

Interval Arithmetic and Its Application to Electrical Circuits

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Abstract: Interval arithmetic is an elegant tool for practical work with inequalities, approximate numbers, error bounds, and more generally with certain convex and bounded sets. In this section we give a number of simple examples showing where intervals and ranges of functions over intervals arise naturally. Interval mathematics is a generalization in which interval numbers replace real numbers, interval arithmetic replaces real arithmetic, and interval analysis replaces real analysis. Interval is limited by two bounds: lower bound and upper bound. The present paper introduces some of the basic notions and techniques from interval analysis needed in the sequel for presenting various uses of interval analysis in electric circuit theory and its applications. In this article we address the representation of uncertain and imprecise information, the interval arithmetic and its application to electrical circuits.

Key words: Interval arithmetic, linear system, uncertain information, propagation of uncertainties, information.

1. Introduction

The interval arithmetic was introduced by Moore [1-5]. An interval is a pair of numbers which represents all the numbers between these two. Intervals are considered closed, so the bounds are included. [6-7].

Interval arithmetic is a computing system that makes it possible to:

- Automatically perform rigorous error analysis by computing mathematical bounds on the set of all possible problem solutions;
- Solve nonlinear problems that were previously thought to be impossible to solve.

In this article we apply interval arithmetic to electrical circuits [8] in order to measure uncertainties in computations with real numbers; we try to calculate

upper and lower endpoints for the range of values of temporal responses and rounding values during the calculation of response characteristics.

Of the various interval-based software packages that are available, we chose INTLAB (interval laboratory) for several reasons. It is fully integrated into the interactive, programmable and highly popular MATLAB (matrix laboratory) system. It is carefully written, with all basic interval computations represented. Finally, both MATLAB and INTLAB code can be written in a fashion that is clear and easy to debug.

The paper is organized as follows: in the first part, we will present definition of an interval and all terms used; Presentation of an interval, interval arithmetic operations and arithmetical expression will be introduced in Sections 3-5 respectively; An application of this approach to a linear system of first order (RC circuit) is described in Section 6; Finally, the effectiveness of the proposed approach is demonstrated via simulation results. We use test signals such as step, ramp and impulse signals. Uncertain parameters of

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system are described by upper and lower bound, and then the responses are computed using interval functions and interval arithmetic.

2. Definition of an Interval

Using a pair $[a, b]$ of computer numbers to represent an interval of real numbers $a \leq x \leq b$, we define an arithmetic for intervals and interval valued extensions of functions commonly used in computing.

In this way, an interval $[a, b]$ has a dual nature. It is a new kind of number pair, and it represents a set of $[a, b] = \{x : a \leq x \leq b\}$.

Recall that the closed interval denoted by $a = [a^-, b^+]$ is the set of real numbers given by:

$$a = [a^-, a^+] = \{x : a^- \leq x \leq a^+\} \quad (1)$$

The left and right endpoints of an interval a will be denoted by a^- and a^+ , respectively.

We use the terms “interval number” and “interval” interchangeably. An interval number can be an interval constant or a value of an interval variable. A few other terms will be useful in the paper.

2.1 Width of an Interval

The width of an interval is defined and denoted by:

$$w(a) = a^+ - a^- \quad (2)$$

2.2 Midpoint

The midpoint of a is given by:

$$M(a) = \frac{a^+ + a^-}{2} \quad (3)$$

2.3 Mignitude and the Magnitude

Mignitude and the magnitude of an interval a are respectively defined by:

$$Mig(a) = \min(|a^+|, |a^-|) \quad (4)$$

$$Mag(a) = \max(|a^+|, |a^-|) \quad (5)$$

2.4 Radius

The radius of an interval a :

$$R(a) = \frac{W(a)}{2} = \frac{a^+ - a^-}{2} \quad (6)$$

3. Representation of Interval

We note:

$$[R] = \{[a^-, a^+] : a^- \leq a^+, a^-, a^+ \in \square\} \quad (7)$$

$$[R^+] = \{[a^-, a^+] : 0 \leq a^- \leq a^+, a^-, a^+ \in \square\} \quad (8)$$

$$[R^-] = \{[a^-, a^+] : a^- \leq a^+ \leq 0, a^-, a^+ \in \square\} \quad (9)$$

$$[R^{-+}] = \{[a^-, a^+] : a^- < 0, a^+ > 0, a^-, a^+ \in \square\} \quad (10)$$

$$[R] = [R^+] \cup [R^-] \cup [R^{-+}] \quad (11)$$

4. Interval Arithmetic Operations

In this subsection, arithmetic operations with intervals will be introduced.

4.1 Definition of Arithmetic Operation

Let $+$, $-$, \times and \div denote the operations of addition, subtraction, multiplication and division respectively. If \circ denotes any one of these operations for arithmetic on real numbers a and b , then the corresponding operation for arithmetic on interval numbers A and B is

$$\{a \circ b \mid a \in A, b \in B\} \quad (12)$$

Thus the interval $A \circ B$ resulting from the operation contains every possible number that can be formed as $a \circ b$ for each $a \in A$ and each $b \in B$. This definition produces the following rules for generating the endpoints of $A \circ B$ from the two intervals $a = [a^-, a^+]$ and $b = [b^-, b^+]$.

4.2 Arithmetic Operations

Let $a = [a^-, a^+]$ and $b = [b^-, b^+]$ are two intervals.

The arithmetic operations (addition, subtraction, multiplication and division) are defined by:

4.2.1 Addition

$$[a^-, a^+] \oplus [b^-, b^+] = [a^- + b^-, a^+ + b^+] \quad (13)$$

Example 1: $[1, 2] \oplus [3, 4]$

$$\begin{aligned} [1, 2] \oplus [3, 4] &= [1+3, 2+4] \\ &= [4, 6] \end{aligned}$$

4.2.2 Subtraction

$$[a^-, a^+] - [b^-, b^+] = [a^- - b^+, a^+ - b^-] \quad (14)$$

Example 2: $[4, 6] - [3, 4]$

$$\begin{aligned} [4, 6] - [3, 4] &= [4-4, 6-3] \\ &= [0, 3] \end{aligned}$$

4.2.3 Multiplication

$$\begin{aligned} [a^-, a^+] \otimes [b^-, b^+] &= \\ \left[\begin{array}{l} \min[a^-b^-, a^-b^+, a^+b^-, a^+b^+] \\ \max[a^-b^-, a^-b^+, a^+b^-, a^+b^+] \end{array} \right] & \quad (15) \end{aligned}$$

Example 3: $[1, 2] \otimes [2, 4]$

$$\begin{aligned} [1, 2] \otimes [2, 4] &= \left[\begin{array}{l} \min[1.2, 1.4, 2.2, 2.4] \\ \max[1.2, 1.4, 2.2, 2.4] \end{array} \right] \\ &= [2, 8] \end{aligned}$$

4.2.4 Division $0 \notin [b^-, b^+]$

$$[a^-, a^+] / [b^-, b^+] = [a^-, a^+] \cdot [1/b^+, 1/b^-] \quad (16)$$

Example 4: $[2, 8] / [2, 4]$

$$\begin{aligned} [2, 8] / [2, 4] &= [2, 8] \cdot [1/4, 1/2] \\ &= [\min[1/2, 1, 2, 4], \max[1/2, 1, 2, 4]] \\ &= [1/2, 4] \end{aligned}$$

4.3 Other Arithmetic Operations

4.3.1 Power Squared

if $0 \notin [a^-, a^+]$

$$[a^-, a^+]^2 = [\min[a^{-2}, a^{+2}], \max[a^{-2}, a^{+2}]] \quad (17)$$

if $0 \in [a^-, a^+]$

$$[a^-, a^+]^2 = [0, \max[a^{-2}, a^{+2}]]$$

Example 5: $[2, 3]^2; 0 \notin [2, 3]$

$0 \notin [2, 3]$

$$\begin{aligned} [2, 3]^2 &= [\min[2^2, 3^2], \max[2^2, 3^2]] \\ &= [\min[4, 9], \max[4, 9]] \\ &= [4, 9] \end{aligned}$$

Example 6: $[-1, 2]^2, 0 \in [-1, 2]$

$0 \in [-1, 2]$

$$\begin{aligned} [-1, 2]^2 &= [0, \max[(-1)^2, 2^2]] \\ &= [0, \max[1, 4]] \\ &= [0, 4] \end{aligned}$$

4.3.2 Square Root

$$\sqrt{[a, b]} = [\sqrt{a^-}, \sqrt{a^+}], 0 \leq a^- \quad (18)$$

Example 7: $\sqrt{[1, 4]}$

$$\begin{aligned} \sqrt{[1, 4]} &= [\sqrt{1}, \sqrt{4}], 0 \leq 1 \\ &= [1, 2] \end{aligned}$$

4.3.3 Absolute Value

The absolute value of a denoted $|a|$ the maximum of the absolute values of its endpoints:

$$|a| = \max\{|a^-|, |a^+|\} \quad (19)$$

4.4 The Intersection Union

The intersection of two intervals a and b is empty if either $b^+ < a^-$ or $a^+ < b^-$. In this case we let \emptyset denote the empty set and write:

$$a \cap b = \emptyset \quad (20)$$

We may define the intersection $a \cap b$ as the interval:

$$\begin{aligned} a \cap b &= \{c : c \in a \text{ and } c \in b\} \\ &= [\max(a^-, b^-), \min(a^+, b^+)] \end{aligned} \quad (21)$$

The union of a and b is also an interval:

$$\begin{aligned} a \cup b &= \{c : c \in a \text{ and } c \in b\} \\ &= [\min(a^-, b^-), \max(a^+, b^+)] \end{aligned} \quad (22)$$

Note: We use INTLAB, a powerful and flexible MATLAB toolbox, for verification of results.

5. Arithmetical Expression

An interval function is an interval-valued function of one or more interval arguments. Thus, an interval function maps the value of one or more interval arguments onto an interval. Consider a real valued function f of real.

We define arithmetical expressions and their interval evaluation in interval arithmetic, we can also define the elementary functions (sin, exp, acoth ... etc.) taking intervals for argument.

Let us apply some other familiar functions to interval arguments. The reasoning is particularly straightforward with functions $f(x)$ that happen to be monotonic, i.e., either increasing or decreasing with increasing x .

Note that it maps an interval $x = [x^-, x^+]$ into the interval $f(x) = [f(x^-), f(x^+)]$. Let us apply some other familiar functions to interval arguments.

$$e^{[a,b]} = [e^a, e^b], \text{ exp is increasing}$$

$$a \coth[a,b] = [a \coth b, a \coth a], 0 \notin [a,b]$$

because acoth is decreasing.

In this subsection, the important notion of an interval function will be introduced. For easier understanding, first an interval function of one variable will be considered; then the generalization to interval functions of several variables will be presented.

Example 8: consider the polynomial

$$\begin{aligned} F(X) &= X^3 - 2X^2 + X - 3, X = [-5, 2] \\ &= [-5, 2]^3 - 2[-5, 2]^2 + [-5, 2] - 3 \\ &= [-183, 7] \end{aligned}$$

Example 9:

$$G(X, Y, Z) = \sin X + 2Xe^Y - Y^2\sqrt{Z},$$

$$X = \left[-\pi, \frac{\pi}{4}\right], Y = [-1, 1], Z = [1, 4]$$

$$G(X, Y, Z) =$$

$$\begin{aligned} &\sin\left[-\pi, \frac{\pi}{4}\right] + 2\left[-\pi, \frac{\pi}{4}\right]e^{[-1,1]} - [-1,1]^2\sqrt{[1,4]} \\ &= \left[-1, \frac{\sqrt{2}}{2}\right] + \left[-2\pi e, \frac{\pi e}{2}\right] - [0, 2] \\ &= \left[-3 - 2\pi e, \frac{\sqrt{2}}{2} + \frac{\pi e}{2}\right] \end{aligned}$$

6. Application on Interval Arithmetic Case of a Linear System of First Order (RC Circuit)

Consider the first order system RC circuit composed of a single resistor and capacitor $u(t)$ is the input

and $y(t)$ is the output.

In the following, we shall analyze the system responses to such inputs as the unit-step, unit-ramp and unit-impulse functions.

Note that all systems having the same transfer function will exhibit the same output in response to the same input. RC circuit will produce a first-order differential Eq. (23):

$$\tau \frac{dy(t)}{dt} + y(t) = kx(t) \quad (23)$$

The transmittance is defined by:

$$G(P) = \frac{Y(p)}{X(p)} = \frac{k}{1 + \tau p} \quad (24)$$

where,

τ : a value known as the "Time Constant" $\tau > 0$;

k : a constant value that corresponds to the source value, static gain of the system;

$G(P)$: transfer function.

We defined the output of system by Eq. (25):

$$Y(P) = X(P)G(P) = \frac{k}{p(1 + \tau p)} \quad (25)$$

Use this example and show the mechanism for calculating the intervals. The parameters chosen are:

$$\tau = [\tau^-, \tau^+] = [9, 10] \quad k = [k^-, k^+] = [0.3, 1]$$

6.1 Unit Step Response

When the system is excited by a unit step, output is called the response index. Since the Laplace transform of the unit-step function is $\frac{1}{p}$, we obtain:

$$Y(p) = \frac{k}{1 + \tau p} \cdot \frac{1}{p} \quad (26)$$

Taking the inverse Laplace transform of Eq. (26), we obtain

$$y(t) = k \left(1 - e^{-\frac{t}{\tau}} \right) \quad (27)$$

In particular, we present step response with MATLAB.

The smaller the time constant τ , the faster the system response. Another important characteristic of the exponential response curve is that the slope of the

tangent line $et \tau$ is $\frac{1}{\tau}$ [8].

The exponential response curve τ given by Eq. (27) is shown in Fig. 1 with parameters $k=1, \tau=1$.

The slope of the response curve decreases monotonically from $\frac{1}{\tau}$ at $t=0$ to zero at $t=\infty$.

The exponential response has gone from 0 to 63.2% of the final value. In two time constants, the response reaches 86.5% of the final value. At $t=3\tau, 4\tau$ and 5τ , the response reaches 95%, 98.2% and 99.3% respectively, of the final value, the steady state is reached mathematically only after an infinite time [8].

Characteristics of unit step response are given in Table 1.

The use of simulation for INTLAB gave Fig. 2.

Note that:

$$\tau = [\tau^-, \tau^+], k = [k^-, k^+], t_s = [t_s^-, t_s^+], t_r = [t_r^-, t_r^+]$$

are interval time constant, final interval value, settling interval time and rise interval time, respectively.

Table 2 gives characteristics and particular values of unit step response respectively.

6.2 Unit Impulse Response

The unit impulse input is the derivative of unit step. The input $X(p)$:

$$x(t) = \delta(t) \rightarrow X(p) = 1 \tag{28}$$

The output $Y(p)$:

$$Y(p) = X(p)G(p) = \frac{k}{1 + \tau p} \tag{29}$$

The response time $y(t)$:

$$y(t) = \frac{k}{\tau} e^{-\frac{t}{\tau}} \tag{30}$$

The simulation results are presented in Fig. 3.

Table 3 shows the lower and upper bound of impulse response.

6.3 Unit Ramp Response

When the system is excited by a unit ramp, the exit is called in response speed. The input is given by:

$$x(t) = tu(t) \rightarrow X(p) = \frac{1}{p^2} \tag{31}$$

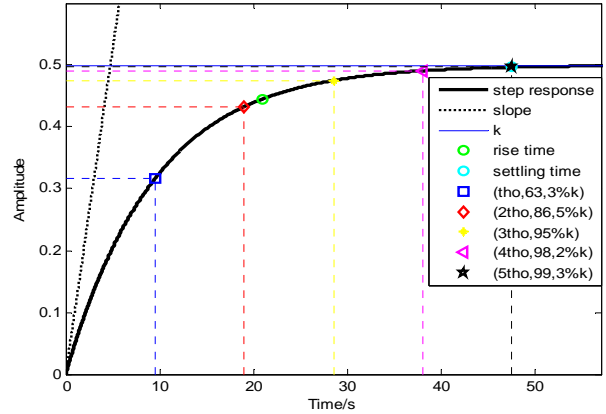


Fig. 1 Unit step response $k = 0.5, \tau = 9.5$ with MATLAB.

Table 1 Characteristics and particular value of unit step response $k = 0.5, \tau = 9.5$.

Characteristic	t	$y(t)$
Rise time	20.90	0.445
Settling time	47.50	0.497
Final value		1
$t = \tau$	9.50	0.316
$t = 2\tau$	19.00	0.432
$t = 3\tau$	28.50	0.475
$t = 4\tau$	38.00	0.491
$t = 5\tau$	47.50	0.497

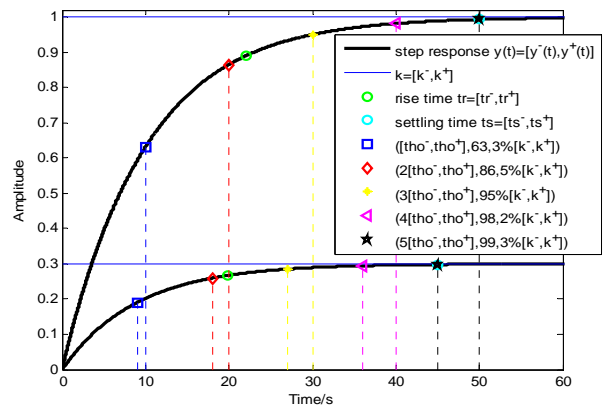


Fig. 2 Unit step response $k = [0.3, 1]$ and $\tau = [9, 10]$.

Table 2 Characteristics and particular values of unit step response $k = [0.3, 1], \tau = [9, 10]$.

Characteristic	$[t^-, t^+]$	$y[t^-, t^+]$
t_r	[19.80, 22]	[0.27, 0.89]
t_s	[45, 50]	[0.30, 0.99]
k		[0.3, 1]
$t = \tau$	[9, 10]	[0.19, 0.63]
$t = 2\tau$	[18, 20]	[0.26, 0.86]
$t = 3\tau$	[27, 30]	[0.28, 0.95]
$t = 4\tau$	[36, 40]	[0.29, 0.98]
$t = 5\tau$	[45, 50]	[0.29, 0.99]

Table 3 Lower and upper bound of impulse response.

	Lower bound	Upper bound
Peak amplitude at $t=0$ s	0, 0333	0, 1
Settling time (s)	35, 2	39.1

Since the Laplace transform of the unit ramp function $\frac{1}{p^2}$, we obtain the output of the system:

$$Y(p) = \frac{k}{p^2(1+\tau p)} \tag{32}$$

A response time is:

$$y(t) = k \left(t - \tau + \tau e^{-\frac{t}{\tau}} \right) \tag{33}$$

We note:

$$\lim_{t \rightarrow \infty} y(t) = y_1(t) = k(t - \tau) \tag{34}$$

At $t = 0$

$$y_1(t = 0) = -k\tau \tag{35}$$

The unit ramp response is given by Fig. 4.

7. Conclusions

The interval arithmetic is based on a very simple concept: replace the real numbers by intervals.

Some properties of mathematical operations are lost:

$$X - X \neq 0 \text{ and } X/X \neq 1$$

However, as illustrated by the following calculations in Table 4:

$$X - X = [1, 2] - [1, 2] = [-1, 1] \neq 0$$

$$X/X = [1, 2]/[1, 2] \neq 1$$

It is trivial that these operations give wrong results.

Each function has a mathematical equivalent elementary interval; there is equivalent interval for all usual mathematical functions.

The main focus in the interval arithmetic is on the simplest way to calculate upper and lower endpoints for the range of values of a function in one or more variables. These barriers need not be necessarily the supremum or infimum, since the precise calculation of those values are often too difficult.

Our contribution in this article is the application of interval arithmetic to electrical circuits. The interval result of numerical computation is an interval guaranteed

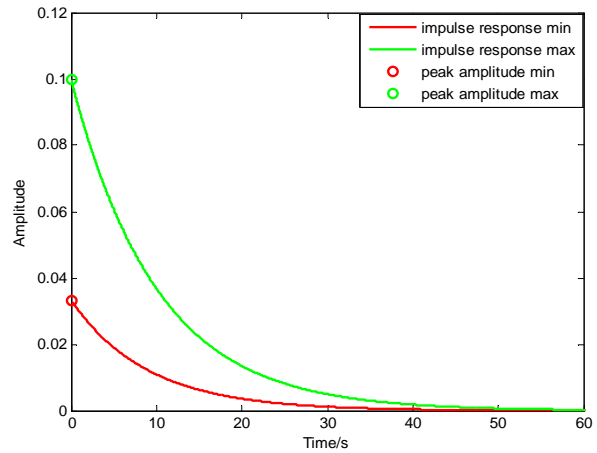


Fig. 3 Unit impulse response $k = [0.3, 1], \tau = [9, 10]$.

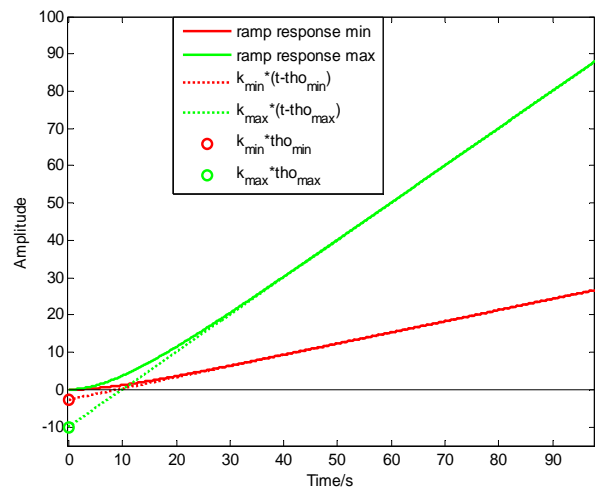


Fig. 4 Unit ramp response $k = [0.3, 1], \tau = [9, 10]$.

Table 4 Lower and upper of ramp response.

	Lower bound	Upper bound
Amplitude	26, 7	87, 9
Time (s)	97, 9	97, 9

to contain the result of the exact computation.

The chosen circuit is described by a differential equation of first order whose coefficients are uncertain value that why we introduce analysis interval, we can generalize the application of interval arithmetic to other systems regardless of their order, so we can study the stability of any system.

With more interval support in education, hardware and software, intervals will become the natural way to think about and compute numerical solutions to physical problems.

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