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Consistent Criteria for checking Hypotheses

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Abstract: This paper we study a consistent criterion for checking Hypotheses. Given definition of consistent criterion for checking hypotheses for family probability measures which were defined by Z. Zerakidze (see 5). We prove the necessary and sufficient conditions are obtained for the existence of consistent criteria of hypotheses. For example we clearly build of a consistent criteria for checking hypotheses.

Keywords: Consistent criterion, orthogonal, separable, hypotheses.

1. Introduction

Let (E, S) be a measurable (selection) space with given family of probability measures $\{\mu_i,\ i \in I\}$.

We us resell other definitions [1-6]

Definition 1. The family of probability measures $\{\mu_i,\ i \in I\}$ is called orthogonal if $i \neq j$ μ_i and μ_j are orthogonal for each $i \neq j$.

Definition 2. The family of probability measures $\{\mu_i,\ i \in I\}$ is called weakly separable if there exists a family of S - measurable sets $\{X_i,\ i \in I\}$ such that relations

$$\mu_i(X_j) = \begin{cases} 1, & if\ \ i = j \\ 0, & if\ \ i \neq j \end{cases}$$

are fulfilled

Definition 3. The family of probability measures $\{\mu_i,\ i \in I\}$ is called strongly separable if there exists a disjoint family of S – measurable sets $\{\mu_i,\ i \in I\}$ such that the relations $(\forall i)(i \in I \Rightarrow \mu_i(X_i) = 1,$ $\forall i \in I)$ are fulfilled.

Remark 1. From strongly separable there follows weakly separable and from weakly separable there follows orthogonal, but not vice versa. (see 3,4)

Example 1. Let $E = [0,1]X[0,1]$. S be Borel σ -algebra of parts of E. Take the S – measurable sets

$$X_i = \{0 \leq x \leq 1,\ \ y = i,\ \ i \in [0,1]\}$$

and assume that l_i are linear Lebesgue probability measures on X_i. That the family $\{l_i, i \in [0,1]\}$ is strongly separable.

Example 2. Let $E = [0,1]X[0,1]$ S be a Borel σ -algebra of parts of E. Take the S – measurable sets.

$$X_i = \begin{cases} 0 \leq x \leq 1,\ \ y = i,\ \ i \in [0,1] \\ x = i - 2,\ \ 0 \leq y \leq 1,\ \ i \in [2,3] \end{cases}$$

Let l_i be linear lebesgue measures on X_i Then a family of probability measures $\{l_i, i \in [0,1] \cup [2,3]\}$ is weakly separable, but not strongly separable.

Example 3. Let $E = [0,1]X[0,1]$. S be a Boral σ – algebra of parts of E. Take the S – measurable sets.

$$X_i = \{0 \leq x \leq 1,\ \ i \in (0,1]$$

and

$$X_i = [0,1] \times [0,1]$$

Assume that l_i , $i \in (0,1]$ are linear Lebesgue measures on X_i, $i \in (0,1]$ and l_0 is plane lebesgue measure on X_0. Then a family of probability measures $\{l_i, i \in [0,1]\}$ is orthogonal, but not weakly separable.

Let H be sets hypotheses and $\beta(\mathrm{H})$ is σ –algebra witch contains of all finite subsets H .

Definition 4. The family of probability measures $\{\mu_H, H \in \mathrm{H}\}$ will be said to admit a consistent

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criterion for checking hypotheses if there exists even though one measurable mapping δ of the Space (E,S) in $(\mathrm{H},\beta(\mathrm{H}))$ such that.

$$\mu_H(x:\delta(x)=H)=1,\ \forall H\in \mathrm{H}$$

Definition 5. The following probability

$$\alpha_i(\delta)=P_{H_i}(x:\delta(x)\neq H_i)$$

Is called the probability of type I error for given δ criterion.

Remark 2. Let the family of probability measures $\{\mu_H, H\in \mathrm{H}\}$ admit a consistent criterion $\delta(x)$ for checking hypotheses then: 1) the probability all of type errors are equal zero; 2) a family of probability measures $\{\mu_H, H\in \mathrm{H}\}$ is strongly separable but not vice versa; 3) if ν is for any measure on $\beta(H)$ and $U\subset H$ is Borel set, then measures

$$\mu_1(x:\delta(x)\in U)=\int_{H-U}\mu_H(x:\delta(x)\in U)\nu(dH)$$

and

$$\mu_2(x:\delta(x)\notin U)=\int_{U}\mu_H(x:\delta(x)\notin U)\nu(dH)$$

are orthogonal.

Example 4. There are two simple H_1 and H_2 hypotheses and criterion $\delta(x)\equiv H_1$, then the probability of error of the first kind is $\alpha_1=P_{H_1}(\delta(x)\neq H_1)=0$ and the probability of error of the second kind is $\alpha_2=P_{H_2}(\delta(x)\neq H_2)=1$.

Example 5. There are two simple H_1 and H_2 hypotheses and criterion $\delta(x)\equiv H_2$, then the probability of error of the first kind is $\alpha_1=P_{H_1}(\delta(x)\neq H_1)=1$ and the probability of error of the second kind is $\alpha_2=P_{H_2}(\delta(x)\neq H_2)=0$.

Theorem1. Let $\mathrm{H}=\{H_1,H_2,..,H_n,...\}$. The family of probability measures $\{\mu_{H_i}, i\in N\}$ $N=\{1,2,...,n,..\}$ admits consistent criterion for checking hypotheses if and only if, when the family of probability measures $\{\mu_{H_i}, i\in N\}$ is orthogonal.

Proof: Necessity. Since the family $\{\mu_{H_i}, i\in N\}$ admits a consistent criterion for checking hypotheses, then there exists a measurable map δ of the space (E,S) in $(\mathrm{H},\beta(\mathrm{H}))$ such that $\mu_{H_i}(x:\delta(x)=H_i)=1,\ \forall i\in I$. Let $X_i=\{x:\delta(x)=H_i\}$, them it is obvious that $X_i\cap X_j=\varnothing$ for all $i\neq j$ and $\mu_{H_i}(X_i)=1,\ \forall i\in N$.

Therefore, the family of probability measures $\{\mu_{H_i}, i\in N\}$ is strongly separable and from strongly separable there follows orthogonal.

Sufficiency. As the family of probability measures $\{\mu_{H_i}, i\in N\}$ is orthogonal implies of family $\{X_i,\ i\in N\}$ of S-measurable sets such that for any $i\neq j$ we have $\mu_{H_k}(X_{i_k})=0$ and $\mu_{H_i}(E-X_{i_k})=0$.

Let us consider the sets

$$X_i=\bigcup_{k\neq i}(E-X_{i_k}),$$

then $\mu_{H_i}(X_i)=\mu_{H_i}\left(\bigcup_{k\neq i}(E-X_{i_k})\right)=0.$

Therefore $\mu_{H_i}(X_i)=0,\ \mu_{H_i}(E-X_i)=1$.

On the other hand

$$\mu_{H_k}(E-X_i)=\mu_{H_k}\left[E-\bigcup_{k\neq i}(E-X_{i_k})\right]=0$$

for $k\neq i$

and $\mu_{H_i}(E-X_i)=1$. Let us consider the sets

$$B_1=(E-X_1)-(E-X_1)\cap\left(\bigcup_{k\neq 1}(E-X_k)\right)$$

$$B_2=(E-X_2)-(E-X_2)\cap\left(\bigcup_{k\neq 2}(E-X_k)\right)$$

....

$$B_n=(E-X_n)-(E-X_n)\cap\left(\bigcup_{k\neq n}(E-X_k)\right)$$

....

It is obvious that $B_1, B_2, ..., B_n, ...$ is a disjunctive family of S-measurable sets and $\mu_{H_i}(B_i) = 1$, $\mu_{H_i}(B_i) = 1, \forall i \in N$.

Let is define δ map $(E, S) \to (\mathrm{H}, \beta(\mathrm{H}))$ like that $\delta(B_i) = H_i$, $\forall i \in N$. We have $\mu_{H_i}(x : \delta(x) = H_i)$ $= 1$, $\forall i \in N$.

The theorem 1 is proved.

Remark 3. From remark 1 and theorem 1 follows that of the countable family of probability measures $\{\mu_k,\ k \in N\}$, $N = \{1,2,...n,...\}$ strongly separable, weakly separable and orthogonal are equivalent notions.

We build clearly a consistent criterion for checking Hypotheses :

Example 6. Let $E = R = (-\infty, +\infty)$, S be a Borel σ -algebra of parts of R and μ_{H_m} are Lebesgue measures on $[m, m+1]$, $m \in Z$. The family of probability measures $\{\mu_{H_m},\ m \in Z\}$ is strongly separable.let is defined δ map $(R, B(R)) \to (\mathrm{H}, \beta(\mathrm{H}))$. Like that $\delta([m,\ m+1]) = H_m$ $\forall m \in Z$. We have $\mu_{H_m}(\delta(x) = H_m) = 1$, $\forall m \in Z$. e.i. $\delta(x)$ is a consistent criteria for checking hypotheses.

From remark 1 and theorem 1 follows that of the countable family of probability measures $\{\mu_k,\ k \in N\}$, $N = \{1,2,...n,...\}$ strongly separable, weakly separable and orthogonal are equivalent notions.

Theorem 2. Let H be a complete separable metric space, $\beta(\mathrm{H})$ be a σ -algebra of Borel subsets of H, $\mu_H(A)$ be a measurable function on H for any $A \in S$.If ν is measure on $\beta(\mathrm{H})$ such that $\nu(U) > 0$ for any open set U and the measures μ_1 and μ_2 defined by

$$\mu_1(A) = \int_U \mu_H(A)\nu(dH),$$
$$\mu_2(A) = \int_{\mathrm{H}-u} \mu_H(A)\nu(dH), \qquad (1)$$

are orthogonal for any open set $U \in H$, then there exist a measurable map $g : (E, S) \to (\mathrm{H}, \beta(\mathrm{H}))$ such that

$$\mu_H(\{x : g(x) = H\}) = 1$$

for almost all with respect to the measure ν. $g(x)$ -is a consistent criterion for checking hypotheses.

Proof: It is easy to see that the family of sets for which the measures μ_1 and μ_2 are orthogonal is a monotone class: if $C_n \uparrow C$, then

$$\int_{C_n} \mu_H(A)\nu(dH) \uparrow \int_C \mu_H(A)\nu(dH),$$
$$\int_{\mathrm{H}-C_n} \mu_H(A)\nu(dH) \downarrow \int_{\mathrm{H}-C} \mu_H(A)\nu(dH).$$

Therefore, choosing $D_n \in H$, $D_n \uparrow D$ such, that

$$\int_{\mathrm{H}-C_n} \mu_H(D_n)\nu(dH) = 0. \quad \int_{C_n} \mu_H(E - D_n)\nu(dH) = 0.$$

We see, that

$$\int_{\mathrm{H}-C_n} \mu_H(D)\nu(dH) = 0 \quad \int_{C_n} \mu_H(E - D)\nu(dH) = 0$$

$$\int_{\mathrm{H}-C_m} \mu_H(D_n)\nu(dH) = 0\ (m > n)$$

$$\int_{C_m} \mu_H(D_n)\nu(dH) = 0,$$

$$\int_{\mathrm{H}-C} \mu_H(D)\nu(dH) = 0, \quad \int_C \mu_H(E - D)\nu(dH) = 0.$$

And hence

$$\int_C \mu_H(E - D)\nu(dH) = 0.$$

However, the monotone class containing all open sets coincides with the σ – algebra of Borel sets. This means that under the conditions of the theorem the measures μ_1 and μ_2 will be orthogonal for all Borel sets $C \in \beta(\mathrm{H})$.

Next we suppose that H is separable and construct a sequence of partitions of H info pair wise disjoint sets

$$\mathrm{H} = \bigcup_{k_1,\ldots,k_n} C^{(n)}_{k_1\ldots k_n}$$

such that the conditions below are fulfilled:

Sets $C^{(n)}_{k_1,\ldots,k_n}$ are defined for all positive integers n, $k_1,\ldots,k_n$ and

$$C^{(n+1)}_{k_1,\ldots,k_{n+1}} \subset C^{(n)}_{k_1,\ldots,k_n}, \bigcup_{k_{n+1}} C_{k_1,\ldots,k_{n+1}} = C^{(n)}_{k_1,\ldots k_n};$$

The measure of the boundary of $C^{(n)}_{k_1,\ldots,k_n}$ is zero and $\nu\left(C^{(n)}_{k_1,\ldots,k_n}\right) > 0;$

The diameter of $C^{(n)}_{k_1,\ldots,k_n}$ is not greater that 2^{-n}.

Since, for various $k_1, k_2, \ldots, k_n$ the measures

$$\int_{C^{(n)}_{k_1,\ldots,k_n}} \mu_H(\cdot)\nu(dH)$$

are pair wise orthogonal, there exist Borel sets $X^{(n)}_{k_1,\ldots,k_n}$ such that

$$\int_{C^{(n)}_{k_1,\ldots,k_n}} \mu_H\left(X^{(n)}_{k_1,\ldots,k_n}\right)\nu(dH) = \begin{cases} 0, \text{ if } \sum_{i-1}^{n}(k_i - k_{i'})^2 > 0 \\ \nu\left(C^{(n)}_{k_1,\ldots,k_n}\right) \text{ if } \sum_{i=1}^{n}(k_i - k_{i'})^2 = 0 \end{cases}$$

where $X^{(n)}_{k_1,\ldots,k_n}$ can be chosen pair wise disjoint.

Indeed, since thrice family of measures is countable and pair wise orthogonal, this family is also strongly separable.

Since $C_{k_1,\ldots,k_{n+1}} \subset C_{k_1,\ldots,k_n}$, $X^{(n+1)}_{k_1,\ldots,k_{n+1}}$ can be chosen so that $X^{(n+1)}_{k_1,\ldots,k_{n+1}} \subset X^{(n)}_{k_1,\ldots,k_n}$. Chose and point $h^{(n)}_{k_1,k_2,\ldots k_n} \in C^{(n)}_{k_1,\ldots,k_{n+1}}$.

Denote by $\widetilde{C}^{(n)}_{k_1,\ldots,k_n}$ the set of those $H \in C^{(n)}_{k_1,\ldots,k_n}$ for which $\mu_H\left(X^{(n)}_{k_1,\ldots,k_n}\right) = 1$.

Since $0 = \int_{C^{(n)}_{k_1,\ldots,k_n}} \left(1 - \mu_H\left(X^{(n)}_{k_1,\ldots,k_n}\right) \nu(dH)\right)$ and the integrand is non-negative, it is equal to zero almost everywhere, i.e.

$$\nu\left(C^{(n)}_{k_1,\ldots,k_n}\right) = \nu\left(\widetilde{C}^{(n)}_{k_1,\ldots,k_n}\right).$$

Setting $C = \bigcap_n \bigcup_{k_1\ldots k_n} C^{(n)}_{k_1,\ldots,k_n}$, we have $\nu(C) = \nu(\mathrm{H})$. If $H \in C$, then the equality $\mu_H\left(X^{(n)}_{k_1,\ldots,k_n}\right) = 1$ will be fulfilled for all n.

Let $g_n(x) = H^{(n)}_{k_1,\ldots,k_n}$ if $x \in X^{(n)}_{k_1,k_2,\ldots k_n}$. On the set $\bigcap_n \bigcup_{k_1\ldots,k_n} X^{(n)}_{k_1,\ldots k_n} = \widetilde{X}, g_n(x)$t converge to some limit $g(x)$, since $\rho(g_n, g_{n+1}) < 2^{-n}$. This follows from the fact that, for $x \in X^{(n+1)}_{k_1,\ldots,k_{n+1}}$ we have $g_n(x) = H^{(n)}_{k_1,k_2,\ldots,k_n}$, $g_{n+1}(x) = H^{(n)}_{k_1,k_2,\ldots,k_{n+1}}$ and these points both belong to $C^{(n)}_{k_1,\ldots,k_n}$ whose diameter is not greater than 2^{-n}. The equality $\mu_H\left(\widetilde{X}\right) = 1$ is satisfied for $H \in C$.

If $H \in \bigcap \widetilde{C}^{(n)}_{k_1,\ldots,k_n}$ than $\mu_H\left(\bigcap X^{(n)}_{k_1,\ldots,k_n}\right) = 1$ and for $x \in \bigcap X^{(n)}_{k_1,\ldots,k_n}$ we get $g_n(x) \in C^{(n)}_{k_1,\ldots,k_n}$.

Since $\bigcap C^{(n)}_{k_1,\ldots,k_n}$ also contains the unique point H, we have $\rho(g_n(x), H) \le 2^{-n}$ and hence, $g(x) = H$.

The theorem 2 is proved.

Remark 4. If H isn't separable this theorem 2 remains true because a Borel measure given on a non separable complete metric space has a separable support [6]

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