

# Application Of Jessen's Type Inequality For Positive $C_0$ -Semigroup Of Operators

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Recently in [4], the Jessen's type inequality for normalized positive  $C_0$ -semigroups is obtained. In this note, we present few results of this inequality, yielding Hölder's Type and Minkowski's type inequalities for corresponding semigroup. Moreover, a Dresher's type inequality for two-parameter family of means, is also proved.

*Keywords:* Mean inequalities, Positive semigroup of operators, Hölder's Type Inequality, Minkowski's Type Inequality, Dresher's Type Inequality.

# Introduction

In last few years the "Type" functional inequalities and their applications have been addressed extensively by several authors like [2, 6, 9]. Researchers have great interest in this field due to vast applications of these inequalities. In tejti, the authors have derived a Jessen's type inequality for normalized positive  $C_0$ -semigroup of operators. The classical Jessen's inequality has a wide theory of its applications in the field of inequalities and analysis.

In the presented note the authors established certain applications of Jessen's type inequality to obtain mean-inequalities and functional inequalities for normalized positive  $C_0$ -semigroup of operators defined on a Banach lattice algebra. These resultstake the form of Hölder's type and Minkowski's type inequalities. Then finally in the last section a Dresher's type inequality is established for two-parameter family of means.

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## **Preliminaries and Definitions**

In this section, we will present some definitions that will be used in the proof of our main results.

**Definition 1.** A (real) vector space V endowed with an ordering  $\geq$ , such that it satisfies

 $O_1: v \le w$  implies  $v + u \le w + u$  for all  $u, v, w \in V$ ,

 $O_2: v \ge 0$  implies  $\lambda v \ge 0$  for all  $v \in V$  and  $\lambda \ge 0$ ,

is known as an ordered vector space (see [8]).

It can be readily seen that  $O_1$  expresses the translation invariance. Therefore, it implies that the ordering of an ordered vector space V can be completely determined by the positive part  $V_+ = \{v \in V : v \ge 0\}$  of V. In other words,  $v \le qw$  if and only if  $w - v \in V_+$ .

The other property  $O_2$ , shows that the positive part of V is a convex set and a cone with vertex 0 (mostly called the *positive cone* of V).

If for any two elements  $v, w \in V$ , a supremum sup(v, w) and an infumum inf(v, w) can be defined, an ordered vector space V becomes a vector lattice. It is understood that the existence of supremum of any two elements in an ordered vectorspace implies the existence of supremum of finite number of elements in V. Moreover,  $v \ge w$  implies  $-v \le -w$ , so the existence of finite infima thus implied.

Here are a few important number of definitions

sup(v, -v) = |v| (absolutevalueofv)  $sup(v, 0) = v^+$  (positivepartofv)  $sup(-v, 0) = v^-$  (negativepartofv).

**Remark 2.** Some compatibility axiom between norm and order is required to move from a vector lattice to a Banach lattice. It is considered in the following short way:

$$|v| \le |w| \quad implies \quad ||v|| \le ||w||. \tag{1}$$

The norm defined on a vector lattice is called as a lattice norm.

Now, we are in position to define a Banach lattice in a formal way.

#### **Definition 3**

A *Banach lattice* is a Banach space V endowed with an ordering  $\leq$ , such that  $(V, \leq)$  is a vector lattice with a lattice norm defined on it.

A Banach lattice transforms to *Banach lattice algebra*, provided  $u, v \in V_+$  implies  $uv \in V_+$ .

A linear mapping T from an ordered Banach space V into itself is *positive* (denoted by:  $T \ge 0$ ) if  $T(v) \in V_+$ , for all  $v \in V_+$ . The set of all positive linear mappings forms a convex cone in the space L(V) of all linear mappings from V into itself, defining the natural ordering of L(V). The absolute value of T, if it

exists, is given by

$$|T|(v) = \sup\{T(u) : |u| \le v\}, (v \in V_+).$$

Thus  $T: V \to V$  is positive if and only if  $|T|(v) \le T(|v|)$  holds for any  $v \in V$ .

**Lemma 4.** [8], *PP-249* A bounded linear operator T on a Banach lattice V is a positive contraction if and only if  $||(Tv)^+|| \le ||v^+||$  for all  $v \in V$ .

An operator A on V satisfies the positive minimum principle if for all  $v \in D(A)_+ = D(A) \cap V_+$ ,  $\varphi \in V_+$ ,

$$\langle v, \varphi \rangle = 0 \quad implies \quad \langle Av, \varphi \rangle \ge 0.$$
 (2)

**Definition 5.** A (one parameter)  $C_0$ -semigroup (or strongly continuous semigroup) of operators on a Banach space X is a family  $\{Z(t)\}_{t\geq 0} \subset B(X)$  such that

(i) Z(s)Z(t) = Z(s+t) for all  $s, t \in \mathbb{R}^+$ .

(ii) Z(0)=I, the identity operator on X.

(iii) for each fixed  $f \in X$ ,  $Z(t)f \to f$  (with respect to the norm on X) as  $t \to 0^+$ .

Where B(X) denotes the space of all bounded linear operators defined on a Banach space X.

**Definition 6.** The (infinitesimal) generator of  $\{Z(t)\}_{t\geq 0}$  is the densely defined closed linear operator  $A: X \supseteq D(A) \to R(A) \subseteq X$  such that

$$D(A) = \{f : f \in X, \lim_{t \to 0^+} A_t f \text{ exists in } X\}$$

$$Af = \lim_{t \to 0^+} A_t f \ (f \in D(A))$$

where, for t > 0,

$$A_t f = \frac{[Z(t) - I]f}{t} \quad (f \in X)$$

A Banach algebra X, with the multiplicative identity element e is called the *unital Banach algebra*.

We shall call the strongly continuous semigroup  $\{Z(t)\}_{t\geq 0}$  defined on X, a normalized semigroup, whenever it satisfies

$$Z(t)(e) = e, \quad for \ all \quad t > 0. \tag{3}$$

The notion of normalized semigroup is inspired from normalized functionals [7].

Let  $\{Z(t)\}_{t\geq 0}$  be a strongly continuous positive semigroup, defined on a Banach lattice V. The positivity of the semigroup is equivalent to

$$|Z(t)v| \le Z(t) |v|, \quad t \ge 0, \quad v \in V.$$

Where for positive contraction semigroups  $\{Z(t)\}_{t\geq 0}$ , defined on a Banach lattice V we have;

$$||(Z(t)v)^+|| \le ||v^+||, \text{ for all } v \in V.$$

The literature presented in [8], guarantees the existence of the strongly continuous positive semigroups and positive contraction semigroups on Banach lattice V with some conditions imposed on the generator of the strongly continuous positive semigroup and the very important amongst them is, that it must always satisfy (2).

A Banach lattice V is said to be Banach Lattice Algebra whenever for  $u, v \in V_+$ ,  $uv \in V_+$  and  $||uv|| \leq ||u|| ||v||$ .

The theory presented in next section, is defined on normalized semigroups of positive linear operators defined on a unital Banach lattice algebra (UBLA) V.

# Hölder's Type and Minkowski's Type Inequalities

In this section, we present several consequences of the Jessen's type inequality for normalized positive  $C_0$ -semigroup defined on a Banach lattice algebra V [4]. The motivation for this paper is from [3], where such results are proved forisotonic linear functionals. These results take the form of Hölder's type and Minkowski's type inequalities.

Let  $D_c(V)$  denotes the set of all differentiable convex operators  $\varphi: V \to V$ .

**Theorem 1.** [4] Let  $\{Z(t)\}_{t\geq 0}$  be the positive  $C_0$ -semigroup on V such that it satisfies (3). For an operator  $\varphi \in D_c(V)$  and  $t \geq 0$ ;

$$\varphi(Z(t)f) \le Z(t)(\varphi f), \quad f \in V.$$
(4)

For a strongly continuous semigroup of linear operators  $\{Z(t)\}_{t\geq 0}$  defined on a Banach lattice X and strictly monotonic continuous operator  $\psi: X \to X$ , we define the generalized mean:

$$M_{\psi}(Z, f, t) := \psi^{-1} \{ Z(t) \psi(f) \}, \quad f \in X.$$
(5)

**Theorem 2.** For a normalized semigroup of positive linear operators  $\{Z(t)\}_{t\geq 0}$  defined on (UBLA) V and strictly monotonic continuous operators  $\psi, \chi: V \to V$ 

$$M_{\mu}(Z, f, t) \le M_{\gamma}(Z, f, t), \quad f \in V, \tag{6}$$

provided either  $\chi$  is increasing and  $\varphi = \chi \circ \psi^{-1}$  is convex or  $\chi$  is decreasing and  $\varphi$  is concave. **Proof:** For  $f \in V$ , we have  $\psi(f), \chi(f) \in V$  and therefore,  $\varphi(\psi(f)) = \chi(f) \in V$ . Thus, if  $\varphi$  is convex, by Jessen's type inequality (4) we have for  $f \in V$ ;

$$\varphi(Z(t)(\psi(f))) \le Z(t)(\varphi(\psi(f)))$$
$$= Z(t)(\chi(f)).$$

Hence, if  $\chi$  is increasing then  $\chi^{-1}$  is also increasing and we finally obtain

$$\chi^{-1}[\varphi(Z(t)(\psi(f)))] \le \chi^{-1}[Z(t)(\chi(f))]$$

and the assertion (6) follows. If  $\varphi$  is concave then  $-\varphi$  is convex and one can obtain the required inequality similarly.

**Definition 3.** [10] Let V be a Banach algebra with unit e. For  $f \in V$ , we define a function log(f) from V to V;

$$log(f) = -\sum_{n=1}^{\infty} \frac{(e-f)^n}{n} = -(e-f) - \frac{(e-f)^2}{2} - \frac{(e-f)^3}{3} - \dots$$

for  $||(e-f)|| \le 1$ .

In correspondence with the usual definition of generalized power means for isotonic functionals [1], we shall define the generalized power means for semigroup of operators, as follows.

**Definition 4.** Let X be a Banach space and  $\{Z(t)\}_{t \in R}$  the  $C_0$ -semigroup of linear operators on X. For  $f \in X$  and  $t \in R_+$ , the genralized power mean is defined as;

$$M_{G_r}(Z, f, t) = \left\{ (Z(t)[f^r])^{1/r}, r \neq 0 exp[Z(t)[log(f)]], r = 0.(7) \right\}$$

As an application of Theorem (2), it follows as a special case that;

$$M_{G_{\omega}}(Z, f, t) \leq M_{G_{\omega}}(Z, f, t), \quad -\infty \leq r \leq s \leq \infty.$$

**Lemma 5.** Let  $\{Z(t)\}_{t\geq 0}$  be the positive  $C_0$ -semigroup defined on V such that it satisfies (3). For a convex operator  $\varphi: V \to V$  and  $t \geq 0$ , we have;

$$\frac{\varphi[Z(t)[f_1h_1]]}{Z(t)[f_1]} \le \frac{Z(t)[f_1\varphi[h_1]]}{Z(t)[f_1]}, \quad f_1, h_1 \in V_+.$$
(8)

**Proof:** For  $f \in V_+$  we have  $\varphi[f] \in V$ . Since V is a lattice algebra,  $f, k \in V_+$  implies  $fk \in V_+$ , therefore the set of operators defined by;

$$F_f(t) := \frac{Z(t)[fk]}{Z(t)[f]}, \quad f \in V_+, t \ge 0,$$

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is a semigroups of positive linear operators satisfying  $F_f(t)[e] = e$ . Thus the assertion (8) follows from (4).

One can observe that when r is any integer (positive or negative), the  $C_0$ -semigroup property implies that  $Z(t)^r = Z(rt)$ . While we can generalize it for  $r \in R_+$ . For example take Z(1/2t)Z(1/2t) = Z(t)and thus we get  $Z(t)^{1/2} = Z(1/2t)$ . For  $r \in R_+$ , the generator of  $\{Z(rt)\}_{t\geq 0}$  is (rA, D(A)). Such semigroups are often called *rescaled semigroups*. (See e.g. [4,8]).

Next, we prove a Hölder's type inequality for positive  $C_0$ -semigroup of operators, assuming the fractional powers of elements in Banach algebra exist.

**Theorem 6.** Hölder's Type Inequality For  $C_0$ -semigroups Let  $\{Z(t)\}_{t\geq 0}$  be the positive  $C_0$ -semigroup defined on V. If p > 1 and  $q = \frac{p}{p-1}$  so  $p^{-1} + q^{-1} = 1$ , then if  $f, g, h \in V_+$  and  $fg^p, fh^q, fgh \in V_+$ , we have for  $t \geq 0$ ;

$$Z(t)[fgh] \le [Z(t)]^{1/p} [gf^{p}] [Z(t)]^{1/q} [fh^{q}]$$
(9)

**Proof:** Since  $fh^q \in V_+$ , we have for  $t \ge 0$ ,  $Z(t)[fh^q] \in V_+$ . For p > 1, (9) follows from (8) by substituting;

$$\varphi(f) = f^p, \quad h_1 = gh^{-q/p}, \quad f_1 = fh^q.$$

**Theorem 7.** Minkowski's Type Inequality For  $C_0$ -semigroups Let  $\{Z(t)\}_{t\geq 0}$  be the positive  $C_0$ -semigroup defined on V. If p > 1 and  $f, g, h \in V_+$  such that  $hf^p, hg^p, h(f+g)^p \in V_+$ , then;

$$Z(t)[h(f+g)^{p}] \leq Z(t)^{1/p}[hf^{p}] + Z(t)^{1/p}[hg^{p}], \quad f \ge 0.$$
<sup>(10)</sup>

**Proof:** For  $f, g, h \in V_+$  and p > 1, we have

$$h(f+g)^{p} = hf(f+g)^{p-1} + hg(f+g)^{p-1}$$

The assertion (10) follows by using (9).

## **Dresher's Type Inequality**

First, we introduce two-parameter family of means in the following way. **Definition 1.** Let  $\{Z(t)\}_{t\geq 0}$  be a strongly continuous semigroup defined on a Banach algebra X. Then the two-parameter family of means  $B_{r,s}(Z, f, t)$  for  $r, s \in R$  is defined by;

$$B_{r,s} = \{\frac{Z(t)[f^r]}{Z(t)[f^s]}\}^{\frac{1}{r-s}, r\neq s}$$

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$$B_{r,r} = exp\{\frac{Z(t)[f^r log f]}{Z(t)[f^r]}\}$$
(11)

**Theorem 2.** Dresher's Type Inequality Let  $\{Z(t)\}_{t\geq 0}$  be a positive  $C_0$ -semigroup defined on a Banach lattice algebra V. Then for  $f \in V_+$  and  $p,q,r,s \in R$ , we have;

$$B_{r,s}(Z,f,t) \le B_{p,q}(Z,f,t) \quad r \le p, s \le q \quad and \quad r \ne s, p \ne q.$$
(12)

**Proof:** Let  $p,q,r,s \in R$  such that  $r \le p, s \le q$  and  $r \ne s, p \ne q$ . When applying the known result for convex functions

$$\frac{\varphi(r) - \varphi(s)}{r - s} \le \frac{\varphi(p) - \varphi(q)}{p - q},\tag{13}$$

to the convex operator  $\varphi(x) = \log Z(t)[f^x]$ , we can obtain (12).

We now show that (12) holds even if r = s or p = q. To prove this we use the fact that  $M_{G_r}(Z, f, t)$ is increasing function of  $r \in R$ . In particular for  $f \in V_+$ ;

$$(Z(t)[f^{s-r}])^{\frac{1}{s-r}} \le \exp[Z(t)logf] \le (Z(t)[f^{r-s}])^{\frac{1}{r-s}}, \quad s < r.$$
(14)

Apply (14) to the positive semigroup (see Lemma 5)  $Z_m(t)g := \frac{Z(t)[f^mg]}{Z(t)[f^m]}$ . By taking m = s the right-hand inequality (14) reduces to

$$B_{s,s}(Z, f, t) \le B_{r,s}(Z, f, t), \quad s < r.$$

Similarly, by taking m = r the left-hand inequality of (14) reduces to

$$B_{r,s}(Z, f, t) \le B_{r,r}(Z, f, t), \quad s < r.$$

By these two inequalities we conclude that the inequality (12) holds for r = s or p = q.

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