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On Nonhomogeneous Singular Elliptic Systems with Critical *C-K-N* Exponent

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Abstract: In this paper, we are interested in the existence and multiplicity results of nontrivial solutions to nonhomogeneous singular elliptic systems with critical *C-K-N* exponent $\left(\mathcal{S}_{\lambda_1,\lambda_2}\right)$. With the help of the Nehari manifold and under sufficient conditions on the parameters λ_1 and λ_2, we prove some existence results.

Keywords: Singular nonhomogeneous elliptic systems, Nehari manifold, critical C-K-N exponent

1. Introduction

This paper deals with the existence and multiplicity of nontrivial solutions to the following system $\left(\mathcal{S}_{\lambda_1,\lambda_2}\right)$

$$\begin{cases} -\operatorname{div}\left(|x|^{-2a}\nabla u\right)-\mu|x|^{-2(a+1)}u= \\ (\alpha+1)|x|^{-2_*b}|u|^{\alpha-1}u|v|^{\beta+1}+\lambda_1 f_1 \text{ in } \Omega \\ -\operatorname{div}\left(|x|^{-2a}\nabla v\right)-\mu|x|^{-2(a+1)}v= \\ (\beta+1)|x|^{-2_*b}|u|^{\alpha+1}|v|^{\beta-1}v+\lambda_2 f_2 \text{ in } \Omega \\ u=v=0 \qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$ $(N\geq 3)$ containing in its interior, $-\infty<a<(N-2)/2$, $a\leq b<a+1$, $2_*=2N/\left(N-2+2(b-a)\right)$ is the critical Caffarelli-Kohn-Nirenberg exponent, $-\infty<\mu<\bar{\mu}_a:=\left(\left(N-2(a+1)\right)/2\right)^2$, α, β are positive real such that $\alpha+\beta=2_*-2$, λ_1, λ_2 are real parameters and f_1, f_2 are functions defined on $\bar{\Omega}$.

The degeneracy and singularity occur in the system $\left(\mathcal{S}_{\lambda_1,\lambda_2}\right)$, thus standard variational methods do not apply.

In recent years much attention has been paid to the existence of nontrivial solutions for problems $\left(\mathcal{P}_{a,\lambda,\mu}\right)$ of the type

$$\begin{cases} -\operatorname{div}\left(|x|^{-2a}\nabla u\right)-\mu|x|^{-2(a+1)}u= \\ h(x)|x|^{-2_*b}|u|^{2_*-2}u+\lambda f(x) \text{ in } \Omega \\ u=0 \qquad \text{on } \partial\Omega. \end{cases}$$

Wang and Zhou [10] have proved that $\left(\mathcal{P}_{0,\mu,1}\right)$, for $h(x)\equiv 1$ and $a=0$, has at least two distinct solutions when $0\leq\mu<\bar{\mu}_0:=\left((N-2)/2\right)^2$ and under some sufficient conditions on f. In [2], Bouchekif and Matallah have showed the existence of two nontrivial solutions of $\left(\mathcal{P}_{a,\lambda,\mu}\right)$ when $0<\mu\leq\bar{\mu}_a$, $-\infty<a<(N-2)/2$, $a\leq b<a+1$, $\lambda\in(0,\Lambda_*)$ with Λ_* a positive constant and under some appropriate conditions on functions f and h.

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Many existence results are available for regular systems which derive from potential, we quote for example [1] and [6]. However, to our knowledge there are few results for singular systems, we can cite for example [8].

By $\mathcal{H}_\mu := \mathcal{H}_\mu(\Omega)$, we denote the completion of the space $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mu,a} = \left(\int_\Omega \left(|x|^{-2a}|\nabla u|^2 - \mu|y|^{-2(a+1)}|u|^2\right)dx\right)^{1/2},$$

for $-\infty < \mu < \overline{\mu}_a$.

Using the Hardy inequality, this norm is equivalent to $\|u\|_{0,a}$. More explicitly, we have

$$\left(1-\max(\mu,0)/\overline{\mu}_a\right)^{1/2}\|u\|_{0,a} \le$$
$$\|u\|_{\mu,a} \le \left(1-\min(\mu,0)/\overline{\mu}_a\right)^{1/2}\|u\|_{0,a}.$$

The space $\mathcal{H} := \mathcal{H}_\mu \times \mathcal{H}_\mu$ is endowed with the norm

$$\|(u,v)\|_{\mu,a} = \left(\|u\|_{\mu,a}^2 + \|v\|_{\mu,a}^2\right)^{1/2}.$$

Since our approach is variational, we define the functional $J := J_{\lambda_1,\lambda_2}$ on $\mathcal{H}$ by

$$J(u,v) := (1/2)\|(u,v)\|_{\mu,a}^2 - P(u,v) - Q(u,v),$$

where

$$P(u,v) := \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}|x|^{-2_*b}\,dx$$

and

$$Q(u,v) := \int_\Omega (\lambda_1 f_1 u + \lambda_2 f_2 v)\,dx.$$

A couple $(u,v)\in\mathcal{H}$ is a weak solution of the system $(\mathcal{S}_{\lambda_1,\lambda_2})$ if it satisfies

$$\langle J'(u,v),(\phi,\psi)\rangle := R(u,v)(\phi,\psi) -$$
$$S(u,v)(\phi,\psi) - T(u,v)(\phi,\psi) = 0,$$
$$\text{for all } (\phi,\psi)\in\mathcal{H},$$

with

$$R(u,v)(\phi,\psi) := \int_\Omega \begin{pmatrix} |x|^{-2a}(\nabla u\nabla\phi + \nabla v\nabla\psi) \\ -\mu|x|^{-2(a+1)}(u\phi + v\psi) \end{pmatrix}$$

$$S(u,v)(\phi,\psi) := \int_\Omega |x|^{-2_*b}\begin{bmatrix} (\alpha+1)|u|^\alpha|v|^{\beta+1}\phi \\ +(\beta+1)|u|^{\alpha+1}|v|^\beta\psi \end{bmatrix}$$

$$T(u,v)(\phi,\psi) := \int_\Omega (\lambda_1 f_1\phi + \lambda_2 f_2\psi).$$

Here $\langle .,.\rangle$ denotes the product in the duality $\mathcal{H}'$, $\mathcal{H}$.

Let

$$S_\mu := \inf_{u\in\mathcal{H}_\mu\setminus\{0\}} \frac{\|u\|_{\mu,a}^2}{\left(\int_\Omega |x|^{-2_*b}|u|^{2_*}\,dx\right)^{2/2_*}}$$

and

$$\tilde{S}_\mu := \inf_{(u,v)\in\mathcal{H}\setminus\{(0,0)\}} \frac{\|(u,v)\|_{\mu,a}^2}{\left(\int_\Omega |u|^{\alpha+1}|v|^{\beta+1}|x|^{-2_*b}\,dx\right)^{2/2_*}}.$$

From [7], S_μ is achieved.

Lemma 1. *Let Ω be a domain (not necessarily bounded), $-\infty < \mu < \overline{\mu}_a$ and $\alpha+\beta \le 2_* - 2$. Then we have*

$$\tilde{S}_\mu := \left[\left(\frac{\alpha+1}{\beta+1}\right)^{(\beta+1)/2_*} + \left(\frac{\beta+1}{\alpha+1}\right)^{(\alpha+1)/2_*}\right]S_\mu.$$

For simplicity of writing, let us note the quantity $\left[\left(\frac{\alpha+1}{\beta+1}\right)^{(\beta+1)/2_*} + \left(\frac{\beta+1}{\alpha+1}\right)^{(\alpha+1)/2_*}\right]$ *by* $K(\alpha,\beta)$.

Proof The proof is essentially given in [1] with minor modifications.

In our work, we research the critical points as the minimizers of the energy functional associated to the problem $(\mathcal{S}_{\lambda_1,\lambda_2})$ on the constraint defined by the Nehari manifold, which are solutions of our system, under some sufficient conditions on the parameters α, β, μ, λ_1 and λ_2.

Let Λ_0 be positive number such that

$$\Lambda_0 := 2_*(2_*-2)\times$$
$$\left[2_*(2_*-1)\right]^{-\frac{(2_*-1)}{(2_*-2)}}\left[K(\alpha,\beta)\right]^{\frac{2_*}{2(2_*-2)}}\left(S_\mu\right)^{\frac{2_*}{2(2_*-2)}}.$$

Then, we obtain the following results.

Theorem 1. *Let be* $f_1, f_2 \in \mathcal{H}_\mu'$ (dual of $\mathcal{H}_\mu$).

Assume that $-\infty < a < (N-2)/2$, $-\infty < \mu < \bar{\mu}_a$, $\alpha+\beta+2=2_*$ *and* λ_1, λ_2 *real parameters satisfying* $0<|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}<\Lambda_0$, *then* $(\mathcal{S}_{\lambda_1,\lambda_2})$ *has at least one nontrivial solution.*

Theorem 2. *In addition to the assumptions of the Theorem 1,* λ_1, λ_2 *verifying*

$$0<|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}<(1/2)\Lambda_0,$$

then $(\mathcal{S}_{\lambda_1,\lambda_2})$ *has at least two nontrivial solutions.*

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proofs of Theorems 1 and 2.

2. Preliminaries

We list here a few integral inequalities. The first one that we need is the Caffarelli-Kohn-Nirenberg inequality [4], which ensures the existence of a positive constant $C_{a,b}$ such that

$$\left(\int_{\mathbb{R}^N}|x|^{-2_*b}|v|^{2_*}dx\right)^{2/2_*}\le C_{a,b}\int_{\mathbb{R}^N}|x|^{-2a}|\nabla v|^2dx, \text{ for all } v\in\mathcal{C}_0^\infty(\mathbb{R}^N). \tag{2.1}$$

In (2.1), as $b=a+1$, then $2_*=2$ and we have the following weighted Hardy inequality [5]:

$$\int_{\mathbb{R}^N}|x|^{-2(a+1)}v^2dx\le\frac{1}{\bar{\mu}_a}\int_{\mathbb{R}^N}|x|^{-2a}|\nabla v|^2dx,$$

for all $v\in\mathcal{C}_0^\infty(\mathbb{R}^N)$.

Definition 1. *Let* $c\in\mathbb{R}$, E *a Banach space and* $I\in C^1(E,\mathbb{R})$.

(i) $(u_n,v_n)_n$ is a Palais-Smale sequence at level (in short $(PS)_c$) in E for I if

$$I(u_n,v_n)=c+o_n(1) \text{ and } I'(u_n,v_n)=o_n, \tag{2.1}$$

where $o_n(1)$ *tends to 0 as* n *goes at infinity.*

(ii) *We say that* I *satisfies the* $(PS)_c$ *condition if any* $(PS)_c$ *sequence in* E *for* I *has a convergent subsequence.*

2.1 Nehari manifold

It is well known that J is of class C^1 in $\mathcal{H}$ and the solutions of $(\mathcal{S}_{\lambda_1,\lambda_2})$ are the critical points of J which is not bounded below on $\mathcal{H}$. Consider the following Nehari manifold

$$\mathcal{N}=\left\{\begin{matrix}(u,v)\in\mathcal{H}\setminus\{0,0\}\ : \\ \langle J'(u,v),(u,v)\rangle=0\end{matrix}\right\}.$$

Thus, $(u,v)\in\mathcal{N}$ if and only if

$$\|(u,v)\|_{\mu,a}^2-2_*P(u,v)-Q(u,v)=0. \tag{2.2}$$

Note that $\mathcal{N}$ contains every nontrivial solution of the problem $(\mathcal{S}_{\lambda_1,\lambda_2})$. Moreover, we have the following results.

Lemma 2. J *is coercive and bounded from below on* $\mathcal{N}$.

Proof. If $(u,v)\in\mathcal{N}$, then by (2.2), the Hölder and Young inequalities, we deduce that

$$\begin{aligned}J(u,v)&=((2_*-2)/2_*2)\|(u,v)\|_{\mu,a}^2\\&\quad-(1-(1/2_*))Q(u,v)\\&\ge((2_*-2)/2_*2)\|(u,v)\|_{\mu,a}^2-(1-(1/2_*))\\&\quad\times\left(|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}\right)\|(u,v)\|_{\mu,a}\\&\ge-C_0,\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}C_0&:=C_0\left(\lambda_1,\ \lambda_2,\ \|f_1\|_{\mathcal{H}_\mu'},\ \|f_2\|_{\mathcal{H}_\mu'}\right)\\&=\left[2(2_*-1)^2/2_*(2_*-2)\right]\\&\quad\times\left(|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}\right)^2>0.\end{aligned}$$

Thus, J is coercive and bounded from below on $\mathcal{N}$.

Define

$$(u,v)=\langle J'(u,v),(u,v)\rangle.$$

Then, for $(u,v)\in\mathcal{N}$

$$\begin{aligned}&\left\langle\varphi'(u,v),(u,v)\right\rangle=\\&2\|(u,v)\|_{\mu,a}^2-(2_*)^2P(u,v)-Q(u,v)\\&=\|(u,v)\|_{\mu,a}^2-2_*(2_*-1)P(u,v)\\&=(2_*-1)Q(u,v)-(2_*-2)\|(u,v)\|_{\mu,a}^2.\end{aligned}\tag{2.4}$$

Now, we split $\mathcal{N}$ in three parts:

$$\mathcal{N}^+=\left\{(u,v)\in\mathcal{N}\ :\ \left\langle\varphi'(u,v),(u,v)\right\rangle>0\right\},$$

$$\mathcal{N}^0=\left\{(u,v)\in\mathcal{N}\ :\ \left\langle\varphi'(u,v),(u,v)\right\rangle=0\right\},$$

and

$$\mathcal{N}^-=\left\{(u,v)\in\mathcal{N}\ :\ \left\langle\varphi'(u,v),(u,v)\right\rangle<0\right\}.$$

We have the following results.

Lemma 3. *Suppose that* (u_0,v_0) *is a local minimizer for* J *on* $\mathcal{N}$. *Then, if* $(u_0,v_0)\notin\mathcal{N}^0$, (u_0,v_0) *is a critical point of* J.

Proof. If (u_0,v_0) is a local minimizer for J on $\mathcal{N}$, then (u_0,v_0) is a solution of the optimization problem

$$\min_{\{(u,v)/\ \varphi(u,v)=0\}}J(u,v).$$

Hence, there exists a Lagrange multipliers $\theta\in\mathbb{R}$ such that

$$J'(u_0,v_0)=\theta\varphi'(u_0,v_0)\ \text{in}\ \mathcal{H}'\ (\text{dual of}\ \mathcal{H})$$

Thus,

$$\left\langle J'(u_0,v_0),(u_0,v_0)\right\rangle=\theta\left\langle\varphi'(u_0,v_0),(u_0,v_0)\right\rangle,$$

But $\left\langle\varphi'(u_0,v_0),(u_0,v_0)\right\rangle\neq0$, since $(u_0,v_0)\notin\mathcal{N}^0$. Hence $\theta=0$. This completes the proof.

Lemma 4. *There exists a positive number* Λ_0 *such that, for all* λ_1, λ_2 *verifying*

$$0<|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}<\Lambda_0,$$

we have $\mathcal{N}^0=\varnothing$.

Proof. Let us reason by contradiction.

Suppose $\mathcal{N}^0=\varnothing$ such that $0<|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}<\Lambda_0$. Then, by (2.4) and for $(u,v)\in\mathcal{N}^0$, we have

$$\begin{aligned}\|(u,v)\|_{\mu,a}^2&=2_*(2_*-1)P(u,v)\\&=((2_*-1)/(2_*-2))Q(u,v).\end{aligned}\tag{2.5}$$

Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\begin{aligned}&\|(u,v)\|_{\mu,a}\ge\\&\left[K(\alpha,\beta)\right]^{\frac{2_*}{2(2_*-2)}}\left(S_\mu\right)^{\frac{2_*}{2(2_*-2)}}\left[2_*(2_*-1)\right]^{\frac{-1}{(2_*-2)}}\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}&\|(u,v)\|_{\mu,a}\le\\&\left[\left((2_*-1)\left(|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}\right)(2_*-2)^{-1}\right)\right].\end{aligned}\tag{2.7}$$

From (2.6) and (2.7), we obtain $|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}\ge\Lambda_0$, which contradicts our hypothesis.

Thus $\mathcal{N}=\mathcal{N}^+\cup\mathcal{N}^-$. Define

$$c:=\inf_{u\in\mathcal{N}}J(u,v),c^+:=\inf_{u\in\mathcal{N}^+}J(u,v)$$

and

$$c^-:=\inf_{u\in\mathcal{N}^-}J(u,v).$$

For the sequel, we need the following Lemma.

Lemma 5.

(i) *For all* λ_1, λ_2 *such that* $0<|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}<\Lambda_0$, *one has* $c\le c^+<0$.

(ii) *For all* λ_1, λ_2 *such that* $0<|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}<(1/2)\Lambda_0$, *one has*

$$c^- > C_1 = C_1\left(\lambda_1,\ \lambda_2,\ S_\mu,\ \|f_1\|_{\mathcal{H}'_\mu},\ \|f_2\|_{\mathcal{H}'_\mu}\right),$$

where

$$\begin{aligned} C_1 :=& \left((2_*-2)/2_*2\right)\left[2_*(2_*-1)\right]^{-2/(2_*-2)} \\ &\times\left[K(\alpha,\beta)\right]^{2_*/(2_*-2)}\left(S_\mu\right)^{2_*/(2_*-2)} + \\ &-\left((2_*-2)/2_*\right)\left(|\lambda_1|\|f_1\|_{\mathcal{H}'_\mu}+|\lambda_2|\|f_2\|_{\mathcal{H}'_\mu}\right). \end{aligned}$$

Proof. (*i*) Let $(u,v)\in\mathcal{N}^+$. By (2.4), we have

$$\left[1/2_*(2_*-1)\right]\|(u,v)\|^2_{\mu,a} > P(u,v)$$

and so

$$\begin{aligned} J(u,v) &= (-1/2)\|(u,v)\|^2_{\mu,a}+(2_*-1)P(u,v) \\ &< -\left((2_*-1)/2_*2\right)\|(u,v)\|^2_{\mu,a}. \end{aligned}$$

We conclude that $c\le c^+<0$.

(*ii*) Let $(u,v)\in\mathcal{N}^-$. By (2.4), we get

$$\left[1/2_*(2_*-1)\right]\|(u,v)\|^2_{\mu,a} < P(u,v).$$

Moreover, by Sobolev embedding theorem, we have

$$P(u,v)\le\left[K(\alpha,\beta)\right]^{-2_*/2}\left(S_\mu\right)^{-2_*/2}\|(u,v)\|^{2_*}_{\mu,a}.$$

This implies

$$\begin{aligned} &\|(u,v)\|_{\mu,a} > \\ &\left[2_*(2_*-1)\right]^{\frac{-1}{(2_*-2)}}\left[K(\alpha,\beta)\right]^{\frac{2_*}{2(2_*-2)}}\left(S_\mu\right)^{\frac{2_*}{2(2_*-2)}}, \end{aligned}$$

for all $u\in\mathcal{N}^-$.

By (2.3), we get

$$\begin{aligned} J(u,v)\ge& \left((2_*-2)/2_*2\right)\|(u,v)\|^2_{\mu,a} + \\ &-\left(1-(1/2_*)\right) \\ &\times\left(|\lambda_1|\|f_1\|_{\mathcal{H}'_\mu}+|\lambda_2|\|f_2\|_{\mathcal{H}'_\mu}\right)\|(u,v)\|_{\mu,a}. \end{aligned}$$

Thus, for all λ_1, λ_2 such that $0<|\lambda_1|\|f_1\|_{\mathcal{H}'_\mu}+|\lambda_2|\|f_2\|_{\mathcal{H}'_\mu}<(1/2)\Lambda_0$, we have $J(u,v)\ge C_1$.

For each $(u,v)\in\mathcal{H}$, we write

$$t_m := t_{\max}(u,v) = \left[\frac{\|(u,v)\|_{\mu,a}}{2_*(2_*-1)\int_\Omega|u|^{\alpha+1}|v|^{\beta+1}|x|^{-2_*b}\,dx}\right]^{1/(2_*-2)} > 0.$$

Lemma 6. *Let* λ_1, λ_2 *such that* $0<|\lambda_1|\|f_1\|_{\mathcal{H}'_\mu}+|\lambda_2|\|f_2\|_{\mathcal{H}'_\mu}<\Lambda_0$. *For each* $(u,v)\in\mathcal{H}$*, one has the following:*

(*i*) *If* $Q(u,v)\le0$, *then there exists a unique* $t^->t_m$ *such that* $(t^-u,t^-v)\in\mathcal{N}^-$ *and*

$$J(t^-u,t^-v)=\sup_{t\ge0}(tu,tv).$$

(*ii*) *If* $Q(u,v)>0$, *then there exist unique* t^+ *and* t^- *such that* $0<t^+<t_m<t^-$, $(t^+u,t^+v)\in\mathcal{N}^+$, $(t^-u,t^-v)\in\mathcal{N}^-$,

$$J(t^+u,t^+v)=\inf_{0\le t\le t_m}J(tu,tv)$$

and

$$J(t^-u,t^-v)=\sup_{t\ge0}J(tu,tv).$$

Proof With minor modifications, we refer to [3].

Taking the idea of the work of Brown-Zhang [3], we prove the following result

Proposition 1.

(*i*) *For all* λ_1, λ_2 *such that* $0<|\lambda_1|\|f_1\|_{\mathcal{H}'_\mu}+|\lambda_2|\|f_2\|_{\mathcal{H}'_\mu}<\Lambda_0$, *there exists a* $(PS)_{c^+}$ *sequence in* $\mathcal{N}^+$.

(*ii*) *For all* λ_1, λ_2 *such that* $0<|\lambda_1|\|f_1\|_{\mathcal{H}'_\mu}+|\lambda_2|\|f_2\|_{\mathcal{H}'_\mu}<(1/2)\Lambda_0$, *there exists a* $(PS)_{c^-}$ *sequence in* $\mathcal{N}^-$.

3. Proof of Theorem 1

Drawing on the works of [3] and [9], we establish the existence of a local minimum for J on $\mathcal{N}^+$.

Proposition 2. *For all* λ_1, λ_2 *such that*

$0<|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}<\Lambda_0$, *the functional* J *has a minimizer* $\left(u_0^+,v_0^+\right)\in\mathcal{N}^+$ *and it satisfies*

(*i*) $J\left(u_0^+,v_0^+\right)=c=c^+$,

(*ii*) $\left(u_0^+,v_0^+\right)$ *is a nontrivial solution of* $\left(\mathcal{S}_{\lambda_1,\lambda_2}\right)$.

Proof If $0<|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}<\Lambda_0$, then by Proposition 1(*i*) there exists a $\left(u_n,v_n\right)_n$ $(PS)_{c^+}$ sequence in $\mathcal{N}^+$, thus it bounded by Lemma 2. Then, there exists $\left(u_0^+,v_0^+\right)\in\mathcal{H}$ and we can extract a subsequence which will denoted by $\left(u_n,v_n\right)_n$ such that

$$\begin{gathered}\left(u_n,v_n\right)\to\left(u_0^+,v_0^+\right)\text{ weakly in }\mathcal{H}\\ \left(u_n,v_n\right)\to\left(u_0^+,v_0^+\right)\text{ weakly in }\left(L^{2_*}\left(\Omega,|x|^{-2_*b}\right)\right)^2\\ u_n\to u_0^+\text{ a.e in }\Omega,\\ v_n\to v_0^+\text{ a.e in }\Omega.\end{gathered}\tag{3.1}$$

Thus, by (3.1), $\left(u_0^+,v_0^+\right)$ is a weak nontrivial solution of $\left(\mathcal{S}_{\lambda_1,\lambda_2}\right)$. Now, we show that $\left(u_n,v_n\right)$ converges to $\left(u_0^+,v_0^+\right)$ strongly in $\mathcal{H}$. Suppose otherwise. By the lower semi-continuity of the norm, then either $\left\|u_0^+\right\|_{\mu,a}<\liminf_{n\to\infty}\left\|u_n\right\|_{\mu,a}$ or $\left\|v_0^+\right\|_{\mu,a}<\liminf_{n\to\infty}\left\|v_n\right\|_{\mu,a}$ and we obtain

$$\begin{aligned}c\le J\left(u_0^+,v_0^+\right)=&\\ &\left((2_*-2)/2_*2\right)\left\|\left(u_0^+,v_0^+\right)\right\|_{\mu,a}^2\\ &-\left(1-(1/2_*)\right)Q\left(u_0^+,v_0^+\right)\\ &<\liminf_{n\to\infty}J\left(u_n,v_n\right)=c.\end{aligned}$$

We get a contradiction. Therefore, $\left(u_n,v_n\right)$ converge to $\left(u_0^+,v_0^+\right)$ strongly in . Moreover, we have $\left(u_0^+,v_0^+\right)\in\mathcal{N}^+$. If not, then by Lemma 6, there are two numbers t_0^+ and t_0^-, uniquely defined so that $\left(t_0^+u_0^+,t_0^+v_0^+\right)\in\mathcal{N}^+$ and $\left(t^-u_0^+,t^-v_0^+\right)\in\mathcal{N}^-$. In particular, we have $t_0^+<t_0^-=1$. Since

$$\frac{d}{dt}J\left(tu_0^+,tv_0^+\right)_{\downarrow t=t_0^+}=0$$

and

$$\frac{d^2}{dt^2}J\left(tu_0^+,tv_0^+\right)_{\downarrow t=t_0^+}>0,$$

there exists $t_0^+<t^-\le t_0^-$ such that $J\left(t_0^+u_0^+,t_0^+v_0^+\right)<J\left(t^-u_0^+,t^-v_0^+\right)$. By Lemma 6, we get

$$\begin{aligned}J\left(t_0^+u_0^+,t_0^+v_0^+\right)&<J\left(t^-u_0^+,t^-v_0^+\right)\\ &<J\left(t_0^-u_0^+,t_0^-v_0^+\right)=J\left(u_0^+,v_0^+\right),\end{aligned}$$

which is a contradiction.

4. Proof of Theorem 2

Next, we establish the existence of a local minimum for J on $\mathcal{N}^-$. For this, we require the following Lemma.

Lemma 7. *For all* λ_1, λ_2 *such that* $0<|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}<(1/2)\Lambda_0$, *the functional* J *has a minimizer* $\left(u_0^-,v_0^-\right)$ *in* $\mathcal{N}^-$ *and it satisfies*

(*i*) $J\left(u_0^-,v_0^-\right)=c^->0$,

(*ii*) $\left(u_0^-,v_0^-\right)$ *is a nontrivial solution of* $\left(\mathcal{S}_{\lambda_1,\lambda_2}\right)$ *in* $\mathcal{H}$.

Proof. If $0<|\lambda_1|\|f_1\|_{\mathcal{H}_\mu'}+|\lambda_2|\|f_2\|_{\mathcal{H}_\mu'}<(1/2)\Lambda_0$, then by Proposition 1(*ii*) there exists a $\left(u_n,v_n\right)_n$, $(PS)_{c^-}$ sequence in $\mathcal{N}^-$, thus it bounded by Lemma 2. Then, there exists $\left(u_0^-,v_0^-\right)\in\mathcal{H}$ and we can extract a subsequence which will denoted by $\left(u_n,v_n\right)_n$ such that

$$\begin{gathered}\left(u_n,v_n\right)\to\left(u_0^-,v_0^-\right)\text{ weakly in }\mathcal{H}\\ \left(u_n,v_n\right)\to\left(u_0^-,v_0^-\right)\text{ weakly in }\left(L^{2_*}\left(\Omega,|x|^{-2_*b}\right)\right)^2\\ u_n\to u_0^-\text{ a.e in }\Omega\\ v_n\to v_0^-\text{ a.e in }\Omega.\end{gathered}$$

This implies

$$P\left(u_n,v_n\right)\to P\left(u_0^-,v_0^-\right),\text{ as }n\to\infty.$$

Moreover, by (2.4) we obtain

$$P(u_n,v_n) > \left[2_*(2_*-1)\right]^{-1}\|u_n,v_n\|^2_{\mu,a}, \qquad (4.1)$$

thus, by (2.6) and (4.1) there exists a positive number

$$C_2 := \left[12_*(2_*-1)\right]^{-2_*/(2_*-2)} \times\left[K(\alpha,\beta)\right]^{\frac{2_*}{(2_*-2)}}\left(S_\mu\right)^{\frac{2_*}{(2_*-2)}},$$

such that

$$P(u_n,v_n) > C_2.$$

This implies that

$$P\left(u_0^-,v_0^-\right) \geq C_2.$$

Now, we prove that $(u_n,v_n)_n$ converges to $\left(u_0^-,v_0^-\right)$ strongly in $\mathcal{H}$. Suppose otherwise. Then, either $\|u_0^-\|_{\mu,a} < \liminf\limits_{n\to\infty}\|u_n\|_{\mu,a}$ or $\|v_0^-\|_{\mu,a} < \liminf\limits_{n\to\infty}\|v_n\|_{\mu,a}$. By Lemma 6 there is a unique t_0^- such that $\left(t_0^-u_0^-,t_0^-v_0^-\right)\in\mathcal{N}^-$. Since

$$(u_n,v_n)\in\mathcal{N}^-,\ J(u_n,v_n) \geq J(tu_n,tv_n), \text{ for all } t\geq 0,$$

we have

$$J\left(t_0^-u_0^-,t_0^-v_0^-\right) < \lim_{n\to\infty} J\left(t_0^-u_n,t_0^-v_n\right) \leq \lim_{n\to\infty} J(u_n,v_n) = c^-,$$

and this is a contradiction. Hence,

$$(u_n,v_n)_n \to \left(u_0^-,v_0^-\right) \text{ strongly in } \mathcal{H}.$$

Thus,

$$J(u_n,v_n) \text{ converges to } J\left(u_0^-,v_0^-\right) = c^- \text{ as } n \text{ tends to } +\infty.$$

By (4.2) and Lemma 3, we may assume that $\left(u_0^-,v_0^-\right)$ is a nontrivial solution of $\left(\mathcal{S}_{\lambda_1,\lambda_2}\right)$.

Now, we complete the proof of Theorem 2. By Propositions 2 and Lemma 7, we obtain that $\left(\mathcal{S}_{\lambda_1,\lambda_2}\right)$ has two nontrivial solutions $\left(u_0^+,v_0^+\right)\in\mathcal{N}^+$ and $\left(u_0^-,v_0^-\right)\in\mathcal{N}^-$. Since $\mathcal{N}^+\cap\mathcal{N}^-=\varnothing$, this implies that $\left(u_0^+,v_0^+\right)$ and $\left(u_0^-,v_0^-\right)$ are distinct.

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