Journal of Physical Science and Application 5 (2) (2015) 163-172
doi: 10.17265/2159-5348/2015.02.011

Five Nontrivial Solutions of p-Laplacian Problems Involving Critical Exponents and Singular Cylindrical Potential

Mohammed el Mokhtar ould el Mokhtar

Departement of Mathemtics, Qassim University, BO 6644, Buraidah 51452, Kingdom of Saudi Arabia

Abstract: In this paper, we establish the existence of at least five distinct solutions to a p-Laplacian problems involving critical exponents and singular cylindrical potential, by using the Nehari manifold, concentration-compactness principle and mountain pass theorem

Key words: Nehari manifold, concentration-compactness principle, critical Hardy-Sobolev exponent, singular cylindrical potential, mountain pass theorem, nontrivial cylindrical solution.

1. Introduction

In this paper, we consider the multiplicity results of nontrivial solutions of the following problem:

$$\begin{cases} -\Delta_p u - \mu|y|^{-p}|u|^{p-2}u = h|y|^{-s}|u|^{p^*(s)-2}u + \lambda f|u|^{q-2}u \text{ in } \mathbb{R}^N,\ y \neq 0 \\ u \in \mathcal{D}_1^p(\mathcal{R}^N), \end{cases}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, 1 < p < k, k and N are integers with N > p, 2 < k < N, $\mathcal{R}^N = \mathcal{R}^k \times \mathcal{R}^{N-k}$, the point x $\in \mathcal{R}^N$ can be written as x = (y, z) $\in \mathcal{R}^k \times \mathcal{R}^{N-k}$, $-\infty < \mu < \bar{\mu}_{k,p} \coloneqq ((k-p)/p)^p$, 0 < s < p, $p^*(s) = p(N-s)/(N-p)$ is the critical Hardy-Sobolev exponent, $1 < q \leq p^* = pN/(N-p)$ is the critical Sobolev exponent, $f \in L^\infty(\mathcal{R}^N)$, h is a bounded positive function on $\mathcal{R}^k$ and λ is a parameter that we will specify later..

When k = N, $\mu = 0$ and p = 2. The fact that the number of positive solutions of equation $(\mathcal{P}_{\lambda,\mu})$ is affected by the nonlinearity terms which has been the focus of a great deal of research in recent years. If the weight functions f ≡ h ≡ 1, the authors Ambrosetti-Brezis-Cerami [1] have investigated equation $(\mathcal{P}_{\lambda,\mu})$. They found that there exists $\mu_0 > 0$ such that equation $(\mathcal{P}_{1,\mu})$ admits at least two positive solutions for $0 < \mu < \mu_0$, and has a positive solution for $\mu = \mu_0$ but no positive solution exists for $\mu > \mu_0$. For more general results, were done by de Figueiredo-Grossez-Ubilla [2], Wu [3], Cao etal. [4], Filippucci et al. [5], Xuan et al. [6], Guo and Niu [7] and the references therein.

In the case of 1 < k < N, equations with cylindrical potentials were also studied by many people [8-14]. For instance, in [15], Xuan studied the multiple weak solutions for p-Laplace equation with singularity and cylindrical symmetry in bounded domains. However, they only considered the equation with sole critical Hardy-Sobolev term.

Let $\mathcal{D}_1^p(\mathcal{R}^N)$ be the space defined as the completion of $C_c^\infty(\mathcal{R}^N)$ with respect to the norm

$$\|\nabla u\|_p = \left(\int_{\mathcal{R}^N}|\nabla u|^p dx\right)^{\frac{1}{p}}.$$

Clearly, the problem $(\mathcal{P}_{\lambda,\mu})$ is related to the following Hardy-Sobolev type inequality with cylidrical weight which first proved in [10]

$$\int_{\mathcal{R}^N}|\nabla u|^p dx \geq C\int_{\mathcal{R}^N}|y|^{-s}|u|^{p^*(s)}dx, \text{ for all } u \in \mathcal{D}_1^p(\mathcal{R}^N) \quad (1)$$

where C > 0, 1 < p < k, 2 < k < N, x = (y,z)∈

Corresponding author: Mohamed El Mokhtar ould El Mokhtar. E-mail: med.mokhtar66@yahoo.fr, M.labdi@qu.edu.sa.

$\mathcal{R}^k \times \mathcal{R}^{N-k}$, $0 < s < p$, $p^*(s)$ = p(N-s)/(N-p), $p^*(s)$ = pN/(N-p), 1 < q < p. In particular, for s = p and 1 < p < k, we have Hardy type inequality:

$$\int_{\mathcal{R}^N} |\nabla u|^p dx \geq \bar{\mu}_{k,p} \int_{\mathcal{R}^N} |y|^{-p}|u|^p dx, \text{ for all } u \in \mathcal{D}_1^p(\mathcal{R}^N) \quad (2)$$

the constant $\bar{\mu}_{k,p} \coloneqq ((k-p)/p)^p$ is sharp but not achieved [10].

When $\mu < \bar{\mu}_{k,p}$, Hardy type inequality implies that the norm

$$\|u\| = \|u\|_{\mu,p} = \left(\int_{\mathcal{R}^N} (|\nabla u|^p - \mu|y|^{-p}|u|^p)dx\right)^{1/p},$$

is will defined in $\mathcal{D}_1^p(\mathcal{R}^N)$ and $\|.\|$ is equivalent to $\|\nabla.\|_p$; since the following inequalities hold:

$$(1-(\max(\mu,0)/\bar{\mu}_{k,p}))^{1/p}\|\nabla u\|_p$$
$$\leq \|u\| \leq (1-(\min(\mu,0)/\bar{\mu}_{k,p}))^{1/p}\|\nabla u\|_p$$

for all u$\in\mathcal{D}_1^p(\mathcal{R}^N)$.

Since our approach is variational, we define the functional J_λ on $\mathcal{D}_1^p(\mathcal{R}^N)$ by

$$J_\lambda(u) \coloneqq (1/p)\|u\|^p - (1/p^*(s))P(u) - (\lambda/q)Q(u),$$

With

$$P(u) \coloneqq \int_{\mathcal{R}^N} |y|^{-s}h|u|^{p^*(s)}dx, Q(u) \coloneqq \int_{\mathcal{R}^N} f|u|^q dx.$$

Let

$$S = S_{(\mu,N,p,0)} \coloneqq \inf_{u\in\mathcal{D}_1^p(\mathcal{R}^N)\setminus\{0\}} \frac{\|u\|^p}{\left(\int_{\mathcal{R}^N}|u|^{p^*}dx\right)^{p/p^*}} \quad (3)$$

and

$$\tilde{S} = S_{(\mu,N,p,s)} \coloneqq \inf_{u\in\mathcal{D}_1^p(\mathcal{R}^N)\setminus\{0\}} \frac{\|u\|^p}{\left(\int_{\mathcal{R}^N}|y|^{-s}|u|^{p^*(s)}dx\right)^{p/p^*(s)}} \quad (4)$$

where 0 < s < p. From [10], $\tilde{S}$ is achieved.

Throughout this work, we consider the following assumption:

$$(H) \quad \lim_{|y|\to 0} h(y) = \lim_{|y|\to\infty} h(y) = h_0 > 0,\ h(y) \geq h_0, y \in \mathcal{R}^k.$$

In our work, we research the critical points as the minimizers of the energy functional associated to the problem $(\mathcal{P}_{\lambda,\mu})$ on the constraint defined by the Nehari manifold, which are solutions of our problem.

Let Λ_0 be positive number

$$\Lambda_0 \coloneqq L(p,q)\left(\tilde{S}\right)^{p^*(s)(p^*-p)/p(p-p^*(s))}(S)^{-p^*/q}$$

where

$$L(p,q) \coloneqq \left[\left(\frac{p-p^*(s)}{(q-p^*(s))|f|_\infty}\right)\right] \times$$
$$\left[|h|_\infty\left(\frac{p^*(s)-q}{(p-q)}\right)\right]^{(p^*-p)/(p^*(s)-p)}$$

and $|f(x)|_\infty = \sup_{x\in\mathcal{R}^N}|f(x)|, |h(y)|_\infty = \sup_{y\in\mathcal{R}^k}|h(y)|.$

Now we can state our main results.

Theorem 1: Let f $\in L^\infty(\mathcal{R}^N)$. Assume that 1 < p < k, N > p, 2 < k < N, 0 < $\mu < \bar{\mu}_{k,p} \coloneqq ((k-p)/p)^p$, 0 < s < p, 1 < q < p, (H) satisfied and λ verifying $0 < \lambda < \Lambda_0$, then the equation $(\mathcal{P}_{\lambda,\mu})$ has at least one positive solution.

Theorem 2: In addition to the assumptions of the Theorem 1, there exists a positive real Λ_1 such that, if λ *satisfy* $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$, then $(\mathcal{P}_{\lambda,\mu})$ has at least two positive solutions.

Theorem 3: In addition to the assumptions of the Theorem 2, there exists a positive real Λ_* such that, if λ satisfy $0 < \lambda < \min(\Lambda_2, \Lambda_*)$, then $(\mathcal{P}_{\lambda,\mu})$ has at least two positive solution and two opposite solutions.

Theorem 4: Let f $\in L^\infty(\mathcal{R}^N)$. Assume that 1 < p < k, N > p, 0 < s < p, $\mu < 0, q = p^*$, (H) satisfied and $\lambda > 0$, then the problem $(\mathcal{P}_{\lambda,\mu})$ has a nontrivial cylindrical weak solution u$\in X_l(\mathcal{R}^N)$ (u satisfying u(y, z) = u(|y|, z)).

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proofs of Theorems 1, 2, 3 and 4.

2. Preliminaries

Definition 1: Let c $\in \mathcal{R}$, E a Banach space and $J_\lambda \in C^1(E, \mathcal{R})$.

(i) $(u_n)_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in E for J_λ if

$$J_\lambda(u_n) = c + o_n(1) \text{ and } J_\lambda^{'}(u_n) = o_n(1),$$

where $o_n(1)$ tends to 0 as n goes at infinity.

(ii) We say that J_λ satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for J_λ has a convergent subsequence.

Lemma 1: Let X Banach space, and $J_\lambda \in C^1(X,\mathcal{R})$ verifying the Palais -Smale condition. Suppose that $J_\lambda(0) = 0$ and that:

(i) there exist R > 0, r > 0 such that if $\| \|u\| \| = R$, then J_λ (u) ≥ r;

(ii) there exist $u_0 \in X$ such that $\| u_0 \| > R$ and $J_\lambda(u_0) \leq 0$;

let $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} (J_\lambda(\gamma(t)))$

where $\Gamma = \left\{ \begin{array}{c} \gamma \in C([0,1];X) \text{ such that} \\ \gamma(0) = 0 \text{ et } \gamma(1) = u_0 \end{array} \right\}$

then c is critical value of J_λ such that c ≥ r.

2.1 Nehari Manifold

It is well known that J_λ is of class C^1 in $\mathcal{D}_1^p(\mathcal{R}^N)$ and the solutions of $(\mathcal{P}_{\lambda,\mu})$ are the critical points of J_λ which is not bounded below on $\mathcal{D}_1^p(\mathcal{R}^N)$. Consider the following Nehari manifold

$$\mathcal{N} = \left\{u \in \mathcal{H}\backslash\{0\} \ : \ \left\langle J'_\lambda(u), u\right\rangle = 0\right\},$$

Thus, u ∈ N if and only if

$$\|u\|^p - P(u) - \lambda Q(u) = 0 \qquad (5)$$

Note that N contains every nontrivial solution of the problem $(\mathcal{P}_{\lambda,\mu})$. Moreover, we have the following results.

Lemma 2: J_λ is coercive and bounded from below on N.

Proof If u ∈ N, then by Eq. (5) and the Sobolev inequality, we deduce that

$$\begin{aligned} J_\lambda(u) &= ((p^*(s)-p)/pp^*(s))\|u\|^p \\ &\quad - \lambda((p^*(s)-q)/qp^*(s))Q(u,v) \\ &\geq ((p^*(s)-p)/pp^*(s))\|u\|^p \\ &\quad - \lambda((p^*(s)-q)/qp^*(s))|f|_\infty S^{(q/p^*)}\|u\|^q \end{aligned} \qquad (6)$$

Thus, J_λ is coercive and bounded from below on N. Define

$$\phi(u) = \left\langle J'_\lambda(u), u\right\rangle.$$

Then, for u ∈ N

$$\begin{aligned} \left\langle \phi'(u), u\right\rangle &= p\|u\|^p - p^*(s)P(u) - \lambda q Q(u) \\ &= (p-q)\|u\|^p - (p^*(s)-q)P(u) \\ &= \lambda(p^*(s)-q)Q(u) - (p^*(s)-p)\|u\|^p \end{aligned} \qquad (7)$$

Now, we split N in three parts:

$$\mathcal{N}^+ = \left\{u \in \mathcal{N} : \left\langle \phi'(u), u\right\rangle > 0\right\}$$
$$\mathcal{N}^0 = \left\{u \in \mathcal{N} : \left\langle \phi'(u), u\right\rangle = 0\right\}$$
$$\mathcal{N}^- = \left\{u \in \mathcal{N} : \left\langle \phi'(u), u\right\rangle < 0\right\}$$

We have the following results.

Lemma 3: Suppose that u_0 is a local minimizer for J_λ on N. Then, if $u_0 \notin N^0$, u_0 is a critical point of J_λ.

Proof If u_0 is a local minimizer for J_λ on N, then u_0 is a solution of the optimization problem

$$\min_{\{u/\ \phi(u)=0\}} J_\lambda(u).$$

Hence, there exists a Lagrange multipliers $\theta \in R$ such that

$$J'_\lambda(u_0) = \theta\phi'(u_0) \text{ in } (\mathcal{D}_1^p(\mathcal{R}^N))'$$

Thus,

$$\left\langle J'_\lambda(u_0,v_0),(u_0,v_0)\right\rangle = \theta\left\langle \phi'(u_0,v_0),(u_0,v_0)\right\rangle.$$

But $\left\langle \phi'(u_0), u_0\right\rangle \neq 0$, since $u_0 \notin N^0$. Hence $\theta = 0$. This completes the proof.

Lemma 4: There exists a positive number Λ_0 such that for all verifying $0 < \lambda < \Lambda_0$, we have $N^0 = \emptyset$.

Proof Let us reason by contradiction. Suppose $N^0 \neq \emptyset$ such that $0 < \lambda < \Lambda_0$, Then, by Eq. (7) and for u ∈ N^0, we have

$$\begin{aligned} \|u\|^p &= (p^*(s)-q)/(p-q)P(u,v) \\ &= \lambda((p^*(s)-q)/(p^*(s)-p))Q(u,v) \end{aligned} \qquad (8)$$

Moreover, by the Holder inequality and the Sobolev embedding theorem, we obtain

$$\|u\| \geq (S)^{p^*/q(p^*-p)}[(p-p^*(s))/\lambda(q-p^*(s))|f|_\infty]^{-1/(p^*-p)} \qquad (9)$$

and

$$\|u\| \leq [h_0((p^*(s)-q)/(p-q))]^{1/(p-p^*(s))}(\tilde{S})^{-p^*(s)/p(p-p^*(s))} \qquad (10)$$

From Eq. (9) and Eq. (10), we obtain $\lambda \geq \Lambda_0$, which contradicts an hypothesis.

Thus $N = N^+ \cup N^-$. Define

$$c := \inf_{u\in\mathcal{N}} J_\lambda(u),\; c^+ := \inf_{u\in\mathcal{N}^+} J_\lambda(u) \text{ and } c^- := \inf_{u\in\mathcal{N}^-} J_\lambda(u).$$

For the sequel, we need the following Lemma.

Lemma 5:

(i) For all such that $0 < \lambda < \Lambda_0$, one has $c \le c^+ < 0$.

(ii) There exists $\Lambda_1 > 0$ such that for all $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$ one has

$$c^- > C_0 = C_0(\lambda, p, q, S, \tilde{S}, p^*(s), h_0, |f|_\infty) = \left(\frac{(p^*(s)-p)}{pp^*(s)}\right)\left[\frac{(p-q)}{(p^*(s)-q)h_0}\right]^{\frac{-p}{(p-p^*(s))}}(\tilde{S})^{\frac{p^*(s)}{(p^*(s)-p)}} + -\lambda\left(\frac{(p^*(s)-q)}{q(p^*(s))}\right)|f|_\infty(S)^{q/p^*}.$$

Proof (i) Let $u \in N^+$. By Eq. (7), we have

$$[(p-q)/(p^*(s)-q)]\|u\|^p > P(u)$$

and so

$$J_\lambda(u) = (-(p-q)/pq)\|u\|^p + ((p^*(s)-q)/q(p^*(s)))P(u) < -[((p-q)/pq) + ((p-q)/q(p^*(s)))]\|u\|^p.$$

We conclude that $c \le c^+ < 0$.

(ii) Let $u \in N^-$. By Eq. (7), we get

$$[(p-q)/(p^*(s)-q)]\|u\|^p < P(u).$$

Moreover, by (H) and Sobolev embedding theorem, we have

$$P(u) \le (\tilde{S})^{-p^*(s)/p}|h^+|_\infty\|u\|^{p^*(s)}.$$

This implies

$$\|u\| > (\tilde{S})^{p^*(s)/p(p^*(s)-p)}\left[\frac{(p-q)}{(p^*(s)-q)|h^+|_\infty}\right]^{\frac{-1}{(p-p^*(s))}}, \text{ for all } u \in \mathcal{N}^- \quad (11)$$

By Eq. (6), we get

$$J_\lambda(u) \ge \left(\frac{(p^*(s)-p)}{pp^*(s)}\right)\left[\frac{(p-q)}{(p^*(s)-q)h_0}\right]^{\frac{-p}{(p-p^*(s))}}(\tilde{S})^{\frac{p^*(s)}{(p^*(s)-p)}} + -\lambda\left(\frac{(p^*(s)-q)}{q(p^*(s))}\right)|f|_\infty(S)^{q/p^*}.$$

Thus, for all λ such that $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$, with

$$\Lambda_1 := \left(\frac{(p^*(s)-p)}{pp^*(s)}\right)\left[\frac{(p-q)(\tilde{S})^{\frac{p^*(s)}{p}}}{(p^*(s)-q)h_0}\right]^{\frac{-p}{(p-p^*(s))}} \times\left[\left(\frac{(p^*(s)-q)|f|_\infty}{q(p^*(s))}\right)(S)^{-q/p^*}\right]^{-1}$$

we have $J_\lambda \ge C_0$.

For each $u \in \mathcal{D}_1^p(\mathcal{R}^N)$, we write

$$t_m := t_{\max}(u) = \left[\frac{\|u\|^p}{(p^*(s)-q)P(u)}\right]^{1/(p^*(s)-p)} > 0.$$

Lemma 5: Let real parameters such that $0 < \lambda < \Lambda_0$. For each $u \in \mathcal{D}_1^p(\mathcal{R}^N)$, one has the following:

(i) If $Q(u) \le 0$, then there exists a unique $t^- > t_m$ such that $t^-u \in N^-$ and

$$J_\lambda(t^-u) = \sup_{t\ge 0}(tu).$$

(ii) If $Q(u) > 0$, then there exist unique t^+ and t^- such that $0 < t^+ < t_m < t^-$, $(t^+u) \in N^+$, $(t^-u) \in N^-$,

$$J_\lambda(t^+u) = \inf_{0\le t\le t_m} J_\lambda(tu) \text{ and } J_\lambda(t^-u) = \sup_{t\ge 0} J_\lambda(tu).$$

Proof With minor modifications, we refer to [16].

Proposition 1 [6]

(i) For all such that $0 < \lambda < \Lambda_0$, there exists a $(PS)_{c^+}$ sequence in N^+.

(ii) For all such that $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$, there exists a a $(PS)_{c^-}$ sequence in N^-.

3. Proof of Theorems

3.1 Proof of Theorem 1

Now, taking as a starting point the work of Tarantello [17], we establish the existence of a local minimum for J_λ on N^+.

Proposition 2 For all such that $0 < \lambda < \Lambda_0$, the functional J_λ has a minimizer $u_0^+ \in N^+$ and it satisfies:

(i) $J_\lambda(u_0^+) = c = c^+$,

(ii) u_0^+ is a nontrivial solution of $(\mathcal{P}_{\lambda,\mu})$.

Proof If $0 < \lambda < \Lambda_0$, then by Proposition1 (i), there exists a $(u_n)_n$ $(PS)_{c^+}$ sequence in N^+, thus, it bounded by Lemma 4. Then, there exists $u_0^+ \in \mathcal{D}_1^p(\mathcal{R}^N)$ and we can extract a subsequence which will denoted by $(u_n)_n$ such that

$$\begin{aligned} &u_n \rightharpoonup u_0^+ \text{ weakly in } \mathcal{D}_1^p(\mathcal{R}^N) \\ &u_n \rightharpoonup u_0^+ \text{ weakly in } (L^{p^*(s)}(\mathcal{R}^N, |y|^{-s})) \\ &u_n \to u_0^+ \text{ strongly in } L^q(\mathcal{R}^N) \\ &u_n \to u_0^+ \text{ a.e in } \mathcal{R}^N \end{aligned} \quad (12)$$

Thus, by Eq. (12), u_0^+ is a weak nontrivial solution of $(\mathcal{P}_{\lambda,\mu})$. Now, we show that $(u_n)_n$ converges to u_0^+ strongly in $\mathcal{D}_1^p(\mathcal{R}^N)$. Suppose otherwise. By the lower semi-continuity of the norm, then $\|u_0^+\| < \liminf_{n\to\infty}\|u_n\|$ and we obtain

$$\begin{aligned} c \le J_\lambda(u_0^+) &= ((p^*(s)-p)/p(p^*(s)))\|u_0^+\|^p \\ &\quad - ((p^*(s)-q)/q(p^*(s)))Q(u_0^+) \\ &< \liminf_{n\to\infty} J(u_n) = c. \end{aligned}$$

We get a contradiction. Therefore, $(u_n)_n$ converge to u_0^+ strongly in $\mathcal{D}_1^p(\mathcal{R}^N)$. Moreover, we have $u_0^+ \in N^+$. If not, then by **Lemma 5**, there are two numbers t_0^+ and t_0^-, uniquely defined so that $(t_0^+u_0^+) \in N^+$ and $(t^-u_0^+) \in N^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_\lambda(tu_0^+)_{\downarrow t=t_0^+} = 0 \text{ and } \frac{d^2}{dt^2}J_\lambda(tu_0^+)_{\downarrow t=t_0^+} > 0,$$

there exists $t_0^+ < t^- \le t_0^-$ such that J_λ $(t_0^+u_0^+) <$ J_λ $(t^-u_0^+)$. By **Lemma 5**, we get

$$J_\lambda(t_0^+u_0^+) < J_\lambda(t^-u_0^+) < J_\lambda(t_0^-u_0^+) = J_\lambda(u_0^+),$$

which contradicts the fact that J_λ $(u_0^+) = c^+$. Since J_λ $(u_0^+) =$ J_λ $(|u_0^+|)$ and $|u_0^+| \in N^+$, then by **Lemma 2**, we may assume that u_0^+ is a nontrivial nonnegative solution of $(\mathcal{P}_{\lambda,\mu})$. By the Harnack inequality, we conclude that $u_0^+ > 0$ and $v_0^+ > 0$, see for example [18].

3.2 Proof of Theorem 2

Next, we establish the existence of a local minimum for on N^-. For this, we require the following Lemma.

Lemma 6 For all such that $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$, the functional J_λ has a minimizer u_0^- in N^- and it satisfies:

(i) $J_\lambda(u_0^-) = c^- > 0$,

(ii) *u_0^- is a nontrivial solution of $(\mathcal{P}_{\lambda,\mu})$ in $\mathcal{D}_1^p(\mathcal{R}^N)$.*

Proof If $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$, then by Proposition 1 (ii) there exists a $(u_n)_n$, $(PS)_{c^-}$ sequence in N^-, thus it bounded by **Lemma 1**. Then, there exists $u_0^- \in$ $\mathcal{D}_1^p(\mathcal{R}^N)$ and we can extract a subsequence which will denoted by $(u_n)_n$ such that

$$\begin{aligned} &u_n \rightharpoonup u_0^- \text{ weakly in } \mathcal{D}_1^p(\mathcal{R}^N) \\ &u_n \rightharpoonup u_0^- \text{ weakly in } L^{p^*(s)}(\mathcal{R}^N, |y|^{-s}) \\ &u_n \to u_0^- \text{ strongly in } L^q(\mathcal{R}^N) \\ &u_n \to u_0^- \text{ a.e in } \mathcal{R}^N \end{aligned}$$

This implies

$$P(u_n) \to P(u_0^-), \text{as } n \text{ goes to } \infty.$$

Moreover, by (H) and Eq. (7) we obtain

$$P(u_n) > (p-q)/(p^*(s)-q)\|u_n\|^p \qquad (13)$$

By Eq. (9) and Eq. (13) there exists a positive number

$$C_1 := [(p-q)/(p^*(s)-q)]^{p^*(s)/(p^*(s)-p)}\left(\tilde{S}\right)^{p^*(s)/(p^*(s)-p)},$$

such that

$$P(u_n) > C_1 \qquad (14)$$

This implies that

$$P(u_0^-) \ge C_1.$$

Now, we prove that $(u_n)_n$ converges to u_0^- strongly in $\mathcal{D}_1^p(\mathcal{R}^N)$. Suppose otherwise. Then, $\|$ $\|u_0^-\| < \liminf_{n\to\infty}\|u_n\|$ $\|$. By **Lemma 5** there is a unique t_0^- such that $(t_0^-u_0^-) \in N^-$. Since

$$u_n \in \mathcal{N}^-, \; J_\lambda(u_n) \ge J_\lambda(tu_n), \text{ for all } t \ge 0,$$

we have

$$J_\lambda(t_0^-u_0^-) < \lim_{n\to\infty} J_\lambda(t_0^-u_n) \le \lim_{n\to\infty} J_\lambda(u_n) = c^-,$$

and this is a contradiction. Hence,

$$u_n \to u_0^- \text{ strongly in } \mathcal{D}_1^p(\mathcal{R}^N).$$

Thus,

$J_\lambda(u_n)$ converges to $J_\lambda(u_0^-) = c^-$ as n tends to $+\infty$.

Since J_λ $(u_0^-) =$ J_λ $(|u_0^-|)$ and $u_0^- \in N^-$, then by Eq. (14) and **Lemma 2**, we may assume that u_0^- is a nontrivial nonnegative solution of $(\mathcal{P}_{\lambda,\mu})$. By the maximum principle, we conclude that $u_0^- > 0$ and $v_0^- > 0$.

Now, we complete the proof of **Theorem 2**. By **Proposition 2** and **Lemma 6**, we obtain that $(\mathcal{P}_{\lambda,\mu})$ has two positive solutions $u_0^+ \in N^+$ and $u_0^- \in N^-$. Since $N^+ \cap N^- = \emptyset$, this implies that u_0^+ and u_0^- are distinct.

3.3 Proof of Theorem 3

In this section, we consider the following Nehari submanifold of N

$$\mathcal{N}_\varrho = \left\{ \begin{array}{c} u \in \mathcal{D}_1^p(\mathcal{R}^N)\backslash\{0\} \; : \; \left\langle J_\lambda^{'}(u), u \right\rangle = 0 \\ \text{and } \|u\| \geq \varrho > 0 \end{array} \right\}$$

Thus, u ∈ $\mathcal{N}_\varrho$ if and only if

$$\|u\|^p - P(u) - \lambda Q(u) = 0 \text{ and } \|u\| \geq \varrho > 0$$

Firstly, we need the following Lemmas

Lemma 7 Under the hypothesis of theorem 3, there exist $\varrho_0, \Lambda_3 > 0$ such that $\mathcal{N}_\varrho$ is nonempty for any $\lambda \in (0, \Lambda_3)$ and $\varrho \in (0, \varrho_0)$.

Proof Fix (u_0) ∈ $\mathcal{D}_1^p(\mathcal{R}^N)\backslash\{0\}$ and let

$$\begin{aligned} g(t) &= \left\langle J_\lambda^{'}(tu_0), tu_0 \right\rangle \\ &= t^p \|u_0\|^p - t^{p^*(s)} P(u_0) - t\lambda Q(u_0) \end{aligned}$$

Clearly g(0) = 0 and g(t)→-∞ as n→+∞. Moreover, we have

$$\begin{aligned} g(1) &= \|u_0\|^p - P(u_0) - \lambda Q(u_0) \\ &\geq \left[\|u_0\|^p - \left(\tilde{S}\right)^{-p^*(s)/p} |h^+|_\infty \|u_0\|^{p^*(s)} \right] - \lambda \|u_0\| \end{aligned}$$

$$\text{If } \|u_0\| \geq \varrho > 0 \text{ for}$$

$$0 < \varrho < \varrho_0 := (|h^+|_\infty (p^*(s) - 1))^{\frac{-1}{(p^*(s)-p)}} \left(\tilde{S}\right)^{\frac{p^*(s)}{p(p^*(s)-p)}},$$

and

$$|h^+|_\infty \in (0, \alpha_0) \text{ with}$$

$$\alpha_0 := \left(\tilde{S}\right)^{p^*(s)/p} / ((p^*(s) - 1))^{(p^*(s)-1)/p^*(s)}$$

then, there exists

$$\Lambda_3 := \left[(|h^+|_\infty (p^*(s) - 1)) \left(\tilde{S}\right)^{-p^*(s)/p} \right]^{-1/(p^*(s)-p)} - \Theta \times \Phi$$

where

$$\Theta := ((p^*(s) - 1))^{p^*(s)-1} \left((|h^+|_\infty)^{p^*(s)/p} \left(\tilde{S}\right) \right)^{-(2_*)^2/2}$$

and

$$\Phi := \left[(|h^+|_\infty (p^*(s) - 1)) \left(\tilde{S}\right)^{-p^*(s)/p} \right]^{-1/(p^*(s)-p)}$$

and there exists $t_0 > 0$ such that $g(t_0) = 0$. Thus, $t_0 u_0 \in \mathcal{N}_\varrho$ and $\mathcal{N}_\varrho$ is nonempty for any $\lambda \in (0, \Lambda_3)$.

Lemma 8 There exist M, Λ_* positive reals such that

$$\left\langle \phi^{'}(u), u \right\rangle < -M < 0, \text{ for } u \in \mathcal{N}_\varrho,$$

and any λ verifying

$$0 < \lambda < \min(\Lambda_2, \Lambda_*).$$

Proof Let u ∈ $\mathcal{N}_\varrho$, then by Eq. (5) and Eq. (7), it allows us to write

$$\begin{aligned} \left\langle \phi^{'}(u), u \right\rangle &= (p - p^*(s))\|u\|^p + \lambda(p^*(s) - q)Q(u) \\ &\leq (p - p^*(s))\|u\|^p + \lambda(p^*(s) - q)|f^+|_\infty S^{q/p^*(s)} \|u\|^q \\ &\leq \max(\|u\|^p, \|u\|^q)[(p - p^*(s)) + \lambda(p^*(s) - q)|f^+|_\infty S^{q/p^*(s)}] \end{aligned}$$

Thus, for any λ verifying

$$0 < \lambda < \Lambda_4 = \left[\frac{(p^*(s)-p)}{(p^*(s)-q)|f^+|_\infty} \right] S^{q/p^*(s)},$$

and choosing $\Lambda_* = \min(\Lambda_3, \Lambda_4)$ with Λ_3 defined in Lemma 1, then we obtain that

$$\left\langle \phi^{'}(u), u \right\rangle < 0, \text{ for any } u \in \mathcal{N}_\varrho \tag{15}$$

Lemma 9 Suppose N > p, 0 < s < p, 1 < q < 2. Then, there exist r and positive constants η such that

(i) we have $J_\lambda(u) \geq \eta > 0$ *for* $\|u\| = r.$

(ii) there exists $\sigma \in \mathcal{N}_\varrho$ when $\|\sigma\| > r,$ *with* $r = \|u\| \;\|$, such that $J_\lambda(\sigma) \leq 0.$

Proof We can suppose that the minima of J_λ are realized by u_0^+ and u_0^-. The geometric conditions of the mountain pass theorem are satisfied. Indeed, we have

(i) By Eq. (7) and Eq. (15), we get

$$\begin{aligned} J_\lambda(u) \geq\; & [(p^*(s) - p)/pp^*(s)]\|u\|^p \\ & - [(p^*(s) - p)/qp^*(s)]\|u\|^{p+q}, \end{aligned}$$

Thus, there exist η, r > 0 such that

$$J_\lambda(u) \geq \eta > 0 \text{ when } r = \|u\| \text{ small.}$$

(ii) Let t > 0, then we have for all $\phi \in \mathcal{N}_\varrho$

$$J_\lambda(t\phi) := (t^p/p)\|\phi\|^p - (t^{p^*(s)})P(\phi) - \lambda(t^q/q)Q(\phi).$$

Letting $\sigma = t\phi$ for t large enough. Since $P(\phi) > 0,$

we obtain $J_\lambda(\sigma) \leq 0$ .For t large enough we can ensure $\|\sigma\| > r.$

Let Γ and c defined by

$$\Gamma := \left\{ \gamma : [0,1] \to \mathcal{N}_\varrho : \gamma(0) = u_0^- \text{ and } \gamma(1) = u_0^+ \right\}$$

and

$$c := \inf_{\gamma \in \Pi} \max_{t \in [0,1]} (J_\lambda(\gamma(t))).$$

Proof of Theorem 3:

If $0 < \lambda < \min(\Lambda_2, \Lambda_*)$ then, by the **Lemma 4** and **Proposition 1** (ii), J_λ verifying the Palais -Smale condition in $\mathcal{N}_\varrho$. Moreover, from the Lemmas 3, 8 and 9, there exists u_c such that

$$J_\lambda(u_c) = c \text{ and } u_c \in \mathcal{N}_\varrho.$$

Thus u_c is the third solution of our system such that $u_c \neq u_0^+$ and $u_c \neq u_0^-$. Since $(\mathcal{P}_{\lambda,\mu})$ is odd with respect u, we obtain that $(-u_c)$ is also a solution of $(\mathcal{P}_{\lambda,\mu})$.

3.4 Proof of Theorem 4

In the part, we consider the case $\mu < 0, q = p^*$ and $\lambda > 0$ and obtain the existence of the solution with cylidrical symmetry for $(\mathcal{P}_{\lambda,\mu})$. First, we list some notations.

Define

$$X := X(\mathcal{R}^N; |y|^{-p}dx) := \left\{u \in \mathcal{D}_1^p(\mathcal{R}^N) : \int_{\mathcal{R}^N} |y|^{-p} h u^p dx < +\infty\right\}$$

$$X_l := X_l(\mathcal{R}^N; |y|^{-p}dx) := \{u \in X : u(y,z) = u(|y|,z)\}$$

Now, we set E(u) as the energy functional of equation $(\mathcal{P}_{\lambda,\mu})$ that is

$$E(u) := (1/p)\int_{\mathcal{R}^N} |\nabla u|^p dx - (\mu/p)\int_{\mathcal{R}^N} |y|^{-p}|u|^p dx - (1/p^*(s))P(u) - (\lambda/p^*)Q(u),$$

With

$$P(u) := \int_{\mathcal{R}^N} |y|^{-s} h |u|^{p^*(s)} dx, Q(u) := \int_{\mathcal{R}^N} f |u|^{p^*} dx.$$

The functional E(u) is belong to $C^1(X, \mathcal{R}^N)$. Following, we can define a group of rescaling operators:

$$T_{\eta,x} u := \eta^{-\left(\frac{N-p}{p}\right)} u(\eta^{-1}.+x).$$

By direct computation, we have

$$T_{\eta,x} u := T_{\frac{1}{\eta},-\eta x} u, T_{\eta_1,x_1} T_{\eta_2,x_2} u := T_{\eta_1\eta_2,x_1+x_2} u$$

and if u $\in$ $L^{p^*}(\mathcal{R}^N)$ and $\mathcal{D}_1^p(\mathcal{R}^N)$, one get $T_{\eta,x} u \in L^{p^*}(\mathcal{R}^N)$ and $\mathcal{D}_1^p(\mathcal{R}^N)$. We know that the mapping $u \in L^{p^*}(\mathcal{R}^N) \mapsto L^{p^*}(\mathcal{R}^N)$ and $u \in \mathcal{D}_1^p(\mathcal{R}^N) \mapsto T_{\eta,x} u \in \mathcal{D}_1^p(\mathcal{R}^N)$ are isometric. As the method we used here is the concentration-compactness principle, and some propositions in [8], we list them first:

Lemma 10 (the concentration-compactness principle of Solimini) If $(u_k) \subset \mathcal{D}_1^p(\mathcal{R}^N)$ is bounded, then up to a subsequence, (u_k) converge strongly to 0 in $L^{p^*}(\mathcal{R}^N)$ or there exists $(\eta_k) \subset (0, +\infty)$ and $(x_k) \subset \mathcal{R}^N$ such that $T_{\eta_k,x_k} u_k \rightharpoonup u$ in $L^{p^*}(\mathcal{R}^N)$, u≠0.

Proposition 3 [8] Let $1 < \gamma < \infty$, assuming $(\eta_k) \subset (0, +\infty)$ and $(x_k) \subset \mathcal{R}^N$ are such that $\eta_k \to \eta, x_k \to x$, then

$$T_{\eta_k,x_k} u_k \to T_{\eta,x} u \text{ in } L^\gamma(\mathcal{R}^N)$$

$$\text{if } u_k \to u \text{ in } L^\gamma(\mathcal{R}^N).$$

Proposition 4 If $u \in X_l$ then for all $\psi \in X$ and g ∈ O(k), we have $E'(u)\psi(g.,.) = E'(u)\psi$, where O(k) is the orthogonal group of $\mathcal{R}^k$.

The proof is similar to the proof of Proposition 10 in [8], we omit it.

By a similar analysis in Proposition 3, we get that for the functional $E\mid_{X_l}$, there exist a bounded sequence $(x_k) \subset X_l$ and c > 0 such that

$E(u_k) \to c$ and $E'(u_k)\mid_{X_l} \to 0$ in X_l' (dual of X_l).

where c is the mountain pass level of $E\mid_{X_l}$ defined by

$$c = \inf_{\delta\in\Gamma} \max_{t\in[0,1]} E(\delta(t)), \Gamma := \{\delta \in ([0,1], X_l) : \delta(0) = 0, E(\delta(1)) < 0\}$$

Now we begin to prove **Theorem 4**. Since the sequence (u_k) is bounded, it satisfies one of the cases in **Lemma 10**, now we show that the first case doesn't occur.

Lemma 11 The case $(u_k) \to 0 \text{ in } L^{p^*}(\mathcal{R}^N)$ doesn't hold.

Proof If not, then

$$\begin{aligned}\int_{\mathcal{R}^N} |y|^{-s}|u_k|^{p^*(s)} dx &\le \int_{\mathcal{R}^N} |y|^{-s}|u_k|^p |u_k|^{p^*(s)-p} dx \\ &\le \left(\int_{\mathcal{R}^N} |y|^{-\frac{sN}{N-p+s}} |u_k|^{\frac{pN}{N-p+s}} dx\right)^{\frac{N-p+s}{N}} \left(\int_{\mathcal{R}^N} |u_k|^{p^*} dx\right)^{\frac{p^*(s)-p}{p^*}} \quad (16) \\ &\le C\left(\int_{\mathcal{R}^N} |\nabla u_k|^p dx\right)\left(\int_{\mathcal{R}^N} |u_k|^{p^*} dx\right)^{\frac{p^*(s)-p}{p^*}}\end{aligned}$$

Hence, when $(u_k) \to 0$ in $L^{p^*}(\mathcal{R}^N)$, we have

$$\int_{\mathcal{R}^N} |y|^{-s}|u_k|^{p^*(s)}dx \to 0.$$

Now, in fact that $\mu < 0$ and

$$E'(u_k)u_k = \int_{\mathcal{R}^N} |\nabla u_k|^p dx - \mu \int_{\mathcal{R}^N} |y|^{-p}|u_k|^p dx$$
$$- \int_{\mathcal{R}^N} |y|^{-s}h|u_k|^{p^*(s)}dx - \int_{\mathcal{R}^N} f|u_k|^{p^*} dx \to 0,$$

one get

$$\int_{\mathcal{R}^N} |\nabla u_k|^p dx \to 0 \text{ and } \int_{\mathcal{R}^N} |y|^{-p}|u_k|^p dx \to 0.$$

Then, we obtain

$$E(u_k) = (1/p)\int_{\mathcal{R}^N} |\nabla u_k|^p dx - (\mu/p)\int_{\mathcal{R}^N} |y|^{-p}|u_k|^p dx$$
$$(1/p^*(s))\int_{\mathcal{R}^N} |y|^{-s}h|u_k|^{p^*(s)}dx - (1/p^*)\int_{\mathcal{R}^N} f|u_k|^{p^*}dx \to 0.$$

It contradict the fact that $E(u_k) \to c > 0$. Therefore, **Lemma 11** is proved.

As conclusion, by Lemma 10 and Lemma 11, one has that there exists $(u_k) \subset (0,+\infty)$ and $(x_k) \subset \mathcal{R}^N$ such that

$$T_{\eta_k,x_k}u_k \rightharpoonup u \text{ in } L^{p^*}(\mathcal{R}^N), u \neq 0 \qquad (17)$$

Setting $x_k = (y_k,z_k) = \bar{y}_k + \bar{z}_k$ where

$$\bar{y}_k = (y_k,0), \bar{z}_k = (0,z_k)$$

and

$$y_k \in \mathcal{R}^d, z_k \in \mathcal{R}^{N-d}.$$

Defining $v_k \coloneqq T_{\eta_k,\bar{z}_k}u_k$, we get

Lemma 12 The sequence (v_k) is bounded in X_l and it satisfies

$$E(v_k) \to c, E'(v_k) \to 0 \; in \; X_l'$$

and

$$v_k(.+_{\eta_k,\bar{y}_k}) \rightharpoonup u \; in \; L^{p^*}(\mathcal{R}^N) \qquad (18)$$

Proof Since (u_k) is bounded in X_l and the operators $T_{\eta_k,\bar{z}_k}$ are isometries of X_l, we get (v_k) is bounded in X_l easily. By Eq. (17) we obtain the formula (18).

Now, we say that

$$E(u_k) = E(v_k) \text{ and } \|E'(v_k)\|_{X_l'} = \|E'(u_k)\|_{X_l'}.$$

In fact, one has

$$E(v_k) \; := (1/p)\int_{\mathcal{R}^N} |\nabla v_k|^p dx - (\mu/p)\int_{\mathcal{R}^N} |y|^{-p}|v_k|^p dx$$
$$- (1/p^*(s))\int_{\mathcal{R}^N} |y|^{-s}h|v_k|^{p^*(s)}dx - (1/p^*)\int_{\mathcal{R}^N} f|v_k|^{p^*}dx$$
$$= (1/p)\int_{\mathcal{R}^N} |\nabla u_k|^p dx - (\mu/p)\int_{\mathcal{R}^N} |y|^{-p}|u_k|^p dx$$
$$- (1/p^*(s))\int_{\mathcal{R}^N} |y|^{-s}h|u_k|^{p^*(s)}dx - (1/p^*)\int_{\mathcal{R}^N} f|u_k|^{p^*}dx$$
$$= E(u_k),$$

and for all $\psi \in X_l'$, we have

$$\langle E'(v_k),\psi\rangle$$
$$= \int_{\mathcal{R}^N} |\nabla T_{\eta_k,\bar{z}_k}u_k|^{p-2}\nabla T_{\eta_k,\bar{z}_k}u_k.\nabla\psi dx - \mu\int_{\mathcal{R}^N} |y|^{-p}|T_{\eta_k,\bar{z}_k}u_k|^{p-2}T_{\eta_k,\bar{z}_k}u_k.\psi dx$$
$$- \int_{\mathcal{R}^N} |y|^{-s}h|T_{\eta_k,\bar{z}_k}u_k|^{p^*(s)-2}T_{\eta_k,\bar{z}_k}u_k.\psi dx - \int_{\mathcal{R}^N} f|T_{\eta_k,\bar{z}_k}u_k.\psi|^{p^*-2}T_{\eta_k,\bar{z}_k}u_k.\psi dx$$
$$= \eta_k^{\frac{N-p}{p}}\int_{\mathcal{R}^N} |\nabla u_k|^{p-2}\nabla u_k.\nabla\psi(\eta_k x - \eta_k\bar{z}_k)dx$$
$$- \mu\eta_k^{\frac{N-p}{p}}\int_{\mathcal{R}^N} |y|^{-p}|u_k|^{p-2}u_k.\psi(\eta_k x - \eta_k\bar{z}_k)dx$$
$$- \eta_k^{\frac{N-p}{p}}\int_{\mathcal{R}^N} |y|^{-s}h|u_k|^{p^*(s)-2}u_k.\psi(\eta_k x - \eta_k\bar{z}_k)dx$$
$$- \eta_k^{\frac{N-p}{p}}\int_{\mathcal{R}^N} f|u_k.\psi|^{p^*-2}u_k.\psi(\eta_k x - \eta_k\bar{z}_k)dx$$
$$= \left\langle E'(u_k), T_{\eta_k^{-1},-\eta_k\bar{z}_k}\psi\right\rangle.$$

So one get $\|E'(v_k)\|_{X_l'} = \|E'(u_k)\|_{X_l'}$.

Proposition 4 [8] Let $(\varphi_m) \subset \mathcal{R}^k$ such that $\lim_{m\to\infty}|\varphi_m| = +\infty$, R > 0 fixed, then for any t ∈ N\{0,1} there exists $m_t \in \mathcal{N}$ such that for any $m > m_t$ one can find $g_1,\ldots,g_t \in O(k)$ satisfying the condition $i \neq j \Rightarrow B_R(g_i\varphi_m) \cap B_R(gj\varphi_m) = \emptyset$.

Lemma 13 Up to a subsequence (v_k), there exists $v \in X_l$ and v ≠ 0 such that

$$v_k \rightharpoonup v \; in \; X_l.$$

Proof: since (v_k) is bounded in X_l, we can assume that $v_k \rightharpoonup v$ in X_l, if v = 0, we will show contradiction. Indeed, from Eq. (17) we know that

$$T_{1,\eta_k\bar{y}_k} \rightharpoonup v \text{ in } L^{p^*}(\mathcal{R}^N).$$

To get contradiction, we first prove that

$$\lim_{m\to\infty}\eta_k\bar{y}_k = +\infty \qquad (19)$$

If not, then up to a subsequence, $\lim_{m\to\infty}\eta_k\bar{y}_k = \bar{y}_0$. Therefore, **Lemma 12** implies

$$v_k = T_{1,0}v_k = T(T_{1,-\eta_k\bar{y}_k}T_{1,-\eta_k\bar{y}_k})v_k \rightharpoonup T_{1,-\bar{y}_0}u \neq 0,$$

it contradicts our assumption $v_k \rightharpoonup v = 0$.

Since u ≠ 0, there exist $\omega > 0$ and $D \subseteq \mathcal{R}^N$

with |D| $\neq$ 0 such that either $u > \omega$ or $u < -\omega$ almost everywhere in D. Given R > 0 such that $|B_R \cap D| > 0$, by weak convergence we get

$$\int_{\mathcal{R}^N} T_{1,\eta_k \bar{y}_k} v_k \chi_{B_R \cap D} dx \to \int_{\mathcal{R}^N} u \chi_{B_R \cap D} dx \geq \omega |B_R \cap D| > 0 \quad (20)$$

On the other hand,

$$\begin{aligned} \left| \int_{\mathcal{R}^N} T_{1,\eta_k \bar{y}_k} v_k \chi_{B_R \cap D} dx \right| &\leq \int_{B_R} |T_{1,\eta_k \bar{y}_k} v_k| dx \\ &= \int_{B_R} |v_k((x + \eta_k \bar{y}_k))| dx \\ &= \int_{B_R(\eta_k \bar{y}_k)} |v_k(x)| dx \\ &\leq C \left(\int_{B_R(\eta_k \bar{y}_k)} |v_k(x)|^{p^*} dx \right)^{1/p^*} \end{aligned} \quad (21)$$

where C only depends on R and N. The relations of Eq. (20) and Eq. (21) imply that

$$\lim_{k \to \infty} \inf \int_{B_R(\eta_k \bar{y}_k)} |v_k(x)|^{p^*} dx > 0.$$

Up to a subsequence, we can assume that for some $\varepsilon > 0$,

$$\inf_k \int_{B_R(\eta_k \bar{y}_k)} |v_k(x)|^{p^*} dx > \varepsilon. \quad (22)$$

Then, from **Proposition 4**, we have that for any t $\in$ N$\setminus\{0,1\}$ and $m > m_t$

$$\begin{aligned} \int_{\mathcal{R}^N} |v_k(x)|^{p^*} dx &\geq \sum_{i=1}^{t} \int_{B_R(\eta_k(g_i \bar{y}_k, 0))} |v_k(x)|^{p^*} dx \\ &= \sum_{i=1}^{t} \int_{B_R(\eta_k \bar{y}_k)} |v_k(x)|^{p^*} dx \geq t\varepsilon \end{aligned}$$

This implies that $\|v_k\|_{L^{p^*}(\mathcal{R}^N)} \to \infty$, which contradicts the fact that (v_k) is bounded in $L^{p^*}(\mathcal{R}^N)$

Proof of Theorem 4 From **Lemmas 11 and 13**, we get $E'(v_k) \to 0$ in X_l' and $v_k \rightharpoonup v \neq 0$ in $X_l(\mathcal{R}^N)$, which implies that v is a nontrivial cylindrical weak solution to the problem $(\mathcal{P}_{\lambda,\mu})$.

4. Conclusions

In our work, we have searched the critical points as the minimizers of the energy functional associated to the problem $(\mathcal{P}_{\lambda,\mu})$ on the constraint defined by the Nehari manifold, which are solutions of our problem. In the sections 3, we have proved the existence of at least four positive solutions by using a Nehari and sub-Nehari manifold and mountain pass theorem. In Section 3.4, we have considered the case $\mu < 0, q = p^*$ *and* $\lambda > 0$ and we have obtained the existence of the solution with cylidrical symmetry for $(\mathcal{P}_{\lambda,\mu})$ on the space

$$X \coloneqq X(\mathcal{R}^N; |y|^{-p} dx) \coloneqq \left\{ u \in \mathcal{D}_1^p(\mathcal{R}^N) : \int_{\mathcal{R}^N} |y|^{-p} h u^p dx < +\infty \right\}$$

by using the concentration-compactness principle.

References

[1] Ambrosetti, A., Brezis, H., and Cerami, G. 1994. “Combined effects of concave and convex nonlinearities in some elliptic problems.” *J. Funct. Anal.* 122: 519-543.

[2] de Figueiredo, D. G., Gossez, J. P., and Ubilla, P. 2003. “Local Superlinearity and Sublinearity for Indefinite Semilinear Elliptic Problems.” *J. Funct. Anal.* 199: 452-467.

[3] Wu, T. F. “Multiplicity Results for a Semilinear Elliptic Equation Involving Sign-Changing Weight Function.” *Rocky Mountain Journal of Mathematics*, in press.

[4] Cao, D. M., Peng, S. J., and Yan, S. S. 2012. “Infinitely Many Solutions for p-Laplacian Equation Involving Critical Sobolev Growth.” *J Funct Anal.* 262: 2861-2902.

[5] Filippucci, R., Pucci, P., and Robert, F. 2009. “On a p-Laplace Equation with Multiple Critical Nonlinearities.” *J Math Pure Appl.* 91: 156-177.

[6] Xuan, B. J., and Wang, J. 2010. “Existence of a Nontrivial Weak Solution to Quasilinear Elliptic Equations with Singular Weights and Multiple Critical Exponents.” *Nonlinear Analysis* 72: 3649-3658.

[7] Li, Y. Y., Guo, Q. Q., and Niu, P. C. 2011. “Global Compactness Results for Quasilinear Elliptic Problems with Combined Critical Sobolev-Hardy Terms.” *Nonlinear Analysis*. 74: 1445-1464.

[8] Badiale, M., Bergio, V., and Rolando, S. 2007. “A Nonlinear Elliptic Equation with Singular Potential and Applications to Nonlinear Field Equations.” *J Eur Math Soc.* 9: 355-381.

[9] Badiale, M., Guida, M., and Rolando, S. 2007. “Elliptic Equations with Decaying Cylindrical Potentials and Power-Type Nonlinearities.” *Adv Diff Equ.* 12: 1321-1362.

[10] Badiale, M., and Tarantello, G. 2002. “A Sobolev-Hardy Inequality with Applications to a Nonlinear Elliptic Equation Arising in Astrophysics.” *Arch Ration Mech*

Anal. 163: 252-293.

[11] Bhakta, M., and Sandeep, K. 2009. "Hardy-Sobolev-Maz'ya Type Equations in Bounded Domains." *J Differ Equ.* 247: 119-139.

[12] Gazzini, M., Ambrosetti, A., and Musina, R. Nonlinear elliptic problems related to some integral inequalities. http://www.digitallibrary.sissa.it/retrieve/4320/PhDThesis Gazzini.pdf

[13] Gazzini, M., and Mussina, R. 2009. "On a Sobolev Type Inequality Related to the Weighted p-Laplace Operator." *J Math Anal Appl.* 352: 99-111.

[14] Musina, R. 2008. "Ground State Solutions of a Critical Problem Involving Cylindrical Weights." *Nonlinear Anal.* 68: 3972-3986.

[15] Xuan, B. J. 2003. "Multiple Solutions to p-Laplacian Equation with Singularity and Cylindrical Symmetry." *Nonlinear Analysis* 55: 217-232.

[16] Brown, K. J., and Zhang, Y. 2003. "The Nehari Manifold for a Semilinear Elliptic Equation with a Signchanging Weight Function." *Journal of Differential Equations* 193: 481-499.

[17] Tarantello, G. 1993. "Multiplicity Results for an Inhomogeneous Neumann Problem Critical Exponent." *Manuscripta Math.* 81: 57-78.

[18] Drabek, P., Kufner, A., and Nicolosi, F. 1997. "Quasilinear Elliptic Equations with Degenerations and Singularities." *Walter de Gruyter Series in Nonlinear Analysis and Applications* 5 (New York, 1997).