

The Beta Power Exponential Distribution

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A new three-parameter beta power distribution is introduced and studied. We derive formal expressions for its moments, generating function and Cumulative density function. The maximum likelihood estimation of the model parameters was also conducted. In the end, the superiority of the new distribution over the exponentiated exponential was made by means of data set.

Key words: Exponential distribution, Beta power exponential, exponentiated exponential, maximum likelihood

Introduction

In recent time probability distributions play a significant role in modeling naturally occurring phenomena. By this, different kinds of distribution functions have been propounded and defined extensively in the statistical literature in modeling many real life phenomena. Various distributions emerge from the need to capture some naturally occurring events. For instance, Pareto distribution has been justified in modeling the size of companies, family income size, and population size among others. Poisson distribution can be used to measure the occurrence of rare events; Weibull and gamma have been identified as good distributions in modeling the amount of rainfall in African among other functions.

Probability distributions revealed lately involve compounded beta family of distribution which include beta-normal Eugene (2002); beta-Gumbel [Nadarajah and Kotz (2004)], beta-Weibull [Famoye (2005)], beta-exponential [Nadarajah and Kotz (2006)]; beta- Rayleigh [Akinsete and Lowe (2008)]; beta-Laplace [Kozubowski and Nadarajah (2008)]; beta-Pareto [Akinsete (2008)]; Kumaraswamy Pareto [Marcelo Bourguignon, etal (2012)], beta Nakagami [Shittu and Adepoju(2013)] among others.

The New Distribution

A random variable X is distributed as the Beta power exponential distribution if it satisfies the probability density function given as follows;

$$f(x) = \frac{\lambda}{\beta(a,b+1)} \left(1 - \ell^{-\lambda x}\right)^{a-1} \left(\ell^{-\lambda x}\right)^{b+1} \tag{1}$$

This can be showed to be a proper density function as below

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If we let $y = \ell^{-\lambda x}$ $\frac{dy}{dx} = -\lambda \ell^{-\lambda x}$; we have $\frac{-\lambda}{\beta(a,b+1)} \int (1-y)^{a-1} \frac{y^b dy}{\lambda}$ $\frac{1}{\beta(a,b+1)} \int_0^1 y^b (1-y)^{a-1} dy$ Let $M = 1-y \Rightarrow Y = 1-M$ $\frac{dm}{dy} = -1 \Rightarrow dy = -dm$

$$\frac{1}{\beta(a,b+1)} \int_{0}^{1} m^{a-1} (1-m)^{b} dm$$
$$\frac{\beta(a,b+1)}{\beta(a,b+1)} = 1$$

This verifies that distribution in Eq. (1) is indeed a proper density function.

Cumulative Function

Here we seek to the obtain the cumulative function of the new distribution, this is as follows from Eq. (1), we have

$$f(x) = \frac{1}{\beta(a,b+1)} \int_{0}^{x} \lambda \, \ell^{-\lambda y} \left(1 - \ell^{-\lambda y}\right)^{a-1} \ell^{-\lambda y b} dy$$
$$= \frac{1}{\beta(a,b+1)} \int_{\alpha}^{x} \left(\ell^{-\lambda y}\right)^{b-1} \left(1 - \ell^{-\lambda y}\right)^{a-1} dy$$

Let $U = \lambda y$ $\frac{du}{dy} = \lambda \Rightarrow dy = \frac{du}{\lambda}$

$$F(x) = \frac{1}{\beta(a,b+1)} \int_{0}^{\lambda x} \left(\ell^{-U}\right)^{b-1} \left(1-\ell^{-U}\right)^{a-1} du \quad .$$
⁽²⁾

If a > 0 is real non-integer we have

$$\left(1-K\right)^{a-1} = \sum_{j=0}^{\infty} \left(-1\right)^j \frac{\left(1-K\right)^j}{\left(1-K\right)^j} \frac{\left(1-K\right)^j}{\left(1-K\right)^j} J_i^{i}$$
(3)

Then using Eq. (3) in Eq. (2), we have

$$=\frac{1}{\beta(a,b+1)}\sum_{j=0}^{\infty}\frac{(-1)^{j}\overline{)(a)}}{\overline{)(a-j)}j!}\frac{\ell^{-u(b-1+j)}}{-(b+j-1)}\begin{vmatrix}\lambda x\\0\end{vmatrix}$$
(4)

.

Finally, using the fact that

$$\beta(a,b) = \frac{\overline{)(a)} \ \overline{)(b)}}{\overline{)(a+b)}}, \quad we have$$

$$F(x) = \frac{\overline{)(a+b+1)}}{\overline{)(b+1)}} \sum_{j=0}^{\infty} (-1)^j \ \frac{\left[1 - \ell^{-\lambda x(b+j-1)}\right]}{\overline{)(a-j)} \ j \ !(b+j-1)} \tag{5}$$

For positive real non-integer a, Eq. (5), can be used for further analytical and / or numerical studies. For integer a, Eq. (4) can be written as

$$F(x) = \frac{1}{\beta(a,b+1)} \sum_{j=0}^{a-1} {a-1 \choose j} (-1)^j \frac{\left[1 - \ell^{-\lambda x(b+j-1)}\right]}{(b+j-1)}$$
(6)

From the wolfram Function site for integer a

$$I_{y}(a,b) = 1 - \frac{(1-y)^{b}}{\overline{j(b)}} \sum_{j=0}^{a-1} \frac{j(b+j)y^{j}}{j!}$$

and for integer b

$$I_{y}\left(a,b\right) = \frac{y^{a}}{\overline{j(a)}} \sum_{j=0}^{b-1} \frac{\overline{j(a+j)^{j}}}{j!}$$

Then, if a is integer Eq. (5) can also be written as

$$F(x) = 1 - \ell \frac{\left(\ell^{-\lambda x}\right)^b}{\overline{i}(b)} \sum_{j=0}^{a-1} \frac{\overline{i}(b+j)\left(1 - \ell^{-\lambda x}\right)^j}{j!}$$
(7)

For integer value of *b*, we have

$$F(x) = \frac{\left(1 - \ell^{-\lambda x}\right)^a}{\overline{i}(a)} \sum_{j=0}^{b-1} \frac{\overline{i}(a+j)\left(\ell^{-\lambda xj}\right)}{j!}$$
(8)

Moments and Generating Function

In this section, we derive the rth moment about the origin and the moment generating function for a random variable X having the density function in Eq. (1). It is convenient to introduce

$$S_{d,b,a} = \int_{0}^{\infty} x^{d-1} \, \ell^{-x(b+1)} \left(1 - \ell^{-x}\right)^{a-1} dx$$

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Let $w = e^{-x}$ |nw = -x|

$$x = |Inw| \quad \frac{dx}{dw} = \frac{1}{w}$$
$$dx = \frac{dw}{w}$$
$$S_{d,b,a} = \int_{0}^{\infty} |Inw|^{d-1} w^{b+1} (1-w)^{a-1} \frac{dw}{w}$$
$$S_{d,b,a} = \int_{0}^{\infty} |Inw|^{d-1} w^{b} (1-w)^{a-1} dw$$

If a > 0 and is non-integer we have

$$= \overline{j(a)} \sum_{j=0}^{\infty} \frac{(-1)^{j}}{\overline{j(a-j)}} \int_{a}^{b} |Inw|^{d-1} w^{b+j} dw$$

Also, for real p>-1, and real q, we have

$$\int_{0}^{1} x^{p} \left| Inx \right|^{q} dx = \frac{\overline{(1+q)}}{(1+p)^{q+1}}$$

Then,

$$\int_{0}^{1} w^{b+j} \left| Inw \right|^{d-1} dw = \frac{\overline{\left(1 + d - 1\right)}}{\left(1 + b + j\right)^{d-1+1}}$$
$$= \frac{\overline{\left(d\right)}}{\left(b + j - 1\right)^{d}}$$
$$\int_{0}^{\infty} x^{d-1} f(x) dx = \overline{\left(a\right)} \sum_{j=0}^{\infty} \frac{(-1)^{j}}{\overline{\left(a - j\right)}} \frac{\overline{\left(d\right)}}{j! \left(b + j + 1\right)^{d}}$$

Finally, we have

$$\int_{0}^{\infty} x^{d-1} e^{-x(b+1)} (1 - e^{-x})^{a-1} dx =$$

$$\overline{i(a)} \overline{i(d)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\overline{i(a-j)} j! (b+j+1)^d}$$

If we let a > 0 be an integer then from ()

$$\int_{0}^{\infty} x^{d-1} f(x) dx = \sum_{j=0}^{d-1} {a-1 \choose j} (-1)^{j} \int_{0}^{1} \left| Inw \right|^{d-1} w^{b+j} dw$$

By using (), we have

$$\int_{0}^{\infty} x^{d-1} f(x) dx = \sum_{j=0}^{a-1} \frac{\binom{a-1}{j} (-1)^{j} \overline{j(d)}}{(b+j+1)^{d}}$$

which simplifies to

$$\int_{0}^{\infty} x^{d-1} e^{-x(b+1)} \left(1 - e^{-x}\right)^{a-1} dx$$
$$= \overline{i(d)} \sum_{j=0}^{a-1} \frac{\binom{a-1}{j} (-1)^{j}}{(b+j+1)^{d}}$$

Then mgf of X can be written

$$M_{(t)} = \frac{\lambda}{\beta(a,b+1)} \int_{0}^{\infty} e^{tx} \left(e^{-\lambda x}\right)^{b+1} \left(1 - e^{-\lambda x}\right)^{a-1} dx$$
$$= \frac{\lambda}{\beta(a,b+1)} \sum_{r=0}^{\infty} \frac{t^{r}}{r!} \int_{0}^{\infty} x^{r} \left(e^{-\lambda x}\right)^{b+1} \left(1 - e^{-\lambda x}\right)^{a-1} dx$$

Considering $\int_{0}^{\infty} x^{r} \left(e^{-\lambda x} \right)^{b+1} \left(1 - e^{-\lambda x} \right)^{ay} dx$

Let $f = e^{-\lambda x}$

$$Inf = -\lambda x$$
$$x = \frac{1}{\lambda} |Inf|$$
$$\frac{dx}{df} = \frac{1}{\lambda f} \Longrightarrow dx = \frac{df}{\lambda f}$$
$$\int_{0}^{\infty} \left[\frac{1}{\lambda} |Inf|\right]^{r} f^{b+1} (1-f)^{a-1} \frac{df}{\lambda f}$$
$$\frac{1}{\lambda^{r+1}} \int_{0}^{\infty} |Inf|^{r} f^{b} (1-f)^{a-1} df$$

If a > 0 is real non-integer

$$(1-f)^{a-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \overline{(a)} f^j}{\overline{(a-j)} j!}$$

$$\frac{1}{\lambda^{r+1}} \frac{\sum_{j=0}^{\infty} (-1)^j \overline{a}}{\overline{a-j} j!} \int_0^1 f^{b+j} \left| Inf \right|^r df$$

-

Using the fact that

$$\int_{0}^{1} x^{p} \left| Inx \right|^{q} dx = \frac{\left| (1+q) \right|}{(1+p)^{q+1}}, \text{ we have}$$
$$\frac{1}{\lambda^{r+1}} \frac{\sum (-1)^{j} \left| (a) \right|}{\left| (a-j) \right| j!} \frac{\left| (1+b+j)^{r+1} \right|}{(1+b+j)^{r+1}}$$

Therefore, mgf of X

$$\begin{split} M_{(t)} &= \frac{\lambda}{\beta(a,b+1)} \sum_{r=0}^{\infty} \frac{t^{r}}{r!} \frac{1}{\lambda^{r+!}} \frac{\sum_{1=\infty}^{\infty} (-1)^{j} \overline{)(a)}}{\overline{)(a-j)} \frac{\overline{)(r+1)}}{(b+j+1)^{r+1}} \\ &= \frac{1}{\lambda^{r}(a,b+1)} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^{r}(-1)^{j} \overline{)(a)} \overline{)(r+1)}}{r! \overline{)(a-j)} \frac{\overline{)(r+1)}}{j! (b+j+1)^{r+1}} \\ M_{(t)} &= \frac{\overline{)a+b+1}}{\lambda^{r} \overline{)a} \overline{)b+1}} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^{r}(-1)^{j} \overline{)(a)} \overline{)(r+1)}}{\overline{)(a-j)} \frac{\overline{)(r+1)}}{j! (b+j+1)^{r+1}} \\ &= \frac{\overline{)a+b+1}}{\lambda^{r} \overline{)a} \overline{)b+1}} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^{r}(-1)^{j} \overline{)(a)} \overline{)(r+1)}}{\overline{)(a-j)} \frac{\overline{)(r+1)}}{j! (b+j+1)^{r+1}} \\ M_{(t)} &= \frac{\overline{)a+b+1}}{\lambda^{r} \overline{)b+1}} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^{r}(-1)^{j} \overline{)(a)} \overline{)(r+1)}}{r! \overline{)(a-j)} (b+j+1)^{r+1}} \end{split}$$

and for integer a > 0, we have

$$M_{(t)} = \frac{1}{\lambda^{r} \beta(a, b+1)} \sum_{r=0}^{\infty} \frac{t^{r}}{r!} \overline{(r+1)} \sum_{j=0}^{a-1} \binom{a-1}{j} \frac{(-1)^{j}}{(b+j+1)^{r+1}} .$$
(9)

Therefore, using the fact that

$$M_{(t)} = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$$
, then

For non-integer *a*, the *r*th moment of X is

$$E\left(X^{r}\right) = \frac{\overline{\left(a+b+1\right)}}{\lambda^{r}\overline{\left(b+1\right)}} \sum_{j=0}^{\infty} \frac{(-1)^{j} \overline{\left(r+1\right)}}{\overline{\left(a-j\right)} (b+j+1)^{r+1}}$$

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$$E(X^{r}) = \frac{\overline{(a+b+1)}(r+1)}{\lambda^{r}(b+1)} \sum_{j=0}^{\infty} \frac{(-1)^{j}}{(a-j)(b+j+1)^{r+1}}$$
(10)

When a > 0 is integer, we have

$$E(X^{r}) = \frac{\overline{(r+1)}}{\lambda^{r}\beta(a,b+1)} \sum_{j=0}^{a-1} \frac{\binom{a-1}{j}(-1)^{j}}{(b+j+1)^{r+1}}$$
(11)

It follows from Eqs. (10) and (11) that when a = 1 and b = 1, and r = 1, it reduces to the mean of power exponential distribution which is

$$E(X^r) = \frac{2\overline{)r+1}}{\lambda^r}$$

Estimation of Parameters

Here we derive the maximum likelihood estimates (MLEs) of the parameters of the Beta Power Exponential Distribution.Let $x_1, x_2, ..., x_n$ be a random sample of size $n \text{ on } X^{\sim}$ BPED (a, b, λ) .

The likelihood function for $\theta = (a, b, \lambda)$ may be written as

$$L = \frac{\lambda^{n}}{\left[\beta(a,b+1)\right]^{n}} \prod_{i=1}^{n} \left(1 - e^{-\lambda x}\right)^{a-1} \left(e^{-\lambda \sum_{i=1}^{n} x}\right)^{b+1}$$
(12)

The values of the parameters that maximize the likelihood function also maximize the log likelihood. Taking the logarithm of equation (), we have

$$l = \log L(x|a,b,\lambda) - (b+1)\lambda \sum_{i=1}^{n} x$$

$$= n \log \lambda - n \log \beta(a,b+1) + (a-1)\sum_{i=1}^{n} \log (1-e^{-dx}) - (b+1)\lambda \sum_{i=1}^{n} x$$
(13)
Since $\beta(a,b+a) = \frac{\overline{j(a)} \overline{j(b+1)}}{\overline{j(a+b+1)}}$

$$= n \log \lambda - n \log \overline{j(a)} - n \log \overline{j(b+1)} + n \log \overline{j(a+b+1)} + (a-1)\sum_{i=1}^{n} \log (1-e^{-\lambda x}) - (b+1)\lambda \sum_{i=1}^{n} x$$
(14)

Differentiating Eq. (14) with respect a, b and λ respectively, and setting results to zero we have

$$\frac{\partial l}{\partial a} = \frac{-n\overline{(a)}^{1}}{(a)} + \frac{n\overline{(a+b+1)}^{1}}{(a+b+1)} + \sum_{i=1}^{n}\log(1-e^{-\lambda x})$$
(15)

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$$\frac{\partial l}{\partial b} = \frac{-n\overline{(b+1)}^{1}}{\overline{(b+1)}} + \frac{n\overline{(a+b+1)}^{1}}{\overline{(a+b+1)}} - \lambda \sum_{i=1}^{n} x$$
(16)

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - (b+1)\sum_{i=1}^{n} x + \lambda(a-1)\sum_{i=1}^{n} \frac{e^{-\lambda x}}{1 - e^{-\lambda x}}$$
(17)

Solving the above equations simultaneously for $a, b, and \lambda$ gives us their respective estimates

 \hat{a} , \hat{b} and $\hat{\lambda}$. Also, finding the derivatives of these equations, gives us the diagonal elements of the Fisher's Information Matrix. They are as follows:

$$\frac{\partial^2 l}{\partial a^2} = -n\Psi^1(a) + n\Psi^1(a+b+1)$$
$$\frac{\partial^2 l}{\partial a^2} = -n\Psi^1(a+b+1)$$
$$\frac{\partial^2 l}{\partial b^2} = -n\Psi^1(b+1) + n\Psi^1(a+b+1)$$
$$\frac{\partial^2 l}{\partial a\partial\lambda} = \lambda \sum_{i=1}^n \frac{e^{-\lambda x}}{1-e^{-\lambda x}}$$
$$\frac{\partial^2 l}{\partial b\partial\lambda} = -\sum_{i=1}^n x$$
$$\frac{\partial^2 l}{\partial a^2} = (a-1)\sum_{i=1}^n \frac{e^{-\lambda x}(1-e^{-\lambda x}-\lambda x)}{(1-e^{-\lambda x})^2}$$

Note that $\Psi(k) = \Gamma(k)' / \Gamma(k)$ the digamma function.

The Fisher's information Matrix is very useful tool for calculating the interval estimates, asymptotic variances, covariance and tests of hypothesis of *a*, *b* and λ .

Application

In this section we compare the fit of Beta power exponential distribution to one real data set. The data set is obtained from Smith and Naylor (1987).

The data are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory England.

The data set is: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

The descriptive statistics, the MLE of the parameters, the log-likelihood values as well as the Akaike





The Qanorm of Glass fibre

Table 1

Descriptive Statistics

Min	Q ₁	Q ₂	Mean	Q ₃	Max	Var
0.550	1.375	1.590	1.507	1.685	2.240	0.105

Table 2

MLES of the Model parameters, the negative log-likelihood values and the statistics AIC

	Estimates			Statistics		
Model	a	b	c	λ	Likelihood	AIC
Beta power exponential	1.00006	1.9999	1	3.0429e-07	-572.5149	1147.03
Exponentiated exponential	2.0006	1.0006	1	2.2944e-07	-611.5369	1227.074

Tables 1 and 2 provide some descriptive statistics and the MLES of the model parameters. Since the values of the Akaike information is smaller for the Beta power distribution compared with the exponentiated

exponential distribution, it

Conclusion

The existing two parameter power exponential distribution is extended with the introduction of two extra shape parameters giving rise to the Beta-power exponential distribution which has a better shape, broader tails and a class of hazard rate functions depending on the parameters.

Detailed studies of the statistical properties of the proposed distribution which include moments, moment generating function among others have been presented.

The parameters of the model were estimated by method of maximum likelihood which actually paves way for the derivation of fisher information matrix. Real life application indicates that Beta-power exponential distribution apart from its flexibility has better representation of data than exponentiated exponential distribution.

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