ,  $(0 < \theta < t)$

$v_1 = 1/2m[|q| + |\beta| + n]$ ,  $K, v_2$  are positive constants and  $|q| < 2m, |\beta| \leq 2m$ .

$$\begin{aligned}
 (C_7) |D_x^q [G(x, t, y, \theta) - G(z, t, y, \theta)]| \\
 \leq \frac{K|x - z|^{v_3}}{(t - \theta)^{v_4}} \Lambda,
 \end{aligned}$$

Where  $\Lambda = \exp[-v_5(|x - y|^{2m}/t - \theta)^{\frac{1}{2m}}] + \exp[-v_5(|z - y|^{2m}/t - \theta)^{\frac{1}{2m}}]$ ,

$v_4 = 1/2m(|q| + v_3 + n)$ ,  $v_3$  is an arbitrary positive number  $\leq 1$ ,  $K$  and  $v_5$  are positive constants and  $|q| < 2m, 0 < \theta < t$ .

$$\text{Let } P = \{p \in R^k : 0 < p_i < v, i = 1, \dots, k\}$$

$$Q_3 = \{(x, t) : x \in R^n, -v < t < T\}$$

We prove now the following uniqueness theorem.

Theorem 2.1: Let  $u$  be a function defined on  $Q_3 \times P$ . Let  $D_t u, D_x^q G \in C(Q_3), |q| \leq 2m$ , for every

fixed  $p \in P$ . Let  $F = 0$  on  $Q_1$ . If  $u$  is a solution of the Cauchy problem (1.1), (1.2) on  $Q_2$ , which satisfies the condition (1.3) on  $Q_1$ . Then the solution is unique

**Proof:** We set  $(x, 0, p) = 0, h(x, t) = 0, x \in R^n, p \in P, t > 0$ .

In this case we must prove  $u(x, t, p) = 0$ , for sufficiently small  $t > 0, x \in R^n, p \in P$

Using  $(C_5), (C_6)$  of the fundamental solution  $G$ , we get

$$\int_{-\infty}^{\infty} G(x, t, y, \theta) D_y^q u(y, \theta, p) dy = \int_{-\infty}^{\infty} (-1)^{|q|} u(y, \theta, p) D_y^q G(x, t, y, \theta) dy$$

And so there exists a number  $c \in (0, 1)$  such that

$$\sup_x \left| \int_{-\infty}^{\infty} G(x, t, y, \theta) D_y^q u(y, \theta, p) dy \right| \leq \frac{K}{(t-\theta)^c} \|u\|, \quad t > \theta \tag{2.4}$$

Where  $\|u\| = \sup_x |u(x, t, p)|, |q| < 2m, K$  is a positive constant. According to

the conditions imposed on the considered problem, we can write

$$u(x, t, p) = \alpha \sum_{j=1}^k \sum_{|q| < 2m} \int_{\eta_j(t)}^t \int_0^{\infty} \int_{-\infty}^{\infty} \theta \zeta_{\alpha}(\theta) (t-\eta)^{\alpha-1} G(x-y, (t-\eta)^{\alpha\theta}) \times b_{qj}(y, \eta) D_y^q u(y, \eta-p_j, p) dy d\theta d\eta$$

Where

$$\eta_j(t) = \begin{cases} t & t \leq p_j \\ p_j & t > p_j \end{cases}$$

Thus

$$u(x, t, p) = \alpha \sum_{j=1}^k \sum_{|q| < 2m} \int_{\eta_j(t)}^t \int_0^{\infty} \int_{-\infty}^{\infty} (-1)^{|q|} u(y, \eta-p_j, p) \theta \zeta(\theta) (t-\eta)^{\alpha-1} \times D_y^q [b_{qj}(y, \eta) G(x-y, (t-\eta)^{\alpha\theta})] dy d\theta d\eta$$

Since  $D_x^q b_{qj} \in C(Q_2), |q| < 2m, j = 1, \dots, k$ .

It follows by using (2.4) that

$$g(t, p) \leq \alpha k \sum_{j=1}^k \int_{\eta_j(t)}^t \int_0^{\infty} \theta (t-\eta)^{-c\alpha} \theta^{-c} (t-\eta)^{\alpha-1} \zeta_{\alpha}(\theta) g(\eta-p_j, p) d\theta d\eta \leq \alpha k \sum_{j=1}^k \int_{\eta_j(t)}^t \left[ \int_0^{\infty} \theta^{1-c} \zeta_{\alpha}(\theta) d\theta \right] (t-\eta)^{(1-c)\alpha-1} g(\eta-p_j, p) d\eta$$

where

$$g(t, p) = \|u\| = \sup_x |u(x, t, p)|$$

Thus

$$\xi(t) \leq \frac{\alpha K}{\Gamma((1-c)\alpha + 1)} \sum_{j=1}^k \int_0^{t-p_j} (t-\eta)^{(1-c)\alpha-1} \xi(\eta) d\eta$$

for

$$t > p_j \tag{2.5}$$

Where  $\xi(t) = \sup_p g(t, p)$

If  $t \leq p_j, j = 1, 2, \dots, k$ , there is nothing to prove.

Suppose  $t > p_j$ , then from equation (2.5)

$$\xi(t) \leq \frac{\alpha K}{\Gamma((1-c)\alpha + 1)} \int_0^t (t-\eta)^{(1-c)\alpha-1} \xi(\eta) d\eta$$

It is easy to get:

$$\xi(t) \leq \frac{K^* M^n t^{n\delta} [\Gamma(\delta)]^{n-1}}{\Gamma(n\delta + 1)}$$

for all

$$n = 1, 2, \dots$$

Where  $M = \sup_t \xi(t), \delta = (1-c)\alpha, K^*$  is a positive integer, letting  $n \rightarrow \infty$ , we get

$$\xi(t) \equiv 0, \text{ for all } t.$$

We prove now the following existence theorem.

Theorem 2.2: There exists a unique function  $u,$

$$u, D_t u, D^q u \in C(Q_3), |q| \leq 2m$$

Such that  $u$  represents the unique solution of the Cauchy problem (1.1), (1.2) on  $Q_2$ , which satisfies the condition (1.3) on  $Q_1$ .

**Proof:** Let us try to find the solution of the integro-partial differential equation.

$$u = H + Eu \tag{2.6}$$

Where  $H$  is a given function defined on  $Q_2$  by,

$$H(x, t, p) = \int_0^\infty \int_{-\infty}^\infty G(x, y, t^\alpha \theta) f(y) \zeta_\alpha(\theta) dy d\theta +$$

$$\alpha \int_0^t \int_0^\infty \int_{-\infty}^\infty \theta \zeta_\alpha(\theta) (t - \eta)^{\alpha-1} G(x - y, (t - \eta)^\alpha \theta) h(y, \eta) dy d\theta d\eta +$$

$$\alpha \sum_{j=1}^k \sum_{|q| < 2m} \int_0^{\eta_j(t)} \int_0^\infty \int_{-\infty}^\infty (-1)^{|q|} F(y, \eta - p_j, p) \theta \zeta_\alpha(\theta) (t - \eta)^{\alpha-1} \times D_y^q [b_{qj}(y, \eta) G(x - y, (t - \eta)^\alpha \theta)] dy d\theta d\eta$$

And E is an integro-partial differential difference operator defined on  $C(Q_2)$  by  $Eu = u^*$ ,

$$u^*(x, t, p) = \alpha \sum_{j=1}^k \sum_{|q| < 2m} \int_0^t \int_0^\infty \int_{-\infty}^\infty (-1)^{|q|} u(y, \eta - p_j, p) \theta \zeta(\theta) (t - \eta)^{\alpha-1} \times D_y^q [b_{qj}(y, \eta) G(x - y, (t - \eta)^\alpha \theta)] dy d\theta d\eta$$

We apply the method of successive approximation to solve (2.7), to do this, set

$$u_{r+1} = H + Eu_r, \quad r = 1, 2, \dots$$

Where the zero approximation  $u_0$  is taken to be zero. The linearity of the operator  $E$  leads to the following equation

$$u_{r+1} - u_r = E(u_r - u_{r-1}),$$

using the conditions imposed on the coefficients and the properties of the fundamental solution  $G$  of equation (2.2), we get by a similar method of the proof of theorem (2.1)

$$\|u_{r+1} - u_r\| \leq \frac{K t^{\delta r} [\Gamma(\delta)]^{r-1}}{\Gamma(\delta r + 1)}$$

where  $K$  is a positive constant,  $\delta = (1 - c)\alpha$ .

Thus the required solution of equation (2.6) is given by

$$u = \sum_{r=1}^\infty (u_r - u_{r-1})$$

This series is absolutely and uniformly convergent on  $Q_2$  to the function  $u \in C(Q_2)$ .

To prove the smoothness of the function  $u$ , we

consider the following equation

$$V_i = D_{x_i} H + E u_i + P_i V_i \tag{2.7}$$

Where

$$E u_i = u_i^*, P_i V_i = V_i^*,$$

$$u_i^*(x, t, p) = \alpha \sum_{j=1}^k \int_{\eta_j(t)}^t \int_0^\infty \int_{-\infty}^\infty \theta \zeta_\alpha(\theta) (t - \eta)^{\alpha-1} b_{0, \dots, 0, j}(y, \eta) u(y, \eta - p_j, p) \times D_{x_i} [G(x - y, (t - \eta)^\alpha \theta)] dy d\theta d\eta$$

$$V_i^*(x, t, p)$$

$$= \alpha \sum_{j=1}^k \sum_{1 \leq |q| \leq 2m} \int_{\eta_j(t)}^t \int_0^\infty \int_{-\infty}^\infty (-1)^{|q|-1} V_i(y, \eta - p_j, p) \theta \zeta(\theta)$$

$$\times (t - \eta)^{\alpha-1} D_{x_i} D_y^{q-e_i} [b_{qj}(y, \eta) G(x - y, (t - \eta)^\alpha \theta)] dy d\theta d\eta$$

$$D_{x_i} = \frac{\partial}{\partial x_i}, e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), e_n = (0, 0, \dots, 1)$$

are the unit vectors

If the solution  $V_i$  of equation(2.7) exists in the space  $C(Q_2)$ , then the partial derivative  $D_{x_i} u$  exists for every  $i = 1, \dots, n$  and

$$V_i = D_{x_i} u \in C(Q_2)$$

Using a similar technique to that used to prove the existence of solution of equation (2.6) in the space  $C(Q_2)$  and noting that  $u$  is a known function in the space  $C(Q_2)$  and using the conditions  $(C_5)$  and  $(C_6)$  of the function  $G$ , we deduce that equation (2.7) has a unique solution  $V_i$  in the space  $C(Q_2)$ , for every  $i = 1, \dots, n$ .

And so  $V_i = D_{x_i} u \in C(Q_2)$ , compare [2]

By induction we can prove that  $D_{x_i}^q u \in C(Q_2)$  for  $|q| < 2m$ .

Now to prove that  $D_x^q \in C(Q_2)$  for  $|q| = 2m$ , we need to prove first all the partial derivatives  $D_x^q u$  satisfy for  $|q| < 2m$ , a Holder continuity condition with respect to  $x$ .

To do this, we notice that

$$D_x^\beta u(x, t, p) - D_z^\beta u(z, t, p) = D_x^\beta H(x, t, p) - D_z^\beta H(z, t, p) +$$

$$\sum_{j=1}^k \sum_{|q| < 2m} \int_{\eta_j(t)}^t \int_0^\infty \int_{-\infty}^\infty \theta \zeta(\theta) (t - \eta)^{\alpha-1} b_{qj}(y, \eta) \times D_y^q u(y, \eta, p) [D_x^\beta G(x - y, (t - \eta)^\alpha \theta) - D_z^\beta G(z - y, (t - \eta)^\alpha \theta)] dy d\theta d\eta$$

Where  $|\beta| < 2m$

Using condition (C7) of the function G and remembering the conditions imposed on the coefficients of equation (1.1) we can prove that there is a constant  $\gamma \in (0, 1)$  such that

$$|D_x^\beta u(x, t, p) - D_z^\beta u(z, t, p)| \leq K |x - z|^\gamma \text{ for all } x, z \in R^n$$

Where the positive constant K, is independent of  $x, z$  and  $t$ .

This completes the proof of the theorem.

### 3. Properties of Smoothness

In this section we consider the dependence on the parameters  $p_1, \dots, p_k$ . We shall deduce some smoothness properties of the solution  $u(x, t, p)$  with respect to  $p$ , compare [10].

Theorem 3.1: If  $F$  has continuous bounded derivatives  $D_i^r F$  on  $Q_1$ , then  $u$  has continuous bounded derivatives

$$D_{p_i}^r u \text{ on } Q_2 \times P, D_{p_i}^r = \frac{\partial}{\partial p_i}, r = 1, 2, \dots, i = 1, \dots, k$$

Proof: For a fixed  $t \in (0, T]$  and a fixed  $x \in R^n$ , we consider the following equation

$$D_t^\alpha V_{r_i}(x, t, p) + \sum_{|q| \leq 2m} a_q(x, t) D_x^q V_{r_i}(x, t, p) = \sum_{j=1}^k \sum_{|q| < 2m} b_{qj}(x, t) D_x^q V_{r_i}(x, t - p_j, p) \quad (3.1)$$

Where  $p \in P$ ,

$$V_{r_i}(x, t - p_j, p) = \begin{cases} D_{p_i}^r F(x, t - p_i), & p_i \geq t, i = j \\ 0, & p_i \geq t, i \neq j \end{cases} \text{ for some fixed } i = 1, \dots, k$$

Assume that

$$V_{r_i}(x, 0, p) = 0 \quad (3.2)$$

Equation (3.1) can be obtained from formally from

equation (1.1) by applying formally the partial differential operator  $D_{p_i}^r$ .

If the Cauchy problem (3.1), (3.2) has a solution, then we deduce immediately that

$$D_{p_i}^r u = V_{r_i} \in C(Q_2 \times P).$$

The Cauchy problem (3.1), (3.2) can be transformed to

$$V_{r_i}(x, t, p) = \alpha \sum_{|q| < 2m} \int_0^{\eta_j(t)} \int_0^\infty \int_{-\infty}^\infty (-1)^{|q|} D_{p_i}^r F(y, \eta - p_i) \theta \zeta_\alpha(\theta) (t - \eta)^{\alpha-1} \times D_y^q [b_{qj}(y, \eta) G(x - y, (t - \eta)^\alpha \theta)] dy d\theta d\eta + \alpha \sum_{j=1}^k \sum_{|q| \leq 2m} \int_{\eta_j(t)}^t \int_0^\infty \int_{-\infty}^\infty (-1)^{|q|} V_{r_i}(y, \eta - p_j, p) \theta \zeta_\alpha(\theta) (t - \eta)^{\alpha-1} \times D_y^q [b_{qj}(y, \eta) G(x - y, (t - \eta)^\alpha \theta)] dy d\theta d\eta \quad (3.3)$$

Similar to theorem (2.2) we see that equation (3.3) satisfies the required. It can be also proved that

$$\frac{\partial u(x, t, p)}{\partial p_i \partial p_j} = 0, \text{ for } i \neq j$$

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