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Relationship Between Classical and Quantum Physics

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Abstract: Schrödinger's equation is one the equations that mark the beginnings of the systematic quantum physics. It was shown that it follows from the Dirac's equation and the relationship with classical physics, i.e. with classical field theory was established. The subject of this work is the relationship between classical relativistic physics and the quantum physics. Investigation carried out in this work, shows that the free electromagnetic field, spinor Dirac's field without mass, spinor Dirac's field with mass, and some other fields are described by the same vibrational formulation. The conditions that a field be described by Maxwell's equations of motion are given in this work, and some solutions of these conditions are also given. Non-relativistic approximation of the equations of the non-quantified field are the Schrödinger's equations. Dirac's equation as a special case, contains Maxwell's equations and the Schrödinger's equation.

Key words: Relationship, quantum physics, classical physics.

1. Introduction

Schrödinger's equation is one the equations that marks the beginnings of the systematic quantum physics. It was shown that it follows from the Dirac's equation and the relationship with classical physics, i.e. with classical field theory was established. The aim of this work is to show the relationship between the classical relativistic physics and the quantum physics. For this purpose, in the second part of this work, the authors have obtained Maxwell's equations from the Dirac's equation, whereas in the third part we brought the Maxwell's equations into the form of the Dirac's equation for leptons. In the fourth part of this work, it is shown that Schrödinger's equation for the free particle is obtained from the Dirac's equation. In this part, the case when the particle is affected by the external electromagnetic field is analyzed as well

2. Relationship between Electromagnetic and Dirac's Field

According to the classical mechanics where the variational description of the law of motion of the physical objects is given, Lagrange's equations are second order differential equations, whereas canonical equations are first order differential equations. Dirac's equation is a first order differential equation. It is in conformity with Maxwell's equations of electrodynamics, which are first order differential equations, as well. This question is not only a formal mathematical problem. It is profoundly physical because it contains the well-founded language of physics. In this language generalized momenta, energy etc., are defined. Therefore, if Dirac's equation represents and describes a physical object, then its character is the character of canonical equations of that object. According to this fact and taking into account the analogy with electromagnetic field, "potentials" of the Dirac's field as Lagrange's variables of that field are introduced, despite the conventional approach where the field itself plays the role of the Lagrange's variables. A correct and consistent formulation of the Dirac's field is built on this basis [1].

Lagrange's density of the linear theory is:

$$\Im = K[(-i\partial_\mu \bar{\Phi}\gamma^\mu)(i\partial_\nu \gamma^\nu \Phi)] \tag{1}$$

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where K is a constant and $\overline{\phi} = \phi^+ \gamma^0$.

Lagrange's equations, given by

$$\frac{\partial \Im}{\partial \eta} - \frac{\partial}{\partial x^\alpha} \frac{\partial \Im}{\partial \left(\frac{\partial \eta}{\partial x^\alpha} \right)} = 0 \qquad (2)$$

are then

$$\partial_\mu \partial^\mu \phi = 0, \qquad \partial_\mu \partial^\mu \phi^+ = 0 \qquad (3)$$

and the corresponding canonical equations are

$$\psi = i\partial_\alpha \gamma^\alpha \Phi, \qquad i\partial_\alpha \gamma^\alpha \psi = 0 \qquad (4)$$

Second of the canonical equations is the Dirac's equation. It is shown, then, that the canonical equation of the (free) Dirac's field is not the Lagrange's equation, but the Dirac's equation. Lagrange's equation Eq. (3) is mathematically equivalent to the Lagrange's equations for potentials of the free electromagnetic field

$$\partial_\mu \partial^\mu A^\alpha = 0 \qquad (5)$$

or

$$\Delta \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0, \quad \Delta \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \qquad (6)$$

Correct canonical formalism for the Maxwell's field must give all four Maxwell's equations that contain first space and time derivatives of the components $\vec{E}$, $\vec{B}$ of the field, whereas the correct Lagrange's formalism must lead to the partial differential Eq. (5) without introducing additional conditions, such as Lorentz's condition. Equivalence of the dynamical equations of these two fields, Dirac's field and the two-vector or tensor-scalar field (that contains standard electromagnetic field) can be shown also by observation of the transformation properties of the Lagrangean Eq. (1).

Components $A_x, A_y, A_z, \varphi, C_x, C_y, C_z, f$ of the potential Φ, which are Lagrange's variables, if they describe tensor-scalar field, must be transformed according to the law [2]

$$\begin{gathered} A'_x = A_x ch\phi - \varphi sh\phi, \quad b'_x = b_x ch\phi - fsh\phi \\ \varphi' = \varphi ch\phi - A_x sh\phi, \quad f' = fch\phi - b'_x sh\phi \qquad (7) \\ A'_y = A_y, \quad b'_y = b_y \\ A'_z = A_z, \quad b'_z = b_z, \quad th\phi = \frac{v}{c} \end{gathered}$$

or

$$\begin{gathered} E'_x = E_x, \quad B'_x = B_x \\ E'_y = E_y ch\phi - B_z sh\phi, \quad B'_y = B_y ch\phi + E_z sh\phi \qquad (8) \\ E'_z = E_z ch\phi + B_y sh\phi, \quad B'_z = B_z ch\phi - E_y sh\phi \\ G' = G, \quad F' = F \end{gathered}$$

On the other hand, if they describe spinor field, then during the transition from one to another Lorentz's system, they must be transformed according to the law

$$\begin{gathered} A'_x = A_x ch\frac{\phi}{2} - \varphi sh\frac{\phi}{2}, \quad b'_x = b_x ch\frac{\phi}{2} - fsh\frac{\phi}{2} \\ A'_y = A_y ch\frac{\phi}{2} + b_z sh\frac{\phi}{2} \\ b'_y = b_y ch\frac{\phi}{2} - A_z sh\frac{\phi}{2} \qquad (9) \\ A'_z = A_z ch\frac{\phi}{2} - b_y sh\frac{\phi}{2}, \ b'_z = b_z ch\frac{\phi}{2} + A_y sh\frac{\phi}{2} \\ \varphi' = \varphi ch\frac{\phi}{2} - A_x sh\frac{\phi}{2}, \ f' = fch\frac{\phi}{2} - b_x sh\frac{\phi}{2} \end{gathered}$$

$$\begin{gathered} E'_x = E_x ch\frac{\phi}{2} - Gsh\frac{\phi}{2}, \\ B'_x = B_x ch\frac{\phi}{2} - Fsh\frac{\phi}{2} \\ E'_y = E_y ch\frac{\phi}{2} - B_z sh\frac{\phi}{2} \\ B'_y = B_y ch\frac{\phi}{2} + E_z sh\frac{\phi}{2} \qquad (10) \\ E'_z = E_z ch\frac{\phi}{2} + B_y sh\frac{\phi}{2} \quad B'_z = B_z ch\frac{\phi}{2} - E_y sh\frac{\phi}{2} \\ G' = Gch\frac{\phi}{2} - B_x sh\frac{\phi}{2} \quad F' = Fch\frac{\phi}{2} - E_x sh\frac{\phi}{2} \end{gathered}$$

Relativistic invariant Lagrangean Eq. (1), through the variational principle, leads to the correct equations of the electromagnetic field. Two four-quantities $\left(A^\alpha, C^\alpha\right)$ appear.If they are four-vectors (one of them pseudo-four-vector), then $F^{\alpha\beta}$ is an antisymmetric tensor of the second range, and G and F are scalars (in relation to the Lorentz's transforms

$$\begin{gathered} \partial_\alpha (F^{\alpha\beta} + g^{\alpha\beta} G) = 0 \\ \partial_\alpha (\tilde{F}^{\alpha\beta} + g^{\alpha\beta} F) = 0 \\ F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha - \frac{1}{2} \varepsilon^{\alpha\beta\xi\zeta} \left(\partial_\xi C_\zeta - \partial_\zeta C_\xi \right) \end{gathered}$$

$$\tilde{F}^{\alpha\beta}=\partial^{\alpha}C^{\beta}-\partial^{\beta}C^{\alpha}+\frac{1}{2}\varepsilon^{\alpha\beta\xi\zeta}\left(\partial_{\xi}A_{\zeta}-\partial_{\zeta}A_{\xi}\right)$$
$$G=\partial_{\alpha}A^{\alpha},\quad F=\partial_{\alpha}C^{\alpha} \tag{11}$$

Taking $F = G = 0$, these equations become:

$$\partial_{\alpha}F^{\alpha\beta}=0,\quad \partial_{\alpha}\tilde{F}^{\alpha\beta}=0 \tag{12}$$

i.e., equations of the free Maxwell's field. The authors see that equations of the free Maxwell's field are, in fact, a special case of the canonical Eq. (11). If the authors chose $C^{\alpha}=0,$ then we obtain equations of the standard electromagnetic theory of the field, which are the equations of the conventional theory of the free electromagnetic field. Therefore, it follows that the same dynamic equations describe free Dirac's field and the electromagnetic field [2]. In the classical theory of the electromagnetic field, Lagrange's variables are the potentials A_{α} of the electromagnetic field, whereas the tensor of the electromagnetic field forms the four-vector ∂_{α} and the potential A_{α}. The authors now discuss the field with the mass term. In that case the density of the Lagrangean is

$$\Im=K\left(\tilde{\psi}\psi-\kappa^{2}\tilde{\Phi}\Phi\right),\quad \psi=i\partial_{\alpha}\gamma^{\alpha}\Phi \tag{13}$$

where κ is a scalar constant. In order that this Lagrangean be a relativistic scalar, $\tilde{\Phi}\Phi$ must be a scalar as well. Using the Euler-Lagrange's equations, the authors obtain Lagrange's equations of motion

$$\left(\partial_{\alpha}\partial^{\alpha}+\kappa^{2}\right)\Phi=0,\quad \left(\partial_{\alpha}\partial^{\alpha}+\kappa^{2}\right)\Phi^{+}=0 \tag{14}$$

According to the definition of the conjugated momentum, the authors have:

$$\pi_{\Phi}=iK\psi^{+},\quad \pi_{\Phi}{}^{+}=-iK\psi \tag{15}$$

Now, the density of the Hamiltonian, is

$$H=\pi_{\Phi}\partial_{0}\Phi+\partial_{0}\Phi^{+}\pi_{\Phi^{+}}-\Im$$
$$=\frac{1}{K}\pi_{\Phi}\gamma^{0}\pi_{\Phi^{+}}-\partial_{i}\Phi^{+}\alpha^{i}\pi_{\Phi^{+}}-\pi_{\Phi}\partial_{i}\alpha^{i}\Phi+K\kappa^{2}\tilde{\Phi}\Phi \tag{16}$$

Canonical equations in covariant notation follow:

$$\partial_{\mu}\gamma^{\mu}\pi_{\Phi^{+}}+K\kappa^{2}\Phi=0$$
$$\frac{1}{K}\pi_{\Phi^{+}}-\partial_{\mu}\gamma^{\mu}\Phi=0 \tag{17}$$
$$\frac{1}{K}\pi_{\Phi}\gamma^{0}-\partial_{\mu}\tilde{\Phi}\gamma^{\mu}=0$$
$$\partial_{\mu}\pi_{\Phi}\gamma^{0}\gamma^{\mu}+K\kappa^{2}\tilde{\Phi}=0$$

After separating the real and imaginary parts of each four components of the Dirac's field function, the authors have

$$\partial_{\alpha}\left(F^{\alpha\beta}+g^{\alpha\beta}G\right)+\kappa^{2}A^{\beta}=0$$
$$\partial_{\alpha}\left(\tilde{F}^{\alpha\beta}+g^{\alpha\beta}F\right)+\kappa^{2}C^{\beta}=0 \tag{18}$$
$$F^{\alpha\beta}=\partial^{\alpha}A^{\beta}-\partial^{\beta}A^{\alpha}-\frac{1}{2}\varepsilon^{\alpha\beta\xi\zeta}\left(\partial^{\xi}C^{\zeta}-\partial^{\zeta}C^{\xi}\right)$$
$$\tilde{F}^{\alpha\beta}=\partial^{\alpha}C^{\beta}-\partial^{\beta}C^{\alpha}+\frac{1}{2}\varepsilon^{\alpha\beta\xi\zeta}\left(\partial^{\xi}A^{\zeta}-\partial^{\zeta}A^{\xi}\right)$$
$$G=\partial_{\alpha}A^{\alpha},\quad F=\partial_{\alpha}C^{\alpha}$$

For $F=G=0,\ \kappa\neq 0$ The authors obtain canonical equation for the real Proca's field, if A^{α},C^{α} are four-vectors,

$$rot\vec{B}=\partial_{0}\vec{E}-\kappa^{2}\vec{A}$$
$$div\vec{B}=-\kappa^{2}f \tag{19}$$
$$rot\vec{E}=-\partial_{0}\vec{B}+\kappa^{2}\vec{C}$$
$$div\vec{E}=-\kappa^{2}\phi$$

For the spinor field, introducing linear combination of spinor Φ and ψ

$$\psi_{I}=\frac{1}{\sqrt{2}}\left(\kappa\Phi+i\psi\right),\quad \psi_{II}=\frac{1}{\sqrt{2}}\left(\kappa\Phi-i\psi\right) \tag{20}$$
$$or\quad \Phi=\frac{1}{\kappa\sqrt{2}}\left(\psi_{I}+\psi_{II}\right),\quad \psi=\frac{1}{i\sqrt{2}}\left(\psi_{I}-\psi_{II}\right)$$

Eq. (19) become,

$$\left(i\partial_{\alpha}\eta^{\alpha}-\kappa\right)\psi_{I}=0,\quad \left(i\partial_{\alpha}\eta^{\alpha}+\kappa\right)\psi_{II}=0 \tag{21}$$

These are Dirac equations with eight - component functions and with positive and negative mass terms. According to this, the authors conclude that we come to the Dirac's field described by the Lagrangean Eq. (13) and both Eq. (21), if we remain in the scope of the classical relativistic theory [2].

3. Conditions of the Maxwell's Field

In the contemporary physics of the electromagnetic fields, Maxwell's operator plays a very important role. The question arises, which physical objects are formulated by the same or analogous equations of physics. We shall discuss this question in this section. Maxwell's system of equations of the free field, we can write in the matrix form [3]

$$\mathrm{D}\psi = 0 \tag{22}$$

$$\psi = \mathrm{D}\Phi \tag{23}$$

where

$$\mathrm{D} = \partial_\alpha \eta^\alpha \tag{24}$$

is called Maxwell's operator, and η^α are matrices that satisfy conditions of the antisymmetry

$$\left\{\eta^\alpha, \eta^\beta\right\} = 2g^{\alpha\beta} \tag{25}$$

$$\eta^0 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}_{8\times 8}, \quad \eta^i = \begin{vmatrix} 0 & a^i \\ -a^{i+} & 0 \end{vmatrix} \tag{26}$$

$$a^1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad a^2 = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix},$$

$$a^3 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} \tag{27}$$

Where ψ and Φ are given in components:

$$\psi = \begin{bmatrix} E_x \\ E_y \\ E_z \\ F \\ B_x \\ B_y \\ B_z \\ G \end{bmatrix}, \quad \Phi = \begin{vmatrix} -\mathrm{A}_x \\ -\mathrm{A}_y \\ -\mathrm{A}_z \\ \mathrm{f} \\ \mathrm{C}_x \\ \mathrm{C}_y \\ \mathrm{C}_z \\ -\varphi \end{vmatrix}, \quad \tilde{\psi} = \psi^+ \eta^0 \tag{28}$$

and F and G are scalars. We operate with the operator D on the field functions ψ, and obtain

$$\begin{gathered} \partial_0 \vec{E} - rot\vec{B} + \nabla G = 0 \\ \partial_0 \vec{B} - rot\vec{E} + \nabla F = 0 \\ div\vec{E} = -\partial_0 G \\ div\vec{B} = -\partial_0 F \end{gathered} \tag{29}$$

Chosing scalars F, G such that $F = G = 0$, the authors obtain Maxwell's equations for the free field. Thus, the authors have shown that we can describe electromagnetic field by Eq. (22) and Eq. (23). These equations are called "Maxwell's equations". We make the question: Is the electromagnetic field the only field that satisfies the Maxwell's Eqs. (22) and (23). The authors are interested in whether any other field exists that can be described by these equations. For this purpose, the authors will seek for the conditions of the relativistic invariance of the field equations. The authors start from the Lorentz's transform given by Ref. [3]

$$x_\alpha = a_\alpha^\beta x'_\beta, \quad a_\alpha^\beta = \begin{vmatrix} \mathrm{ch}\Phi & \text{-}\mathrm{sh}\Phi & 0 & 0 \\ \text{-}\mathrm{sh}\Phi & \mathrm{ch}\Phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{30}$$

Corresponding field transform, the authors will write in the form:

$$\Psi = R\Psi', \ \Phi = S\Phi', \ \left(\Phi \to 1, R \to 1, S \to 1\right) \tag{31}$$

where R and S are operators of the transform for Ψ and Φ, respectively.

Requirement for the relativistic invariance of the Lagrangean,

$$\Im = K\tilde{\Psi}\Psi = K\Psi'^{+}R^{+}\eta^0 R\Psi' = K\tilde{\Psi}'\Psi' \tag{32}$$

$$\text{gives} \quad R^+\eta^0 R = \eta^0 \tag{33}$$

whereas relativistic invariance of the Eq. (23) gives

$$R^{-1} a_\alpha^\beta \eta^\alpha S = \eta^\beta \tag{34}$$

Therefore, the conditions of the relativistic invariance of the "Maxwellian" Eqs. (22) and (23) are given by Eqs. (33) and (3). One of the solutions of the set of Eqs. (33) and (34) is

$$R = \begin{vmatrix} a & b \\ b^+ & a \end{vmatrix}_{8\times 8}, \quad S = \begin{vmatrix} c & d \\ d^+ & c \end{vmatrix}_{8\times 8} \tag{35}$$

where

$$a = \begin{vmatrix} ch\left(\Phi - \varepsilon\right) & 0 & 0 & 0 \\ 0 & ch\varepsilon & 0 & 0 \\ 0 & 0 & ch\varepsilon & 0 \\ 0 & 0 & 0 & ch\left(\Phi - \varepsilon\right) \end{vmatrix}$$

$$b = \begin{vmatrix} 0 & 0 & 0 & sh\left(\Phi - \varepsilon\right) \\ 0 & 0 & sh\varepsilon & 0 \\ 0 & -sh\varepsilon & 0 & 0 \\ sh\left(\Phi - \varepsilon\right) & 0 & 0 & 0 \end{vmatrix}$$

$$c=\begin{vmatrix} ch\varepsilon & 0 & 0 & 0 \\ 0 & ch(\Phi-\varepsilon) & 0 & 0 \\ 0 & 0 & ch(\Phi-\varepsilon) & 0 \\ 0 & 0 & 0 & ch\varepsilon \end{vmatrix}$$

$$d=\begin{vmatrix} 0 & 0 & 0 & sh\varepsilon \\ 0 & 0 & sh(\Phi-\varepsilon) & 0 \\ 0 & -sh(\Phi-\varepsilon) & 0 & 0 \\ sh\varepsilon & 0 & 0 & 0 \end{vmatrix} \quad (36)$$

ε is a parameter confined by the condition:

$R\to 1$, $S\to 1$ for $\Phi\to 0$ i.e. $\varepsilon=\varepsilon(\Phi)$, $\varepsilon(0)=0$

4. Some of the Solutions and Their Interpretation

The authors will seek for the list of solutions of the Eqs. (33) and (34) for the values of the parameters

$$\varepsilon=\lambda\Phi,\ \lambda=1,\frac{1}{2},0,-\frac{1}{2},-1.$$

If we chose λ = 1, $\varepsilon=\Phi$, operating with the operator S on Φ'

$$S=\begin{vmatrix} ch\Phi & 0 & 0 & 0 & 0 & 0 & 0 & sh\Phi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ch\Phi & sh\Phi & 0 & 0 & 0 \\ 0 & 0 & 0 & sh\Phi & ch\Phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ sh\Phi & 0 & 0 & 0 & 0 & 0 & 0 & ch\Phi \end{vmatrix}$$

the authors obtain

$$A_x=A'_x\text{ch}\Phi+\varphi'\text{sh}\Phi$$

$$C_x=C'_x\text{ch}\Phi+f'\text{sh}\Phi$$

$$A_y=A'_y,\quad C_y=C'_y \quad (37)$$

$$A_z=A'_z,\quad C_z=C'_z$$

$$\varphi=\varphi'\text{ch}\Phi+A'_x\text{sh}\Phi,\qquad f=f'\text{ch}\Phi+C'_x\text{sh}\Phi$$

In this case Φ is consisted by four-vectors

$(\phi,\vec{A})$ and $(f,\vec{C})$.

Operating with the operator R on ψ',

$$R=\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & ch\Phi & 0 & 0 & 0 & 0 & sh\Phi & 0 \\ 0 & 0 & ch\Phi & 0 & 0 & -sh\Phi & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -sh\Phi & 0 & 0 & ch\Phi & 0 & 0 \\ 0 & sh\Phi & 0 & 0 & 0 & 0 & ch\Phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

we obtain

$$E_x=E'_x,\qquad B_x=B'_x$$

$$E_y=E'_y\text{ch}\Phi+B'_z\text{sh}\Phi,\quad B_y=B'_y\text{ch}\Phi-E'_z\text{sh}\Phi$$

$$E_z=E'_z\text{ch}\Phi-B'_y\text{sh}\Phi,\quad B_z=B'_z\text{ch}\Phi+E'_y\text{sh}\Phi$$

$$F=F',\quad G=G' \quad (38)$$

In this case Ψ is consisted from the antisymmetric tensor $F^{\alpha\beta}$ and the scalars G and F. That means that for the choice $\lambda=1,\varepsilon=\Phi$, we have two-vector fiel (A^α,C^α), or tensor-scalar field $(F^{\alpha\beta},G,F)$. For $F=G=0,C^\alpha=0$ this field is reduced into the classic standard electromagnetic field.

If the authors chose $\lambda=\frac{1}{2},\varepsilon=\frac{\Phi}{2}$, than $R=S$.

Therefore, Φ and ψ are of the same magnitude in relation to the Lorentz's transform. Operating with operator R on ψ', the authors obtain:

$$E_x=E'_x\text{ch}\frac{\Phi}{2}+G'\text{sh}\frac{\Phi}{2}$$

$$E_z=E'_z\text{ch}\frac{\Phi}{2}+B'_y\text{sh}\frac{\Phi}{2}$$

$$G=G'\text{ch}\frac{\Phi}{2}+E'_x\text{sh}\frac{\Phi}{2}$$

$$B_z=B'_y\text{ch}\frac{\Phi}{2}-E'_z\text{sh}\frac{\Phi}{2} \quad (39)$$

$$E_y=E'_y\text{ch}\frac{\Phi}{2}+B'_z\text{sh}\frac{\Phi}{2}$$

$$F=F'\text{ch}\frac{\Phi}{2}+B'_x\text{sh}\frac{\Phi}{2}$$

$$B_z=B'_z\text{ch}\frac{\Phi}{2}+E'_y\text{sh}\frac{\Phi}{2}$$

$$B_x=B'_x\text{ch}\frac{\Phi}{2}+F'\text{sh}\frac{\Phi}{2}$$

The authors see that we have to deal here with the real spinors' transform (E_x, G), (E_y, B_z), $(E_z, -B_y)$, (F, B_x).

Therefore, Φ and Ψ are spinors, and the field is a spinor field, and the equations of motion Eqs. (22) and (23) satisfy the spinor field as well. This field is without the rest mass.

For the case $\lambda = 0$, $\varepsilon(\Phi) = \lambda\Phi$, $\varepsilon = 0$, we apply R on ψ' and S on Φ'; we have:

$$E_x = E'_x \mathrm{ch}\Phi + G'\mathrm{sh}\Phi,\ B_x = B'_x \mathrm{ch}\Phi + F'\mathrm{sh}\Phi$$

$$E_y = E'_y,\quad B_y = B'_y$$

$$E_z = E'_z,\quad B_z = B'_y$$

$$F = F'\mathrm{ch}\Phi + B'_x\mathrm{sh}\Phi,\quad G = G'\mathrm{ch}\Phi + E'_x\mathrm{sh}\Phi$$

$$A_x = A'_x,\quad C_x = C'_x \tag{40}$$

$$A_y = A'_y\mathrm{ch}\Phi - C'_z\mathrm{sh}\Phi$$

$$C_y = C'_y\mathrm{ch}\Phi + A'_z\mathrm{sh}\Phi$$

$$A_z = A'_z\mathrm{ch}\Phi + C'_y\mathrm{sh}\Phi$$

$$C_z = C'_z\mathrm{ch}\Phi - A'_y\mathrm{sh}\Phi$$

$$f = f',\quad \varphi = \varphi'$$

In this case $\vec{A}$ and $\vec{C}$ form antisymmetric tensor of the second rank, φ and f are scalars, and (E_x, E_y, E_z, G) and (B_x, B_y, B_z, F) form four-vectors analogous electrodynamics where $(\vec{E}, \vec{B})$ form antisymmetric tensor of the electromagnetic field. Dirac's equation $(iD - \kappa)\psi = 0$, without mass term is $(\kappa = 0)$, where Ψ is a field function with components

$$\left(E_x, E_y, E_z, F, B_x, B_y, B_z, G\right)$$

Operating with the operator D on the field function Eq. (28), we obtain Eq. (29). For the values $F = G = 0$ Eq. (29) are the Maxwell's equations for the free field.

5. Description of the Maxwell's Equations in the Form of the Dirac's Equation

We will show that Maxwell's equations in the vaccum can be written in the form of the Dirac's equation for massless particles [4]

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\sum_k \alpha_k \frac{\partial\psi}{\partial x^k} + \beta mc^2\psi \tag{41}$$

$$\text{for } m = 0\quad i\hbar\frac{\partial\psi}{\partial t} = c\left[\sum_k \alpha_k\left(-i\hbar\right)\frac{\partial}{\partial x^k}\psi\right] \tag{42}$$

$$\text{or } i\hbar\frac{\partial\psi}{\partial t} = c\vec{\alpha}\vec{P}\psi \tag{43}$$

Eq. (43) represents the Dirac's equation for leptons.

The authors begin with the expressions

$$\left(\vec{\nabla}\times\vec{E}\right)_j = \varepsilon_{jkl}\frac{\partial}{\partial x_k}E_l \tag{44}$$

$$\left(\vec{\nabla}\times\vec{B}\right)_j = \varepsilon_{jkl}\frac{\partial}{\partial x_k}B_l \tag{45}$$

and the components $j = 1, 2, 3$ of the Maxwell's equations will be

$$\varepsilon_{jkl}\frac{\partial}{\partial x_k}E_l + \frac{1}{c}\frac{\partial B_j}{\partial t} = 0 \tag{46}$$

$$\varepsilon_{jkl}\frac{\partial}{\partial x_k}B_l - \frac{1}{c}\frac{\partial E_j}{\partial t} = 0 \tag{47}$$

$$\frac{\partial B_j}{\partial x_j} = 0 \tag{48}$$

$$\frac{\partial E_j}{\partial x_j} = 0 \tag{49}$$

Substitute the Crammers' complex vector $\psi_j = E_j + iB_j$ in these equations, multiplying Eqs. (48) and (49) by $i\varepsilon_{jkl}$ and thereafter, adding Eq. (46) with Eq. (47), and Eq. (48) with Eq. (49), the authors obtain:

$$\varepsilon_{jkl}\frac{\partial\psi_l}{\partial x_k} - \frac{i}{c}\frac{\partial\psi_j}{\partial t} = 0 \tag{50}$$

$$\frac{\partial\psi_j}{\partial x_j} = 0 \tag{51}$$

Differentiate the Eq. (50) according to x_j

$$\varepsilon_{jkl}\frac{\partial^2\psi_l}{\partial x_j\partial x_k} = \frac{i}{c}\frac{\partial}{\partial t}\left(\frac{\partial\psi_j}{\partial x_j}\right) = 0 \tag{52}$$

We will now try to bring the expression

$$\varepsilon_{jkl}\frac{\partial\psi_l}{\partial x_k} = \frac{i}{c}\frac{\partial\psi_j}{\partial t} = 0 \tag{53}$$

into the form of the Dirac's equation for leptons (43)

$$i\hbar\frac{\partial\psi_j}{\partial t}=c\left(i\varepsilon_{jkl}\right)\left(-i\hbar\frac{\partial}{\partial x_k}\right)\psi_l \tag{54}$$

The authors use the matrix

$$S\left(k\right)jl=i\varepsilon_{jkl} \tag{55}$$

to obtain

$$i\hbar\frac{\partial\psi_j}{\partial t}=c\left[\left(S_k\right)_{jl}P_k\right]\psi_l \tag{56}$$

i.e., Maxwell's equations in the form of the Dirac's equation. Matrix $\left(S_k\right)_{jl}$ is 3×3, $k=1,2,3$ matrix

$$\left(S_1\right)_{jl}=i\varepsilon_{j1l}=i\begin{pmatrix}\varepsilon_{111}&\varepsilon_{112}&\varepsilon_{113}\\\varepsilon_{211}&\varepsilon_{212}&\varepsilon_{213}\\\varepsilon_{311}&\varepsilon_{312}&\varepsilon_{313}\end{pmatrix}$$

$$=S_x=i\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}$$

$$\left(S_2\right)_{jl}=S_y=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix}$$

$$\left(S_3\right)_{jl}=S_z=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix} \tag{57}$$

These matrices have the properties

$$\left[S_i,S_j\right]=i\varepsilon_{ijk}S_k \tag{58}$$

$$S^2=S_1^2+S_2^2+S_3^2=2 \tag{59}$$

so that the authors have:

$$S^2\psi=S\left(S+1\right)\psi=2\psi \tag{60}$$

$$\text{i.e. } S=1 \quad i\hbar\frac{\partial\psi_j}{\partial t}=c\left(S_k\right)_{jl}P_k\psi_l \tag{61}$$

and the Eq. (61) describes the particle with a spin $S=1$. But, this particle is the quant of the electromagnetic field—a photon.

6. Non-Relativistic Boundary and the Schrödinger's Equation

From the Dirac's equations with the positive and negative mass term

$$\left(iD-\kappa\right)\psi_I=0,\ \left(iD+\kappa\right)\psi_{II}=0 \tag{62}$$

The authors will pass to the non-relativistic Schrödinger's field.

The authors begin with $\psi_{II}=0,\ \psi_I\neq0,$ properties of the matrices Eqs. (25), (26) and (27), and the authors will take that [5, 6]

$$\psi_I=e^{-ikx_0}\begin{vmatrix}\chi\\\eta\end{vmatrix} \tag{63}$$

Then the authors will have:

$$\begin{vmatrix}i\partial_0-\kappa & i\partial_j a^j\\-i\partial_j a^{j+} & -i\partial_0-\kappa\end{vmatrix}e^{-ikx_0}\begin{vmatrix}\chi\\\eta\end{vmatrix}=0 \tag{64}$$

$$i\partial_0\chi+i\partial_j a^j\eta=0 \tag{65}$$

$$\left(-2\kappa-i\partial_0\right)\eta-i\partial_j a^{j+}\chi=0 \tag{66}$$

In the non-relativistic limit:

$$\left|i\partial_0\eta\right|\langle\langle2\kappa\left|\eta\right| \tag{67}$$

From the equation (66), we take for η that

$$\eta=-\frac{i}{2\kappa}\partial_j a^{j+}\chi \tag{68}$$

Substitute η in Eq. (33) and obtain

$$i\partial_0\chi=-i\partial_j a^j\left(-\frac{i}{2\kappa}\partial_j a^{j+}\chi\right)$$

$$=-\frac{1}{2\kappa}\partial_j\partial_i\frac{1}{2}\left(a^ja^{i+}+a^ia^{j+}\right)\chi \tag{69}$$

From the assymmetrical condition Eq. (25) the authors have:

$$\begin{vmatrix}0&a^j\\-a^{j+}&0\end{vmatrix}\begin{vmatrix}0&a^i\\-a^{i+}&0\end{vmatrix}+\begin{vmatrix}0&a^i\\-a^{i+}&0\end{vmatrix}\begin{vmatrix}0&a^j\\-a^{j+}&0\end{vmatrix}=2g^{ji}$$

$$a^ja^{i+}+a^ia^{j+}=2\delta^{ij} \tag{70}$$

$$a^{j+}a^i+a^{i+}a^j=2\delta^{ij}$$

Using Eq. (70), the Eq. (69) takes the form

$$i\partial_0\chi=-\frac{1}{2\kappa}\Delta\chi \tag{71}$$

which is the Schrödinger's equation for the free particle. Now we take the case when the particle is under the influence of the external electromagnetic field. The motion of a free relativistic particle of mass

m and charge q is described by the Dirac's equation [7-9]

$$i\hbar \frac{\partial}{\partial t}\psi = \left[-i\hbar c\vec{\alpha}\nabla + \beta mc^2\right]\psi \tag{72}$$

where the 3 matrices α^i and the matrix β are given by

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{73}$$

σ^i, are the Pauli spin matrices, and I is the unit 2×2 matrix. Describe how the Dirac's equation should be modified if particle comes under the influence of an external electromagnetic field described by the four-potential $A^\mu = \left(A^0, \vec{A}\right)$. Hence deduce that the Dirac equation for a particle of charge q in the presence of an electrostatic central potential $V\left(r\right) = qA^0$, with $\vec{A} = 0$, may be written as

$$i\hbar \frac{\partial}{\partial t}\psi = \hat{H}\psi \text{ where } \hat{H} = -i\hbar c\vec{\alpha}\nabla + \beta mc^2 + V\left(r\right) \tag{74}$$

By seeking energy eigenstates of the form

$$\psi\left(\vec{r},t\right) = \psi\left(\vec{r}\right)\exp\left(-\frac{iEt}{\hbar}\right) \tag{75}$$

and writing the Dirac wave function $\psi\left(\vec{r}\right)$ in terms of two-component spinors

$$\psi\left(\vec{r}\right) = \begin{pmatrix} \chi\left(\vec{r}\right) \\ \eta\left(\vec{r}\right) \end{pmatrix} \tag{76}$$

show that the Dirac equation may be written in the form

$$\left(E - V - mc^2\right)\chi\left(\vec{r}\right) - c\left(\vec{\sigma}\vec{p}\right)\eta\left(\vec{r}\right) = 0$$

$$\left(E - V + mc^2\right)\eta\left(\vec{r}\right) - c\left(\vec{\sigma}\vec{p}\right)\chi\left(\vec{r}\right) = 0 \tag{77}$$

Show that for a weak potential in the non-relativistic limit, these equations are reduced to an energy eigenvalue equation of Schrodinger form

$$E'\psi\left(\vec{r}\right) = \left[\frac{\left|\hat{\vec{p}}\right|^2}{2m} + V\right]\psi\left(\vec{r}\right) \tag{78}$$

where $E' = E - mc^2$, $\left[\sigma_i, \sigma_j\right] = 2i\varepsilon_{ijk}\sigma_k$ and $\left(\vec{\sigma}\vec{p}\right)\left(\vec{\sigma}\vec{p}\right) = \left|\vec{p}\right|^2 I.$

7. Conclusions

Investigation carried out in this work shows that the free electromagnetic field, spinor Dirac's field without mass, spinor Dirac's field with mass, and some other fields are described by the same variational formulation. The conditions that a field be described by Maxwell's equations of motion are given in this work, and some solutions of these conditions are also given. Non-relativistic approximation of the equations of the non-quantified field are the Schrödinger's equations. Dirac's equation as a special case, contains Maxwell's equations and the Schrödinger's equation. The authors conclude that the basis of the quantum physics lie on the classical physics, including special theory of relativity.

References

[1] K. Ljolje, S. Vobornik, Canonical formulations of fields satisfying the D'Alambert's equation, Fizika 11 (4) 171 (1979) 171-180.

[2] K. Ljolje, Some remarks on variational formulations of physical fields, Fortschr. Phys. 36 (1) 1988 9-32.

[3] S. Ahmetaj, Physical Fields Described by Maxwell`s Equations © 2007: American Institute of Physics.

[4] D. Bjorken, S.D. Drell, Relativistic Quantum Field, McGraw-Hill, New York, 1965.

[5] E.D Hoker, D. Gieser, Quantum Field Theory, Los Angeles, 2004

[6] P.J. Mulders, Introduction to Quantum Field Theory, Amsterdam, 2004

[7] R. Casalbuoni, Advanced Quantum Field Theory, Firenze, 1999.

[8] H. Hees, Introduction to Relativistic Quantum Field Theory, Texas University, 2004.

[9] J. Norbury, Quantum Field Theory, Milwankee, 2000.