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Dynamics of Collapse of a High Building

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Abstract: This paper is a contribution to the discussion about the fall of the New York World Trade Center (WTC) towers in 2001. The differential equation of the collapse of a high building is derived by taking into account many influences. A computer simulation of the collapse of the WTC building is presented using two independent programs with parameter variations. The results of both, differential equation and computer simulation, are compared resulting correspondences. The authors consider certain probable parameters which would have lessened both the observable speed of the collapse and its extent, however uncertainty regarding the magnitudes of the parameters remains.

Key words: Dynamics, collapse, WTC (World Trade Center).

1. Introduction

This paper deals with the fall of a high building. Its aim is not to investigate the cause of the fall as Bazant did in Ref. [1], but to examine the falling process itself. The paper is based on a final thesis put forward by Juranova [2] and introduces theory of the fall of a high building. The process of the fall is investigated from the point of view of the basic laws of mechanics and considers only the dynamics of the collapse in terms of speed and the extent of the fall. A differential equation of a high building collapse is derived and all major influences of the falling process are included. Results of both, the numerical solution of the differential equation and the computer simulation are presented for the various parameters along with the correspondence found between the approaches. The presented solution was also compared with a study by Kuttler [3] based on a discrete approach, which achieved a very positive concordance. The paper is organized as follows: Section 2 presents the derivation of the differential equation of the fall of a high building; section 3 introduces the solution of the differential equation derived in section 2; section 4 discusses the magnitudes of the parameters used in the differential equation presented above; section 5 puts forward a computer simulation of the fall of a high building and its comparison with the numerical solution of the differential equation; section 6 presents the study of K. Kuttler and its comparison with the solution of the above presented differential equation; in section 7, conclusions are stated.

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2. Derivation of a Differential Equation of the Fall of a High Building

A simple scheme of a falling building is illustrated in Fig. 1.

The authors assume that columns in the location between the coordinates x' and x_0 lose stability and a top part of the building above x' starts to fall and hits the still undamaged lower part of the building under the location x_0 with velocity v_0. The equation of dynamical equilibrium for the location x is

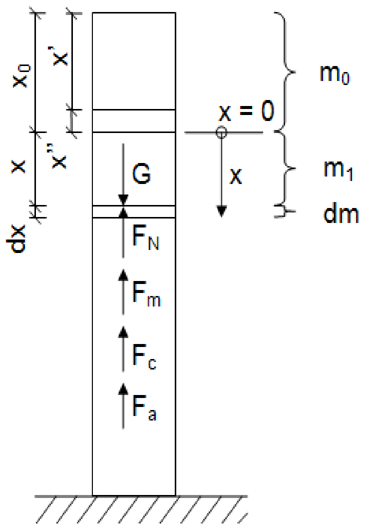

Fig. 1 The scheme of falling building.

$$G - F_N - F_m - F_C - F_a = 0 \quad (1)$$

where:

G is weight of a part of the building above the location x, for which equilibrium equation is formulated;

F_N is resistance put up by the columns against the collapse;

F_m is resistance originated by impact of the falling part of the building into a motionless mass;

F_C is a viscous damping;

F_a is an inertial force of a falling mass.

Following is a discussion of the individual parts of the equilibrium equation.

- The weight of the building above the location x:

$$G = mg\beta \quad (2)$$

where m is the mass of the building above the location x, g is acceleration of gravity, β is portion of the total mass above the location x which pushes to a lower part of the building. Mass which falls outside of the building is subtracted.

- The columns resistance:

$$F_N = mgs\kappa \quad (3)$$

where s is a rate of the ultimate force of columns to the current force in columns in the moment of the collapse, κ is the factor of the ultimate force of columns which represents average column resistance during its deformation related to an ultimate force.

Computation of column pressing was done using the method of controlled deformation in order to investigate this factor and receive its operational chart (See Fig. 2). Factor κ is then the rate of the ultimate force to its median value.

- The resistance of a motionless mass:

It is inertial force of a still mass dm accelerated in time dt to the speed v ($a = dv/dt$). The term $a = v/dt$ can be used for acceleration in the equation due to acceleration starting from zero up to the speed v.

The force F_m can be then expressed in this way:

$$F_m = dm \cdot a = dm \cdot \frac{v}{dt} \quad (4)$$

When considering $v = dx/dt$, Eq. (4) can be rewritten as follows:

$$F_m = dm \cdot \frac{v^2}{dx} = \mu v^2 \quad (5)$$

where $\mu = dm/dx$ is a line density of the building.

- The viscous damping:

$$F_c = C \cdot v = m\alpha v \quad (6)$$

where C is a factor of the viscous damping. The authors are considering Rayleigh damping here which depends on mass quantity $C = m\alpha$ only.

- The inertial force of a falling mass:

$$F_a = \beta m \cdot a = \beta m \frac{dv}{dt} = \beta m v \frac{dv}{dx} \quad (7)$$

Again, only the inertial force of a mass which does not fall outside of the building is considered here.

3. Differential Equation of the Building Collapse

The following equation can be gotten by substituting relations derived above into Eq. (1):

$$mg\beta - mgs\kappa - \mu v^2 - m\alpha v - \beta m v \frac{dv}{dx} = 0 \quad (8)$$

The authors divide the equation with speed v and mass m and adjust

$$\frac{b}{v} - \frac{v}{x + x_0} - \alpha - \frac{\beta dv}{dx} = 0 \quad (9)$$

where $b = g(\beta - s\kappa)$. The relation $\mu(x + x_0) = m$ is used when adjusting the equation.

Analytical solution of the differential equation was found only when the influence of damping was omitted:

$$v\left(x\right)=\sqrt{\frac{2b\left(x+x_0\right)}{2+\beta}+\left(x+x_0\right)^{\frac{-2}{\beta}}\cdot C_1} \tag{10}$$

To specify the constant C_1, the magnitude of the speed v_0 is needed. This is the speed of the mass m_0 above x_0 falling into the undamaged part of the building. The authors start from the same differential equation where factor b is modified into $b_0=\left(\beta-s_0\kappa\right)g$, whereas $s_0<1$:

$$\frac{b_0}{v\left(x\right)}-\frac{v\left(x\right)}{x+x'}-\frac{\beta dv\left(x\right)}{dx}=0 \tag{11}$$

The solution will thus have a similar form. Boundary conditions $v\left(0\right)=0$ will be used for finding the magnitude of the integration constant. Then we are looking for $v\left(x''\right)=v_0\left(x\right)$. After that the authors can return to a search of the integration constant C_1:

$$v_0\left(x\right)=\sqrt{\frac{2b_0\left(x+x_0\right)}{2+\beta}+\left(x+x_0\right)^{\frac{-2}{\beta}}\cdot C_1} \tag{12}$$

$$C_1=\frac{v_0^{\,2}\left(x\right)\left(2+\beta\right)-2b_0\left(x+x_0\right)}{\left(x+x_0\right)^{\frac{-2}{\beta}}\left(2+\beta\right)} \tag{13}$$

The author will establish C_1 in (10) and obtain the solution of Eq. (9) without the influence of damping:

$$v\left(x\right)=\sqrt{\frac{2b\left(x+x_0\right)}{2+\beta}+\left(x+x_0\right)^{\frac{-2}{\beta}}\cdot\frac{v_0^2\left(x\right)\left(2+\beta\right)-2b_0\left(x+x_0\right)}{\left(x+x_0\right)^{\frac{-2}{\beta}}\left(2+\beta\right)}} \tag{14}$$

Then it can be found out that when the collapse of the building will stop by assuming the speed to be zero $v\left(x\right)=0$. A solution of this equation was not found in the closed form so the authors are going to return to treat Eq. (9) with a numerical method. The Euler implicit method of solving a differential equation was found to be the most suitable method. Its principle is

$$v_{i+1}=v_i+h\cdot f\left(x_i,v\left(x_i\right)\right) \tag{15}$$

The following equation can be gotten:

$$v\left(x_{i+1}\right)=v\left(x_i\right)+\frac{h}{\beta}\left(\frac{b}{v\left(x_{i+1}\right)}-\frac{v\left(x_{i+1}\right)}{x_{i+1}+x_0}-\alpha\right) \tag{16}$$

After deduction of the speed $v\left(x_{i+1}\right)$, the following relation will be gotten:

$$v\left(x_{i+1}\right)=\frac{1}{2\left(h+\beta x_{i+1}+\beta x_0\right)}\cdot\left\{\begin{array}{l}-\alpha hx_{i+1}+\beta v\left(x_i\right)x_{i+1}-\alpha hx_0+\beta v\left(x_i\right)x_0+\\+\sqrt{-4\left(h+\beta x_{i+1}+\beta x_0\right)\left(-bhx_{i+1}-bhx_0\right)+\left(\alpha hx_{i+1}-\beta v\left(x_i\right)x_{i+1}+\alpha hx_0-\beta v\left(x_i\right)x_0\right)^2}\end{array}\right\} \tag{17}$$

The speed $v\left(0\right)$ will be computed in a similar way.

It is also important to say that this equation will stand only if the following conditions are present:

- When the mass is evenly spread out over the height of the building (appropriate for high-rise buildings);
- When the critical strength of the columns (including their resistance) is proportional to the weight of the building above the respective columns.

4. Discussion of the Magnitude of Damping, the Safety Factor S and the Ultimate Force Ratio κ

Before the authors began to seek a solution of the equation in a numerical way, they are going to clarify the magnitude of the damping α:

$$\alpha=\frac{C}{m}=\frac{2m\omega_n\xi}{m}=2\omega_n\xi \tag{18}$$

For the ratio of damping, the authors will consider the value 10-30% and the limiting value 0 that yields the damping ratios $\alpha=0.147, \alpha=3.18, \alpha=0$. Value 0 will be considered in order to solve the collapse without the effect of damping.

There is less certainty regarding the value of the parameter s. It is assumed to be around 2.5-3.

The coefficient of the ultimate strength in columns κ was computed from simulation of pressing of the columns by the method of controlled deformation. The pertinent numerical analyses were performed utilizing the RFEM program [4]. A deformed shape and pertinent response diagram is shown in Fig. 2.

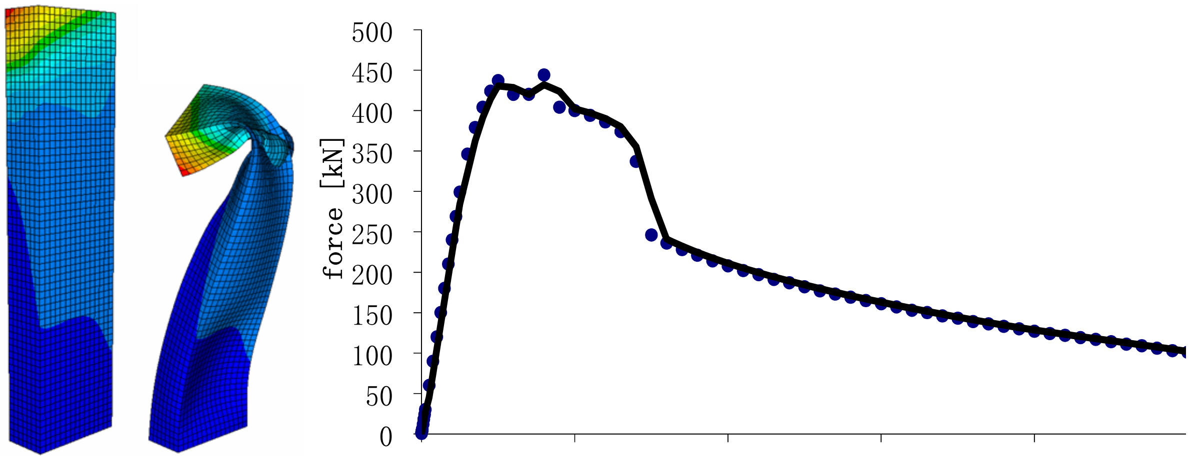

Fig. 2 Response diagram and the deformed shape of a box column. It is clear from the graph that the value of the κ coefficient will be around 0.25.

5. Computer Simulation

For the computer simulation of the fall of a high building, the RFEM and FyDiK programs were used. Both programs use the explicit method. The RFEM is a finite element program whereas the FyDiK uses an inverse approach, mass points connected by elastoplastic springs. The differences between the results of both programs were small. Figs. 3-4 show the resulting deformation of the building after the fall has stopped. For the chosen parameters, which the authors believe could be probable, the fall of the building would stop after falling cca 80 m. The main numerical results of the computer simulation and its comparison with the solution of the differential equation is presented in Table 1.

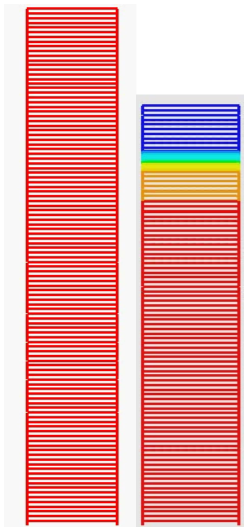

Fig. 3 Deformed building from RFEM (α = 0.5; s = 3).

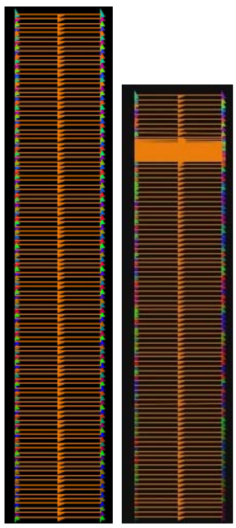

Fig. 4 Deformed building from FyDiK (α = 0.5; s = 3).

6. A Study by Professor Kenneth Kuttler and Its Comparison with the Above Presented Solution of the Differential Equation

In 2006, professor Kuttler of Birgham Young University also carried out a study of the WTC collapse [3]. In his study, he considers only the impact of the falling mass onto the motionless mass as if the floors were floating in the air unsupported by columns. The collapse itself is slowed only by the crash of falling floors onto the motionless mass (hitting the lower floors).

Table 1 Times and extends of the fall for various parameters.

s (-)	α	The theory with falling away of a mass x (m)	RFEM x (m)	FyDiK x (m)	The theory with falling away of a mass t (s)	RFEM t (s)	FyDiK t (s)
2.0	0.147	330	331.0	278.7	25.8	15.7	19.0
2.0	3.18	330	18.2	63.2	357.1	7.4	31.5
3.0	0	330	259.8	287.1	36.1	17.9	20.4
3.0	0.147	330	324.5	304.2	103.0	28.6	33.1
3.0	0.5	330	79.6	72.3	330.5	12.1	28.1
3.0	1.06	330	64.6	66.5	707.3	15.9	37.4
3.2	0.147	36.5	69.0	6.0	29.0	12.5	76.0

His solution is based on the law of the conservation of inertia and is discrete:

$$t_{celk} = \frac{h}{\left(\frac{\sqrt{2gh}}{2}\right)} + \frac{h}{\frac{1}{2}\left(\sqrt{\left(\left(\frac{1}{2}\right)^2+1\right)+2gh}+\frac{1}{2}\sqrt{2gh}\right)} + \sum_{k=3}^{n}\frac{h}{\left(\frac{1}{2}\sqrt{\left(1+\frac{1}{k^2}\sum_{j=1}^{k-1}j^2\right)2gh}+\frac{k-1}{k}\sqrt{\left(1+\frac{1}{(k-1)^2}\sum_{j=1}^{k-2}j^2\right)2gh}\right)} \tag{19}$$

As an example, the authors present a demonstration of Kuttler's solution in comparison with this approach. A building of 411 m high with 110 stories is taken. h will be 411/110 = 3.74 m, g = 9.81 m/s^2 and n = 110. From Eq. (19), it will be calculated that such a building would fall according to this approach for 14.96 s.

In this approach, the authors have utilized the same assumptions as professor Kuttler, the differential equation taking the form:

$$\frac{g}{v(x)} - \frac{v(x)}{x+x_0} - \frac{dv}{dx} = 0 \tag{20}$$

It is possible to arrive at a solution in this case in closed form. The authors will still take a quantity x_A, which will be the place, where the fall will stop. Marginal conditions are clear, the speed is zero at the point of fall cessation. The result is

$$v(x) = \frac{\sqrt{6g\left(x^3 - x_A^3\right)}}{3x} \tag{21}$$

It is possible to determine from this result the fall time by an integration such as:

$$t = \int_0^{x_A}\frac{dx}{v(x)} = \int_0^{x_A}\frac{3x}{\sqrt{6x^3 g}}dx = \frac{x_A^2\sqrt{6}}{\sqrt{x_A^3 g}} = \frac{x_A\sqrt{6}}{\sqrt{x_A g}} \tag{22}$$

Provided considering the same input data as above, fall time is 15.85s.

In fact the fall of the buildings took about 11 s (the time differs in various reports, it is the accuracy of the time span can be verified from video recordings). The material presented in this section is of course greatly oversimplified. In actuality the situation is more complex. Other more complicating influences may be present, for instance relating to column resistance, which might bring into being a substantial increase in fall time. Furthermore consideration should be given to the possibility that the fall could even cease before the whole building destructs.

7. Conclusions

It was possible to solve a differential equation of the building collapse in closed form only when the damping was omitted. Real damping must be a necessary element when the massive destruction of all the components of the building structure. Therefore this solution represents limit to the speed of the fall and the extent of the collapse. The general form of the differential equation was solved only in a numerical way.

Two independent computer programs were used for the simulation, named RFEM [4] and FyDik (by P. Frantík). Despite the difference in the approaches both computer programs gave comparatively similar solutions. The difference between the solution of the differential equation and that of the computer simulation is greater. The differences have two sources. The first cause is the fact, that in the differential equation it is assumed that the average resistance of the columns is the mean value from the diagram in Fig. 2. On the other hand the computer simulation works with a more general approximation of the column response diagram. In this case the computer simulation is better. The computer simulation however did not take into account the ratio β assuming a unit value whereas the differential equation can assume an arbitrary value of this parameter.

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