Journal of Mathematics and System Science 3 (2013) 146-149

Mean Ergodic Theorems in Jordan Banach Weak Algebras

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Received: January 26, 2013 / Accepted: February 21, 2013 / Published: March 25, 2013.

Abstract: The purpose of this paper is to study mean ergodic theorems concerning continuous or positive operators taking values in Jordan-Banach weak algebras and Jordan C*-algebras, making use the topological and order structures of the corresponding spaces. The results are obtained applying or extending previous classical results and methods of Ayupov, Carathéodory, Cohen, Eberlein, Kakutani and Yosida. Moreover, this results can be applied to continious or positive operators appearing in diffusion theory, quantum mechanics and quantum probability theory.

Key words: Jordan Banach weak algebras, Krein spaces, mean ergodic operators.

1. Introduction

Many ergodic theorems have been extended to the von Neumann algebras context, motivated by the theory of quantum dynamical systems. From the physical point of view the most important are the study of the averages of continuous or positive maps on C*- or W*-algebras but in the context of this paper it seems to be also natural to consider the ergodic behavior of maps acting on Jordan Banach and Jordan C*-algebras.

A Jordan Banach (JB-) algebra is a pair $(A,\circ)$ with A a real Banach space endowed with a product $\circ$ such that A is a Jordan algebra with identity e and for any two elements $a, b \in A$ holds $\|a \circ b\| = \|a\|\|b\|$, $\|a^2\| = \|a\|^2$ and $\|a^2\| \le \|a^2 + b^2\|$. Also a Jordan Banach weak (JBW-) algebra is a JB-algebra which is a Banach dual space.

These axioms are closely related to those classical of Segal [1] for general quantum mechanics and the JB-algebras include the finite dimentional formally real algebras studied by Jordan, von Neumann and Wigner [2].

It is well-known that if A is JB-algebra then the set $\{u^2 : u \in A\}$ is a proper closed convex cone under which A becomes a partially ordered Banach space with e as an order unit norm and $-e \le u \le e$ implies $0 \le u^2 \le e$, whenever $u \in A$. Hence a JB-algebra A is a partially ordered vector space with a positive cone A^+.

The motivating example for JB- and JBW-algebras are respectively the self-adjoint part of a C*-algebra and a von Neumann algebra with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. More generally any Jordan operator algebra (norm closed Jordan algebra of self-adjoint operators on a Hilbert space, with identity and the above product) is a JB-algebra.

A JB-algebra A is called monotone complete if whenever $(x_a)_{a \in I}$ is a increasing and order bounded net in A then $x = \sup\{x_a : a \in I\}$ exists in A. As usual we then write $x_a \uparrow x$.

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A state on A is a positive linear functional $f: A \to \mathbf{R}$, with $f(e)=1$. The state space K of A is the w*-compact convex set of all states.

As shown in Ref. [3] the predual of a JBW-algebra is unique and will be a base norm space. The base of the positive cone in the predual can be identified with the set of normal states i.e., the states f such that $f(a)=\ell im_a f(x_a)$, whenever (x_a) is increasing net in A with $x_a \uparrow x$. Also in Ref. [3] proved by Shultz that a JB-algebra is a JBW-algebra if and only if A is monotone complete and admits a seperating set of normal states.

For more details about JB-algebras and partially ordered spaces we refer to Refs. [3-5].

As usual by $L(E,E)$ it is denoted the Banach space of all bounded linear operators mapping an ordered Banach space E into itself. Also a cone C is called generating if $E=C-C$ and C is called normal if there exists a $\delta>0$ such that for all $u,v\in C$, $\|u+v\|\ge \delta\cdot\max\{\|u\|,\|v\|\}$.

We also recall the following classical results (c.f. [6]).

Theorem 1.1 Let $T: E\to E$ be a linear operator on an ordered Banach space E such that E^+ is generating and normal. Then T is bounded.

Theorem 1.2 Let E be an ordered Banach space such that E^+ is generating and normal. Then the cone $L^+(E,E)$ of positive operators in $L(E,E)$ is normal.

2. Main Results

2.1 Strong Ergodic Theorem Concerning Continuous Operators

Let A be a JB-algebra and let $T: A\to A$ be a linear operator. T is called stochastic if it is positive and whenever (x_a) is an increasing net in A with $x_a\uparrow x$ implies $T(x_a)\uparrow T(x)$.

Besides T is called mean ergodic if the sequence of Cesáro averages: $T_n(x):=\frac{1}{n}\sum_{k=1}^{n}T^k(x)$, $n=1,2,\ldots$ converges in A, for every $x\in A$.

In what follows this section A will denote a JBW-algebra.

Theorem 2.1 Let $T\in L(A,A)$ such that the following Eberlein's condition holds.

(E): The sequence $(\|T_n\|)$ is bounded and $\ell im_{n\to\infty}\frac{1}{n}T^n(x)=0$, whenever $x\in A$.

Then T is mean ergodic and if $y=\ell im_{n\to\infty}T_n(x)$, whenever $x\in A$ it follows $T(y)=y$.

Proof: Since A is a JBW-algebra there exists a Banach space X such that $A=X'$. Combining the hypothesis about the sequence $(\|T_n\|)$ and the theorems of Alaoglu-Bourbaki and Eberlein's concerning weakly compactness we deduce the existence of a subsequence (T_{k_n}) of (T_n) and an element y in the unit sphere $S'(0,1)$ of $X'=A$ such that $T_{k_n}\xrightarrow{w^*}y \Leftrightarrow T_{k_n}(x)\to y(x),\ \forall x\in A$.

Therefore by $\|TT_{k_n}-T_{k_n}\|\le\frac{1}{k_n}\|T^{k_n+1}\|+\frac{1}{k_n}\|T\|$, for every $n\in N$ it follows that $\|TT_{k_n}-T_{k_n}\|\to 0$. Hence $TT_{k_n}\xrightarrow{w^*}T_{k_n}\Leftrightarrow TT_{k_n}(x)\to T_{k_n}(x),\ \forall x\in A$. In other words $T_{k_n}(x)\to T(y)(x),\ \forall x\in A$, that is $T(y)=y$.

Inductively we conclude that $T^n(y)=y,\ \forall n\in\mathbf{N}$.

Next writing $x=y+(x-y)$ we have $T_n(x)=y+S_n$, where $S_n:=\frac{1}{n}\sum_{k=1}^{n}T^k(x-y)$. Now following similar arguments as in Ref. [7] we show that $S_n\to 0$ and $(x-y)\in A_0$, where A_0 is the closed linear subspace of A which spanned by the elements of the form $a-T(a),\ a\in A$.

Corollary 2.2 Let $T\in L(A,A)$ be a power bounded operator. Then T is mean ergodic and if $y=\ell im_{n\to\infty}T_n(x)$, whenever $x\in A$ it follows

$T(y) = y$.

Proof: Since T is power bounded operator, that is $\sup\{\|T^n\| : n \in \mathbf{N}\} < +\infty$, it follows the Eberlein's condition (E). Then the claim follows by Theorem 2.1.

Next we recall that a matrix of numbers $(a_{n,j})$, $n = 1,2,\dots$, $j = 1,2,\dots$ is called a Toeplitz matrix if $\sum_{j=1}^{\infty} |a_{n,j}| < +\infty$, for every $n = 1,2,\dots$, $\lim_{n\to\infty} a_{n,j} = 0$, $j = 1,2,\dots$ and $\lim_{n\to\infty} \sum_{j=1}^{\infty} a_{n,j} = 1$. Also a matrix $(a_{n,j})$ is said to be regular if for every convergent sequence of numbers (c_j), $\lim_{n\to\infty} \sum_{j=1}^{\infty} a_{n,j} c_j = \lim_{j\to\infty} c_j$.

Theorem 2.3 Let $T \in L(A, A)$ and let $(a_{n,j})$ be a regular Toeplitz matrix such that uniformly in n, $\lim_{k\to\infty} \sum_{j=k}^{\infty} |a_{n,j+1} - a_{n,j}| = 0$ $L_n(x) := \sum_{j=1}^{\infty} a_{n,j} T^j(x)$ and $\|T^n\| \le C$, for all $n \in \mathbf{N}$. Then the sequence $(L_n(x))$ converges and if

$y = \lim_{n\to\infty} L_n(x)$, whenever $x \in A$ it follows $T(y) = y$.

Proof: The assertion follows applying similar arguments as in Theorem 2.1 and in Ref. [8].

2.2 Strong Ergodic Theorem Concerning Positive Operators

Theorem 2.4 Let A^+ be generated and normal cone and let $T: A \to A$ be a positive linear operator satisfying the following Eberlein-type condition:

(*) The sequence $T_n(x)$ is order bounded and $\lim_{n\to\infty} \frac{1}{n} T^n(x) = 0$, whenever $x \in A$.

Then T is mean ergodic and if $y = \lim_{n\to\infty} T_n(x)$, whenever $x \in A$ it follows $T(y) = y$.

Proof: By Theorem 1.1 the operator T is bounded. Therefore the rest of the proof is consequence of Theorem 2.1.

Corollary 2.5 Let A be a JBW-algebra which is also a Krein space ordered by a normal cone A^+ and let $T: A \to A$ be a positive linear operator satisfying condition (*). Then T is mean ergodic and if $y = \lim_{n\to\infty} T_n(x)$, whenever $x \in A$ it follows $T(y) = y$.

Proof: Since A is a Krein space, that is $IntA^+ \neq \emptyset$, it follows that the ordering is generating. Hence the result follows by presenting theorem.

Theorem 2.6 Let A be a JBW-algebra such that A^+ is generating and normal and let $T: A \to A$ be a positive linear operator satisfying the following Yosida-Kakutani-type condition:

(**) The sequence of operators T_n is order bounded and $\lim_{n\to\infty} \frac{1}{n} T^n(x) = 0$, whenever $x \in A$.

Then T is mean ergodic and if $y = \lim_{n\to\infty} T_n(x)$, whenever $x \in A$ it follows $T(y) = y$.

Proof: Since A is a JBW-algebra and the positive cone A^+ is generating and normal it follows that the sequence $(\|T_n\|)$ is bounded. Therefore, T is bounded by Theorem 1.2. Thus the assertion follows by Theorem 2.4.

Corollary 2.7 Let A be a JBW-algebra which is also a Krein space ordered by a normal cone A^+ and let $T: A \to A$ be a positive linear operator satisfying condition (**). Then T is mean ergodic and if $y = \lim_{n\to\infty} T_n(x)$, whenever $x \in A$ it follows $T(y) = y$.

Proof: The result is obtained by uniform boundedness theorem and following similar arguments as in the proof of Corollary 2.5.

3. Concluding Remarks

Kaplansky introduced in 1976 the following concept of a Jordan algebra which is also a C*-algebra (see Ref. [9]). Let A be a complex Banach space which is also a complex Jordan algebra equipped with an involution $*$. Then A is a Jordan C*-algebra if the following four conditions are satisfied:

(1) $\|a \circ b\| = \|a\| \, \|b\|$, whenever $a, b \in A$

(2) $\|a\| = \|a^*\|$, whenever $a \in A$

(3) $\|\{a, a^*, a\}\| = \|a\|^3$, whenever $a \in A$

Where,
$\{a, b, c\} := (a \circ b) \circ c - (c \circ a) \circ b + (b \circ c) \circ a$ is by definition the Jordan triple product of $a, b, c \in A$.

(4) Each norm-closed associative *-subalgebra of A is a C*-algebra.

Moreover, Wright proved in Ref. [9] that conversly each JB-algebra is the self-adjoint part of a unique Jordan C*-algebra.

Now a Jordan C*-algebra is said to be a JC*-algebra if it is isometrically *-isomorphic to a norm-closed Jordan *-subalgebra of the Jordan *-algebra of all bounded linear operators on a complex Hilbert space.

On the other hand it is well known that the JB-algebra M_3^8 of 3×3 hermitian matrices over the Cayley numbers is not isomorphic to any Jordan operator algebra.

Thus it is further established a connection between Jordan algebras, ergodic theory, stochastic analysis and quantum mechanics.

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