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Some Mathematical Properties of the Dynamically Inconsistent Bellman Equation: A Note on the Two-Sided Altruism Dynamics

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Abstract: This article describes some dynamic aspects on dynastic utility incorporating two-sided altruism with an OLG setting. The special case is analyzed where the weights of two-sided altruism are dynamically inconsistent. The Bellman equation for two-sided altruism proves to be reduced to one-sided dynamic problem, but the effective discount factor is different only in the current generation. It is shown that a contraction mapping result of value function cannot be achieved in general, and that there can locally exist an infinite number of self-consistent policy functions of the class C^n with distinct steady states (indeterminacy of self-consistent, differentiable policy functions).

Keywords: Bellman equation, Two-sided altruism, Dynamic inconsistency, Self-consistent policy functions, Indeterminacy, Overlapping generations model.

1. Introduction

This paper analyzes some mathematical aspects of two sided altruism dynamics especially under dynamic inconsistency, with constant fertility and no saving. The model is based on so-called Buiter-Carmichael-Burbidge (BCB) type two-sided utility, which we modify for the three stage OLG model, so that each generation might hold, in general, two chances of intergenerational linkage, firstly through fertility and capital investment decision planned by middle age parent during young adulthood, and secondly through transfer (compensation/bequest) during old stage. As explained later, this modification proves to induce some interesting, but puzzling behaviors in macro-dynamics, especially under dynamic inconsistency.

As references, two-sided altruism dynamics are treated, for example, in Abel (1987), Kimball (1987), Hori and Kanaya (1989), Altig and Davis (1993), Hori (1997), Aoki (2011). Furthermore, the differentiability of value functions is discussed in Benveniste and Scheinkman (1979), Santos (1991), Araujo (1991), Montrucchio (1987). Mathematical treatments regarding the principle of optimality appear, for example, in Bellman (1957), Pontryagin (1962), Blackwell (1965), Stokey and Lucas (1989), and Mitra (2000). Boldrin and Montrucchio (1986), and Geanakoplos and Brown (1985) are located at the earlier stage among the “indeterminacy of equilibrium” literature. Furthermore, Krusell P., Kuruşçu B. and Smith A. A.

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(2002), and Krusell P. and Smith A. A. (2003, 2008) analyze the quasi-geometric discounting model.

The organization of this paper is as follows. We describe the model in section 2, and theoretical results in section 3, and finally concluding remarks in section 4.

2. Model

We assume a typical OLG model consisting of three life stages, C, Y, O (childhood, young adulthood (working age) and old adulthood (retirement stage)). Generation t, who spends its young adulthood (stage Y) at period t, shares the adjacent life stages with generation $t+1$ and $t-1$. (For example, stage O of old parents and stage Y of young children are shared simultaneously.) The whole life utility of generation t is defined as $u_t = u_t^{(y)} + \delta u_{t+1}^{(o)}$, where $u_t^{(y)}$ and $u_{t+1}^{(o)}$ are the young and old adulthood utility of generation t, respectively, and δ is a time preference discount factor for old (retirement) stage. At period t, generation t decides some of its life strategies, fertility (n_t) and capital investment for children (k_{t+1}) and saving for forthcoming retirement stage O (s_t), and gift for old parents (b_t). Just for simplicity, we assume that fertility is constant ($n_t = 1$) and there is no saving ($s_t = 0$), and that only the gift for old parents is controllable.

2.1 Representative Agent Problem

Now we consider the following type of two sided altruism, where u_t and V_t and is an individual life utility and a (two-sided) dynastic utility of generation t, respectively.

$$V_t = \alpha\delta u_t^{(o)} + \sum_{s=0}^{\infty} \beta^s u_{t+s} = \alpha\delta u_t^{(o)} + \sum_{s=0}^{\infty} \beta^s \left(u_{t+s}^{(y)} + \delta u_{t+1+s}^{(o)} \right) =$$

$$\alpha\delta u_t^{(o)} + \left(u_t^{(y)} + \delta u_{t+1}^{(o)} \right) + \beta \left(u_{t+1}^{(y)} + \delta u_{t+2}^{(o)} \right) + \beta^2 \left(u_{t+2}^{(y)} + \delta u_{t+3}^{(o)} \right) + \cdots.$$

This representation is an OLG version of Buiter-Carmichael-Burbidge (BCG) type utility of the form, $V_t = \alpha u_{t-1} + u_t + \sum_{s=1}^{\infty} \beta^s u_{t+s}$. (As for the BCG utility, see Abel (1987)). We assume $\beta < 1$.

Now denote consumptions at stage Y (period t) and O (period $t+1$) of generation t, by $c_{1,t}$ and $c_{2,t+1}$, respectively. Then $c_{1,t} = f(k_t) - k_{t+1} - b_t$ and $c_{2,t+1} = b_{t+1}$, where $f(\cdot)$ is a production function, k_t is a human capital of generation t with a full depreciation in one period, and b_{t+1} is a gift from young adult generation $t+1$ to old adult generation t. Assuming the intertemporally separable utility form, $u(c) = c^{1-\sigma}/(1-\sigma)$ for $\sigma \neq 1$, and $\ln c$ for $\sigma = 1$, we have $u_t^{(y)} = u(c_{1,t})$ and $u_{t+1}^{(o)} = u(c_{2,t+1})$.

If all the period t and subsequent strategies $\{b_{t'}, k_{t'+1}\}_{t'=t}^{\infty}$ are independently determined by generation t, then the generation solves the following *representative agent problem*.

$$\max_{\{k_{t'+1}, b_{t'}\}_{t'=t}^{\infty}} V_t \qquad \text{(RA1)}$$

However, in this OLG linkage, the inconsistent motive for intergenerational transfer between young children and old parents makes each generation behave differently from RA1. To see why, we rewrite V_t as

$$V_t = \left(u_t^{(y)} + \alpha\delta u_t^{(o)} \right) + \beta \left(u_{t+1}^{(y)} + (\delta/\beta) u_{t+1}^{(o)} \right) + \beta^2 \left(u_{t+2}^{(y)} + (\delta/\beta) u_{t+2}^{(o)} \right) + \cdots$$

$$= \left(u_t^{(y)} + \alpha\delta u_t^{(o)} \right) + \sum_{s=1}^{\infty} \beta^s \left(u_{t+s}^{(y)} + (\delta/\beta) u_{t+s}^{(o)} \right).$$

Maximizing V_t in RA1 necessarily assures that the ratio of marginal utility in consumption between young and old adults be $1 : \alpha\delta$ at period t, while $1 : \delta/\beta$ at period t' $(t' = t+1, t+2, \cdots)$. As a matter of fact, however, all generations $t'(\geq t)$ are to adjust their gifts and allocate their consumptions with old parents by $1:\alpha\delta$ in the ratio of marginal utility. Therefore, the corresponding representative agent problem should be rather the following sequential problem.

$$\max_{\{k_{t'+1},\hat{b}_{t'}\}_{t'=t}^{\infty}} V_t \qquad \text{(RA2)}$$

$$s.t. \quad \hat{b}_{t'} = b_{t'} \operatorname{argmax}\{u_{t'}^{(y)} + \alpha\delta u_{t'}^{(o)}\} \qquad given$$

$k_{t'}$ *and* $k_{t'+1}$, $t' = t, t+1, t+2, \cdots$.

A solution of the constraint, $\hat{b}_{t'}$ $(t' = t, t+1, t+2, \cdots)$, can be derived explicitly as $\hat{b}_{t'} = \frac{1}{1+(\alpha\delta)^{-1/\sigma}}\left\{f(k_{t'}) - k_{t'+1}\right\}$, so defining $\hat{c}_{1,t'} \equiv f(k_{t'}) - k_{t'+1} - \hat{b}_{t'} = \frac{1}{1+(\alpha\delta)^{1/\sigma}}\left\{f(k_{t'}) - k_{t'+1}\right\}$ and $\hat{c}_{2,t'} \equiv \hat{b}_{t'} = \frac{(\alpha\delta)^{1/\sigma}}{1+(\alpha\delta)^{1/\sigma}}\left\{f(k_{t'}) - k_{t'+1}\right\}$, we have $u(\hat{c}_{1,t'}) + \alpha\delta u(\hat{c}_{2,t'}) \equiv Au(\tilde{C}_{t'})$ and $u(\hat{c}_{1,t'}) + (\delta/\beta)u(\hat{c}_{2,t'}) \equiv Bu(\tilde{C}_{t'})$, where $A \equiv \left\{1 + (\alpha\delta)^{1/\sigma}\right\}^{\sigma}$, $B \equiv \frac{\left\{1+(\alpha\delta)^{1/\sigma}(\alpha\beta)^{-1}\right\}}{\left\{1+(\alpha\delta)^{1/\sigma}\right\}^{1-\sigma}}$, and $\tilde{C}_{t'} \equiv f(k_{t'}) - k_{t'+1}$.

Thus defining $\breve{V}_t$, which internalizes the old age support by each generation according to the $1:\alpha\delta$ rule, $\hat{b}_{t'} = b_{t'} \operatorname{argmax}\left\{u_{t'}^{(y)} + \alpha\delta u_{t'}^{(o)}\right\}$,

$$\begin{aligned}\breve{V}_t &= Au(\tilde{C}_t) + \beta Bu(\tilde{C}_{t+1}) + \beta^2 Bu(\tilde{C}_{t+2}) + \\ &\beta^3 Bu(\tilde{C}_{t+3}) + \cdots = A\{u(\tilde{C}_t) + \beta\mu u(\tilde{C}_{t+1}) \\ &+ \beta^2\mu u(\tilde{C}_{t+2}) + \beta^3\mu u(\tilde{C}_{t+3}) + \cdots\},\end{aligned}$$

where $\mu = B/A$. Here $\mu = 1$ if $\alpha\beta = 1$ (*dynamically consistent*). Thus the effective discount factor is $\beta\mu$ at the present period t, but β from the next period $t+1$. Therefore the objective function proves to hold a sort of the quasi-geometric discounting time structure, as discussed in Krusell et al. (2002, 2003 and 2008).

Finally RA2 can be simply rewritten as

$$\max_{\{k_{t'+1}\}_{t'=t}^{\infty}} \breve{V}_t. \qquad \text{(RA2')}$$

It is obvious that RA2' or RA2 are equivalent with RA1 if and only if $\mu = 1$ $(i.e., \alpha\beta = 1)$. We call this case ($\mu = 1$) *dynamic consistency*, and otherwise *inconsistency*. From *time consistency* requirement $\alpha = \frac{1-\sqrt{1-4ab}}{2b}$, $\beta = \frac{1-\sqrt{1-4ab}}{2a}$ $(ab \leq 1/4)$, we have $\alpha\beta < 1$, therefore $\mu > 1$, that is, the model is dynamically inconsistent. See Kimball (1987), Hori et al. (1989) and Hori (1997).

2.2 Functional Bellman Equation

Now we try to rewrite the representative agent problem represented in RA2', in the form of recursive functional Bellman equation.

At first, we define another objective function $\tilde{V}_t$,

$$\begin{aligned}\tilde{V}_t &= Bu(\tilde{C}_t) + \beta Bu(\tilde{C}_{t+1}) + \beta^2 Bu(\tilde{C}_{t+2}) + \\ &\beta^3 Bu(\tilde{C}_{t+3}) + \cdots = B\{u(\tilde{C}_t) + \beta u(\tilde{C}_{t+1}) + \\ &\beta^2 u(\tilde{C}_{t+2}) + \beta^3 u(\tilde{C}_{t+3}) + \cdots\},\end{aligned}$$

where

$$\tilde{V}_t = \breve{V}_t + (B - A)u(\tilde{C}_t).$$

Then two sided altruism dynamics is described as the following one sided functional equation, where dual value functions, $\breve{W}_t(\cdot)$ and $\tilde{W}_t(\cdot)$, correspond with objective functions $\breve{V}_t$ and $\tilde{V}_t$, respectively.

$$\breve{W}_t(k_t) = \max_{0 \leq k_{t+1} \leq f(k_t)} \left(Au(f(k_t) - k_{t+1}) + \beta\tilde{W}_{t+1}(k_{t+1})\right), \qquad \text{(BE1)}$$

where

$$\tilde{W}_t(k_t) = \breve{W}_t(k_t) + (B - A)u(\hat{C}_t)$$

and $\hat{C}_t = f(k_t) - \hat{k}_{t+1}$,

where

$$\hat{k}_{t+1} = \arg\max_{0 \leq k_{t+1} \leq f(k_t)} \left(Au(f(k_t) - k_{t+1}) + \beta\tilde{W}_{t+1}(k_{t+1})\right).$$

$\hat{k}_{t+1} = g_t(k_t)$ is a policy function of generation t, given next generation $t+1$'s value functions $\breve{W}_{t+1}(\cdot)$ and $\tilde{W}_{t+1}(\cdot)$. BE1 is a simpler version of two sided altruism model examined by Hori (1997), eqs. (4.2)-(4.6).

Under dynamic inconsistency $(\mu \neq 1)$, the maximization problem of representative agent's utility (RA2' or RA2) is not equivalent with the corresponding Bellman equation (BE1) with the same objective function $\breve{V}_t$, even though some transversality conditions are appropriately assumed. As a matter of fact, there exist two effects caused by dynamic inconsistency: A. *intra-temporal direct effect on intergenerational transfer*, and B. *inter-temporal indirect effect on preceding generations' policy functions*. And Bellman equation proves to internalize both effects A and B, while representative agent utility maximization does only effect A. In addition, the principle of optimality does not hold in general under dynamic inconsistency, so that Blackwell's contraction mapping theorem (Stokey & Lucas (1989) Theorem 3.3) cannot be applied. However, this complete internalization induces some perplexing aspects in BE1, which does not appear in case of dynamic consistency.

To see this, we now rewrite BE1 in a backwardly recursive fashion.

$$\breve{W}_{n+1}(k) = \max_{y \in Y(k)} \left(Au(f(k)-y) + \beta \tilde{W}_n(y) \right), \tag{BE1'}$$

where

$$\tilde{W}_{n+1}(k) = \breve{W}_{n+1}(k) + (B-A)u(\hat{C}).$$

n is a time distance from the future terminal period $n=0$. Here $\hat{C} = f(k) - \hat{y}$, where $\hat{y} = \arg\max_{y} \left(Au(f(k)-y) + \beta \tilde{W}_n(y) \right) \equiv g_{n+1}(k)$. $Y(k)$ is *a feasible correspondence* defined as $Y(k) = \{\, y \mid 0 \le y \le f(k) \}$.

In case of dynamic consistency $(\mu = B/A = 1)$, the above equation reduces to a regular Bellman equation,

$$W_{n+1}(k) = \max_{y \in Y(k)} \left(Au(f(k)-y) + \beta W_n(y) \right),$$

where $W_n(\cdot) = \tilde{W}_n(\cdot) = \breve{W}_n(\cdot)$.

Then, under quite general conditions, the principle of optimality is known to assure a uniform convergence of $W_n(\cdot)$ to time-independent value function $W(\cdot)$, which satisfies

$$W(k) = \max_{y \in Y(k)} \left(Au(f(k)-y) + \beta W(y) \right).$$

In case of dynamic inconsistency $(\mu \neq 1)$, such a contraction mapping result, for example, by Blackwell (1965), cannot be automatically expected. However roughly dare to consider, at any events, the following time-independent functional equation.

$$\breve{W}(k) = \max_{y \in Y(k)} \left(Au(f(k)-y) + \beta \tilde{W}(y) \right), \tag{BE2}$$

where

$$\tilde{W}(k) = \breve{W}(k) + (B-A)u(\hat{C}),$$

and $\hat{C} = f(k) - \hat{y}$ and $\hat{y} = \arg\max_{y \in Y(k)} \left(Au(f(k)-y) + \beta \tilde{W}(y) \right) \equiv g(k)$.

So far we assume *a priori* the existence of policy function $g_n(k)$ instead of policy correspondence, implicitly its differentiability, the uniform convergence of convergence of $g_n(k)$ to $g(k)$, and so on. See, for example, Stokey et al. (1989). Some of these conditions prove to hold even under dynamic inconsistency, but some do not. In the next section, we will investigate the analytical properties of BE2 and BE1' from various viewpoints.

3. Results

Let be $R_+ = \{x \in R \mid x \ge 0\}$ and define $K \subset R_+$, the domain of capital k, so that $k \in K$. Also assume that $f: R_+ \to R_+$ and $u: R_+ \to R_+$ are differentiable and satisfy the following properties:

Assumptions:

F0: Production function $f(k)$ is C^∞, *i.e.*, infinitely continuously differentiable.

F1: $f(0) = 0$.

F2: f is strictly concave.

F3: $f'(k) > 0$.

F4: $\lim_{k \to 0} f'(k) = \infty$, $\lim_{k \to \infty} f'(k) = 0$.

U0: Utility function $u(c)$ is C^{∞}, *i.e.*, infinitely continuously differentiable. Specifically $u(c)=c^{1-\sigma}/(1-\sigma)$ for $\sigma \neq 1$, where σ is a relative risk aversion or an inverse of elasticity of intertemporal substitution.

U1: $u(0)=0$.

U2: u is strictly concave.

U3: $u'(c)>0$ for $c>0$.

U4: $\lim_{c\to 0} u'(c)=\infty$, $\lim_{c\to\infty} u'(c)=0$.

Theorem 1 derives a modified Euler equation corresponding with two-sided altruism.

Theorem 1: In BE2 assume that value function $\tilde{W}(k)$ and policy function $\hat{y}=g(k)$ are C^1, *i.e.*, once continuously differentiable, and that $\hat{y}=g(k)$ is an interior of the feasible correspondence $Y(k)$. Then BE2 satisfies the following Euler equation EE1, modified for case of dynamic inconsistency.

$$-u'(f(k)-y)+ \\ \beta u'(f(y)-g(y))\left[\mu f'(y)-(\mu-1)g'(y)\right]=0, \tag{EE1}$$

where $y=\hat{g}(k)$ is a solution of EE1.

Proof is trivial and left for Appendix. EE1 (Euler equation under dynamic inconsistency) is equivalent with the "generalized Euler equation", as derived in Krusell et al. (2002). This is plausible, because the OLG model under dynamic inconsistency proves to be equivalent with a sort of the quasi-geometric discounting model. Then time consistency requires that it happens to be $\hat{g}(k)=g(k)$. Therefore EE1 can be rewritten as

$$-u'((f-g)(k))+\beta u'((f-g)\circ g(k))\times \\ \left[(\mu f'-(\mu-1)g')\circ g(k)\right]=0, \tag{EE2}$$

where $a\circ b(x)\equiv a(b(x))$ denotes a composite function of x, and $(a\pm b)(x)\equiv a(x)\pm b(x)$.

If $g(k)$ satisfies EE2, then we say $g(k)$ is *self-consistent*, in the sense that if next generation's policy function is $g(k)$, then the current generation necessarily takes the same policy.

At a fixed point, $k=k^*$, $g(k^*)=k^*$ and $-u'(c^*)+\beta u'(c^*)[\mu f'(k^*)-(\mu-1)g'(k^*)]=0$, where $c^*=f(k^*)-k^*$. Thus $\beta[\mu f'(k^*)-(\mu-1)g'(k^*)]=1$ (★) holds. Note that $\beta f'(k^*)=1$ implies $g'(k^*)=1/\beta$, and that $\beta\mu f'(k^*)=1$ implies $g'(k^*)=0$.

3.1 Indeterminacy of self-consistent policy functions

Next theorem relates dynamic inconsistency with indeterminacy of self-consistent policy functions.

Theorem 2: Assume $\mu\neq 1$. Let $\bar{k}$ be a point such that $\beta\mu f'(\bar{k})=1$, and take any point such that $k^*\neq\bar{k}$. Then, there exists uniquely a self-consistent policy function $g(k)$ satisfying EE2, such that it has a fixed point at $k=k^*$, and is C^n, *i.e.*, infinitely continuously differentiable, for any positive integer n on some open ball around k^*, $k\in B(k^*,\varepsilon)$, *i.e.*, $k^*-\varepsilon<k<k^*+\varepsilon$, with $\varepsilon>0$.

Proof is given in Appendix. This theorem says that if a fixed point k^* is determined, then a corresponding self-consistent policy function is also uniquely determined and C^n in an open ball around k^*. Since there exists a trade-off and degree of freedom between the values of $g(k^*)$ and $g'(k^*)$, it is possible to construct an infinite number of distinct self-consistent policy functions for distinct k^*. While Krusell et al. (2003) find out the indeterminacy of saving rules (policy functions), which is a step function, non-differentiable at and converging to some arbitrary stationary point, this theorem provides a rigorous proof that an infinite number of distinct C^n self-consistent policy functions exist with distinct stationary points.

3.2 Existence of $\mathbf{C}^n$ Self-Consistent Value Functions with Uniform Convergence

Given a self-consistent policy function $g(k)$ satisfying EE2, self-consistent value functions $\tilde{Z}(k)$ and $\breve{Z}(k)$ in duality, which correspond with $\tilde{W}(k)$ and $\breve{W}(k)$, respectively, are defined as the following infinite functional series.

$$\begin{aligned}\tilde{Z}(k) &= B\{u((f-g)(k))+ \\ &\quad \beta u(f-g)\circ g(k))+ \qquad \text{(SV1)} \\ &\quad \beta^2 u((f-g)\circ g\circ g(k))+\cdots\}\end{aligned}$$

$$= B\sum_{s=0}^{\infty} \beta^s u((f-g)s\circ \underbrace{g\circ\cdots\circ g}_{s}(k))$$

$$= B\sum_{s=0}^{\infty} \beta^s u((f-g)\circ g_{(s)}(k))$$

and

$$\breve{Z}(k) = \tilde{Z}(k) - (B-A)u((f-g)(k)).$$

Here we define $g_{(s)}(k) \equiv s\,\underbrace{g\circ\cdots\circ g}_{s}(k)$. It is easy to verify $\tilde{Z}(k^*) = (B/(1-\beta))u(f(k^*)-k^*)$. Now we claim the following theorem.

Theorem 3: Assume $\mu \neq 1$. Let S be a set such that

$$S = \left\{\, k \mid |(\mu f'(k) - 1/\beta)/(\mu - 1)| < 1 \right\},$$

and take any point such that $k^* \in S$ and $k^* \neq \overline{k}$. Then:

(i) Self-consistent value functions $\tilde{Z}(k)$ and $\breve{Z}(k)$, which are represented as an infinite series SV1, where $g(k)$ satisfies EE2, uniformly converges and once continuously differentiable C^1 on some open ball around k^*, $k \in B(k^*, \varepsilon')$, with $\varepsilon' > 0$.

(ii) In BE2, replace $\tilde{W}(\cdot)$ and $\breve{W}(\cdot)$ with y and $\breve{Z}(\cdot)$, respectively. Then $\tilde{Z}(\cdot)$ and $\breve{Z}(\cdot)$ satisfy BE2 with a unique self-consistent policy function $g(k)$.

(iii) Let S' be a set such that $S' = \left\{\, k \mid 0 < (\mu f'(k) - 1/\beta)/(\mu - 1) < 1 \right\}$, and take any point such that $k^* \in S'$, instead of S. Then $\tilde{Z}(k)$ is strictly concave at k^*.

Proof is given in Appendix. By the proof $\tilde{Z}(k)$ and $\breve{Z}(k)$ are also shown to be C^n. Thus BE2 is now formally justified as the following Bellman equation.

$$\breve{Z}(k) = \max_{y\in Y(k)}\left(Au(f(k)-y) + \beta\tilde{Z}(y)\right), \quad \text{(BE3)}$$

where

$$\tilde{Z}(k) = \breve{Z}(k) + (B-A)u(\hat{C}),$$

and $\hat{C} = f(k) - \hat{y}$ and $\hat{y} = y \in \arg\max_{Y(k)}\left(Au(f(k)-y) + \beta\tilde{Z}(y)\right)$.

Theorem 3 says that $\hat{y}$ in BE3 must be the same as $g(k)$, if c and $\breve{Z}(k)$ are defined as SV1.

3.3 Instability Against Perturbation of Self-Consistent Policy/Value Functions

In section 3.2, at first we searched out the policy functions, which satisfies Euler equation EE2 locally around a fixed point k^*, and then calculate the corresponding value functions. Therefore it is not still verified if deviated policy functions would necessarily converges to some of self-consistent ones in a global sense. So we go back to a recursive Bellman equation BE1'. At the terminal stage $n = 0$,

$$\breve{W}_1(k) = \max_{y\in Y(k)}\left(Au(f(k)-y) + \beta\tilde{W}_0(y)\right),$$

where

$$\tilde{W}_1(k) = \breve{W}_1(k) + (B-A)u(\hat{C}).$$

Assume a log utility $u(c) = \ln c$ and a Cobb-Douglus form production function $f(k) = ak^b$, although this utility does not satisfy condition U1. Assuming $\tilde{W}_0(k) = 0$ (therefore $g_0(k) = 0$), it is easy to verify, by recursive calculation, that $g_n(k) = \gamma_n f(k)$, where $\gamma_n = \frac{\beta\mu b\{1-(\beta b)^n\}}{(1-\beta b)+\beta\mu b\{1-(\beta b)^n\}}$, and $g_n(k) \underset{k\to\infty}{\to} g(k) = \gamma f(k)$, where $\gamma = \frac{\beta\mu b}{1-\beta b(1-\mu)}$. ($g(k) = \gamma f(k)$ satisfies self-consistent Euler equation EE2.) As a matter of fact, if $g_0(k)$ belongs to a family of functions of a Cobb-Douglus form, $g_0(k) = \gamma_0 f(k)$ $(0 \le \gamma_0 < 1)$, then it is proved that $g_n(k) \underset{k\to\infty}{\to} g(k) = \gamma f(k)$, the same destination function. However, in general, every possible initial policy function $g_0(k)$ might not necessarily attain a uniform convergence to $g(k)$. This point is totally different from case of dynamic consistency $\mu = 1$. (See Blackwell (1965) for a contraction result in case of $\mu = 1$.)

Here we limit our focus on the local stability against temporal perturbation of self-consistent policy and value functions.

Theorem 4: Let $h(k)$ be a C^{∞} function, which is bounded in an open ball around k^*, and $h(k^*) \neq 0$. Also let $g(k)$ be a self-consistent policy function, which satisfies EE2, and let $\tilde{Z}(k)$ be a corresponding self-consistent value function generated by SV1. Assume both $g(k)$ and $\tilde{Z}(k)$ are C^n in an open ball around k^*.

(i) Assume that the next generation's policy function is subject to a perturbation of the form: $g(y) \to \tilde{g}(y,\eta) = g(y) + \eta h(y)$, and the current generation's policy function changes $g(k) \to \hat{g}(k,\eta)$.

Then the condition for the policy function's contraction in an open neighborhood around k^*, $k \in B(k^*, \varepsilon'')$ with some $\varepsilon'' > 0$, is

$$\left| \frac{g'(k^*)}{f'(k^*)} \left\{ 1 + \beta(\mu - 1) \frac{h'(k^*)}{h(k^*)} \frac{u'(c^*)}{u''(c^*)} \right\} \right| < 1.$$

(ii) Assume that the next generation's value function is subject to a perturbation of the form: $\tilde{Z}(y) \to \tilde{Z}(y,\eta) = \tilde{Z}(y) + \eta h(y)$, and the current generation's value function changes $\tilde{Z}(k) \to \hat{Z}(k,\eta)$.

Then the condition for the value functions's contraction in an open neighborhood around k^*, $k \in B(k^*, \varepsilon'')$ with some $\varepsilon'' > 0$, is

$$\left| \beta \left\{ 1 + (\mu - 1) \frac{h'(k^*)}{h(k^*)} \frac{g'(k^*)}{f'(k^*)} \frac{u'(c^*)}{u''(c^*)} \right\} \right| < 1.$$

Proof is given in Appendix. Both results (i) and (ii) are similar. In case of dynamic consistency $\mu = 1$, the contraction can be achieved under quite general conditions, $| g'(k^*) / f'(k^*) | < 1$ or $\beta < 1$, in which the local convergence in sup norm $\left\| g_n - g \right\|_K \underset{n\to\infty}{\to} 0$ or $\left\| \tilde{Z}_n - \tilde{Z} \right\|_K \underset{n\to\infty}{\to} 0$ are attained, whatever the first order or the higher orders of perturbation $(h'(k),\ h''(k),\ h'''(k) \cdots)$ might be. However, in case of $\mu \neq 1$, the first order perturbation $h'(k)$ or the first order nondifferentiability directly affects the possibility of $0'th$ order contraction (in sup norm), and so do the second or higher perturbation $(h''(k),\ h'''(k),\ \cdots)$, or the nondifferentiability in these orders, indirectly. So finally in the next theorem we state the first order effect on $\hat{g}(k,\eta)$ of perturbation $h(k)$ around k^*, which is measured by $\hat{g}_{12}(k^*, 0)$.

Theorem 5: Under the same assumptions as in (i) of Theorem 4, the $\hat{g}_{12}(k^*, 0)$, the first order effect of $\hat{g}(k^*, \eta)$ for a small change in $\eta h(y)$, is given in the following formula.

$$\hat{g}_{12}(k^*, 0) = X_0 h(k^*) + X_1 h'(k^*) + X_2 h''(k^*),$$

Where

$$X_0 = -\frac{g'(k^*)}{f'(k^*)} \left(\frac{g'(k^*)}{f'(k^*)} \{ f''(k^*) g'(k^*) - f'(k^*) g''(k^*) \} - \frac{u''(c^*)}{u'(c^*)} \{ f'(k^*) - g'(k^*) \} \{ 1 - g'(k^*) \} \right),$$

$$X_1 = \frac{g'(k^*)}{f'(k^*)} \left(g'(k^*) + \beta(\mu - 1) \left[\begin{array}{c} g'(k^*) \{ f'(k^*) - g'(k^*) \} \\ -\dfrac{u'(c^*)}{u''(c^*)} \dfrac{g'(k^*)}{f'(k^*)} \left(\begin{array}{c} \{ f''(k^*) g'(k^*) - f'(k^*) g''(k^*) \} \\ + \dfrac{u'''(c^*)}{u''(c^*)} \dfrac{g'(k^*)}{f'(k^*)} \{ f'(k^*) - g'(k^*) \} \end{array} \right) \end{array} \right] \right),$$

$$X_2 = \beta(\mu - 1)\frac{\{g'(k^*)\}^2}{f'(k^*)}\frac{u'(c^*)}{u''(c^*)}.$$

Proof is given in Appendix. Let $\eta h_n(k)$ denote a functional deviation from self-consistent policy function $g(k)$ at stage n. In case of dynamic consistency $\mu = 1$, then $X_1 = \frac{\{g'(k^*)\}^2}{f'(k^*)}$ and $X_2 = 0$. From Theorem 4, $h_n(k^*) \to 0$, as $n \to \infty$. Since $\frac{\{g'(k^*)\}^2}{f'(k^*)} < 1$, $h_n'(k^*)$ proves to converge to zero. That is, qualitatively speaking, the order-by-order derivative contraction operates in general. However, in case of dynamic inconsistency $\mu \neq 1$, the higher order derivative coefficient affects the lower one, and the lower one, if failing in contraction, remains an obstacle for contraction in the higher one, as $n \to \infty$.

4. Concluding Remarks

Thus this paper describes a dynamics of one-sector growth model under two sided altruism. Here we derived a modified Euler equation for dynamic inconsistency.

From viewpoints of macrodynamics & game theory, one important implication of this paper is that even under this perfect foresight setting with a perfectly rational representative agent (in the sense that each generation takes account of and internalizes all the predictable reaction by the subsequent generations), dynamic inconsistency still induces indeterminacy of self-consistent policy functions, and possibly cause some dynamic fluctuation of policy function generated in recursive fashion.

This aspect is crucial not only in this two-sided altruism dynamics, but also in other models incorporating irregular structures of variable effective discount factors, as in hyperbolic discount factor model, endogenized (so variable) discount factor model, or fertility endogenized model.

This paper focused on self-consistency, differentiability, and fragility against recursive perturbation of policy/value functions, in a local area around any arbitrary fixed point k^*. Investigation on global transition in BE1 or BE1', characterized by dynamic fluctuation, will be left for future work.

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Appendix

Proof of Theorem 1

From the assumption, $\hat{y}$ is an interior of $Y(k)$. Differentiating BE2 with k, we have $-Au'(f(k)-\hat{y})+\beta\tilde{W}'(\hat{y})=0$ (♠). Then, since $\hat{y}=g(k)$ and $g(k)$ is once continuously differentiable,

$$\begin{aligned}\breve{W}'(k) &= Au'(f(k)-\hat{y})f'(k)+\left(\frac{d\hat{y}}{dk}\right)\underbrace{\left[-Au'(f(k)-\hat{y})+\beta\tilde{W}'(\hat{y})\right]}_{=0}\\ &= Au'(f(k)-g(k))f'(k).\end{aligned}$$

Then, from $\tilde{W}(k)=\breve{W}(k)+(B-A)u(f(k)-g(k))$,

$$\begin{aligned}\tilde{W}'(k) &= Au'(f(k)-g(k))f'(k)+\left(B-A\right)u'(f(k)-g(k))\left\{f'(k)-g'(k)\right\}\\ &= u'(f(k)-g(k))[Bf'(k)-\left(B-A\right)g'(k)]\\ &= Au'(f(k)-g(k))[\mu f'(k)-\left(\mu-1\right)g'(k)].\end{aligned}$$

Plugging this into (♠),

$-Au'(f(k)-y)+\beta Au'(f(y)-g(y))\left[\mu f'(y)-(\mu-1)g'(y)\right]=0$. Divide this by A, finally we get EE1.

Proof of Theorem 2

At a fixed point $g(k^*)=k^*$, $\beta[\mu f'(k^*)-(\mu-1)g'(k^*)]=1$ holds. In case of $\mu\neq 1,$ there exist an infinite number of combination of $g(k^*)$ and $g'(k^*)$. Take any arbitrary point such that $k^*\neq\bar{k}$. Then $g'(k^*)\neq 0$, and since $g(k)$ is C^1, there exists an open neighborhood around k^*, $B(k^*,\varepsilon)$, such that ε is enough small, and $g'(k)>0$ or $g'(k)<0$ for all $k\in B(k^*,\varepsilon)$. $u(\cdot)$ and $f(\cdot)$ are C^∞, then applying the implicit function theorem to EE2, $g'(k)$ is C^1 on $B(k^*,\varepsilon)$ (that is, $g(k)$ is C^2 (twice continuously differentiable)). Differentiating EE2 with k, we get

$$\begin{aligned}&-u''(f(k)-g(k))\{f'(k)-g'(k)\}\\ &+\beta\left\{\begin{matrix}u''(f(y)-g(y))\left[\mu f'(y)-(\mu-1)g'(y)\right]\{f'(y)-g'(y)\}\\ +u'(f(y)-g(y))\left[\mu f''(y)-(\mu-1)g''(y)\right]\end{matrix}\right\}g'(k)\\ &=0.\end{aligned}\tag{EE2-2}$$

Here $y=g(k)$. Then at a fixed point k^* with $c^*=f(k^*)-k^*$,

$$\begin{aligned}&-u''(c^*)\{f'(k^*)-g'(k^*)\}\\ &+\beta\left\{\begin{matrix}u''(c^*)\left[\mu f'(k^*)-(\mu-1)g'(k^*)\right]\{f'(k^*)-g'(k^*)\}\\ +u'(c^*)\left[\mu f''(k^*)-(\mu-1)g''(k^*)\right]\end{matrix}\right\}g'(k^*)\\ &=0.\end{aligned}$$

Arranging this with $\beta[\mu f'(k^*)-(\mu-1)g'(k^*)]=1$ (★), we have the following equality.

$$\left\{1-g'(k^*)\right\}u''(c^*)\{f'(k^*)-g'(k^*)\}=\beta g'(k^*)u'(c^*)\left[\mu f''(k^*)-(\mu-1)g''(k^*)\right] \quad (\blacklozenge)$$

Thus, considering $\mu \neq 1$, $u'(c^*)\neq 0$ and $g'(k^*)\neq 0$, $g''(k^*)$ is uniquely determined.

Again applying the implicit function theorem to EE2-2, $g''(k)$ is C^1 on $B(k^*,\varepsilon)$ (that is, $g(k)$ is C^3 (three times continuously differentiable). Differentiating EE2-2 with k, and setting at a fixed point k^*,

$$\begin{aligned}
&-u''(c^*)\{f''(k^*)-g''(k^*)\}-u'''(c^*)\{f'(k^*)-g'(k^*)\}^2 \qquad (\blacktriangledown)\\
&\beta\left\{\begin{array}{c} u'''(c^*)\left[\mu f'(k^*)-(\mu-1)g'(k^*)\right]\{f'(k^*)-g'(k^*)\}^2 \\ +2u''(c^*)\left[\mu f''(k^*)-(\mu-1)g''(k^*)\right]\{f'(k^*)-g'(k^*)\} \\ +u''(c^*)\left[\mu f'(k^*)-(\mu-1)g'(k^*)\right]\{f''(k^*)-g''(k^*)\} \\ +u'(c^*)\left[\mu f'''(k^*)-(\mu-1)g'''(k^*)\right] \end{array}\right\}\{g'(k^*)\}^2 \\
&+\beta\left\{\begin{array}{c} u''(c^*)\left[\mu f'(k^*)-(\mu-1)g'(k^*)\right]\{f'(k^*)-g'(k^*)\} \\ +u'(c^*)\left[\mu f''(k^*)-(\mu-1)g''(k^*)\right] \end{array}\right\}g''(k^*) \\
&=0.
\end{aligned}$$

Similarly, $g'''(k^*)$ is uniquely determined. Thus, by induction, $g(k)$ is C^n for any positive integer n, and the n'th order coefficient of derivative at k^*, $g^{(n)}(k^*)$, say, is uniquely determined at any positive integer n.

Proof of Theorem 3

(i) It is easy to verify that $k^*\neq \overline{k}$ and $k^*\in S$ imply $g'(k^*)\neq 0$ and $|g'(k^*)|<1$. Then there exists an open neighborhood around k^*, $B(k^*,\varepsilon')$, such that ε' is enough small, and $1>g'(k)>0$ or $-1<g'(k)<0$ for all $k\in B(k^*,\varepsilon')$. Then obviously,

$$g_{(s)}(B(k^*,\varepsilon'))\subset g_{(s-1)}(B(k^*,\varepsilon'))\subset\cdots\subset g(B(k^*,\varepsilon'))\subset B(k^*,\varepsilon'). \quad (\clubsuit)$$

Since $u((f-g)(k))$ is positive and upper bounded on $B(k^*,\varepsilon')$, it holds that $\left\|u((f-g)\circ g_{(s)}(k))\right\|_{B(k^*,\varepsilon')}\leq M_0$ for all $s\geq 0$, where $\|f\|_K\equiv\sup_{k\in K}|f(k)|$. In addition, $\sum_{s=0}^{\infty}\beta^s M_0$ converges, therefore, from Weierstrass's M test, an infinite functional series $B\sum_{s=0}^{\infty}\beta^s u((f-g)\circ g_{(s)}(k))$ uniformly converges to $\tilde{Z}(k)$ on $B(k^*,\varepsilon')$. Each term $\beta^s u((f-g)\circ g_{(s)}(k))$ is continuous on $B(k^*,\varepsilon')$, so is $\tilde{Z}(k)$ on $B(k^*,\varepsilon')$.

Next, differentiating $u((f-g)\circ g_{(s)}(k))$ with k,

$$(u((f-g)\circ g_{(s)}(k)))'=u'((f-g)\circ g_{(s)}(k))[(f'-g')\circ(g_{(s)}(k))][g'(g_{(s-1)}(k))]\cdot\cdot[g'(g(k))][g'(k)].$$

$u((f-g)\circ g_{(s)}(k))$ is C^1, so $(u((f-g)\circ g_{(s)}(k)))'$ is continuous on $B(k^*,\varepsilon')$. Considering ($\clubsuit$), $\left\|u'((f-g)\circ g_{(s)}(k))\right\|_{B(k^*,\varepsilon')}\leq M_1$, $\left\|(f'-g')\circ(g_{(s)}(k))\right\|_{B(k^*,\varepsilon')}\leq M_2$ and $\left\|g'(g_{(u-1)}(k))\right\|_{B(k^*,\varepsilon')}\leq 1$

$(0 \le u \le s)$. Therefore, $\|(u((f-g)\circ g_{(s)}(k)))'\|_{B(k^*,\varepsilon')} \le M_1 M_2$. Furthermore $\sum_{s=0}^{\infty} \beta^s M_1 M_2$ converges, so $B\sum_{s=0}^{\infty} \beta^s (u((f-g)\circ g_{(s)}(k)))'$ uniformly converges and is continuous on $B(k^*,\varepsilon')$.

Summarizing the above, (1) $B\sum_{s=0}^{\infty} \beta^s u((f-g)\circ g_{(s)}(k))$ converges to $\tilde{Z}(k)$, (2) $u((f-g)\circ g_{(s)}(k))$ is C^1, (3) $B\sum_{s=0}^{\infty} \beta^s (u((f-g)\circ g_{(s)}(k)))'$ uniformly converges. From (1), (2) and (3), the conditions for the term-by-term differentiability, $\tilde{Z}(k)$ is C^1, and $\tilde{Z}'(k) = B\sum_{s=0}^{\infty} \beta^s (u((f-g)\circ g_{(s)}(k)))'$. The proof of the uniform convergence and C^1 (once differentiability) of $\breve{Z}(k)$ is now straightforward.

(ii) By the proof of (i), $\tilde{Z}'(k) = B\sum_{s=0}^{\infty} \beta^s (u((f-g)\circ g_{(s)}(k)))'$. Then at a fixed point $k = k^*$,

$$\begin{aligned}\tilde{Z}'(k^*) &= B\sum_{s=0}^{\infty} [\beta^s u'(c^*)\cdot(f-g)'(k^*)\cdot\{g'(k^*)\}^s] \\ &= Bu'(c^*)\cdot(f-g)'(k^*)\cdot\sum_{s=0}^{\infty}\{g'(k^*)\}^s \\ &= Bu'(c^*)\cdot(f-g)'(k^*)\cdot\frac{1}{1-g'(k^*)},\end{aligned}$$

where $c^* = f(k^*) - k^*$. $g(k)$ is, by definition, a solution of EE2 with a fixed point $g(k^*) = k^*$ and (★). Then

$$\beta[\mu f'(k^*) - (\mu-1)g'(k^*)] = 1 \Longleftrightarrow -Au'(c^*) + \beta Bu'(c^*)\cdot(f-g)'(k^*)\cdot\frac{1}{1-g'(k^*)} = 0$$

$$\Longleftrightarrow -Au'(c^*) + \beta\tilde{Z}'(k^*) = 0 \qquad (▲).$$

Now define $g(k) \equiv \operatorname{argmax}_{y\in Y(k)}(Au(f(k)-y) + \beta\tilde{Z}(y))$. Then $g(k)$ is the only candidate solution of BE3. Since $g(k^*) \equiv \operatorname{argmax}_{y\in Y(k^*)}(Au(f(k^*)-y) + \beta\tilde{Z}(y))$, and by (▲), we have $g(k^*) = k^*$. If $g(k)$ is a solution of BE3, then by similar calculation as in Theorem 1, $g(k)$ proves to be a solution of EE2. Since $g(k)$ has a fixed point at k^*, by Theorem 2, $g(k)$ is uniquely determined at the neighborhood around k^*, therefore it must be that $g(k) = g(k)$ on $B(k^*,\varepsilon)$. It is obvious that if $g(k) = g(k)$, then $\breve{Z}(k)$, generated as $\tilde{Z}(k) = \breve{Z}(k) + (B-A)u(f(k)-g(k))$ in BE3, coincides with $\breve{Z}(k)$, as defined in SV1. Now we have proved that $\tilde{Z}(k)$ and $\breve{Z}(k)$ satisfy BE3 with a unique self-consistent policy function $g(k)$.

(iii) Next we prove a strict concavity of $\tilde{Z}(k)$ at $k = k^*$. From the proof of Theorem 1, $\tilde{Z}'(k) = \tilde{W}'(k) = Au'(f(k)-g(k))[\mu f'(k) - (\mu-1)g'(k)]$. (It is easy to verify that $\tilde{Z}'(k^*) = (A/\beta)u'(c^*) > 0$, where $c^* = f(k^*) - k^*$.) As $g(k)$ is C^n on $B(k^*,\varepsilon')$, so $\tilde{Z}(k)$ and $\breve{Z}(k)$ are also C^n on it. Again differentiating $\tilde{Z}'(k)$ with k,

$$\tilde{Z}''(k) = A\begin{bmatrix} u''(f(k)-g(k))\{f'(k)-g'(k)\}\{\mu f'(k)-(\mu-1)g'(k)\} \\ +u'(f(k)-g(k))\{\mu f''(k)-(\mu-1)g''(k)\}\end{bmatrix}.$$

At $k = k^*$, using (★),

$$\tilde{Z}''(k^*) = A\begin{bmatrix} u''(c^*)\{f'(k^*) - g'(k^*)\}\{\mu f'(k^*) - (\mu - 1)g'(k^*)\} \\ +u'(c^*)\{\mu f''(k^*) - (\mu - 1)g''(k^*)\} \end{bmatrix}$$

$$= \frac{A}{\beta}\left[u''(c^*)\{f'(k^*) - g'(k^*)\} + \beta u'(c^*)\{\mu f''(k^*) - (\mu - 1)g''(k^*)\}\right]$$

$$= \frac{A}{\beta}\left[u''(c^*)\{f'(k^*) - g'(k^*)\} + \frac{(1 - g'(k^*))}{g'(k^*)}u''(c^*)\{f'(k^*) - g'(k^*)\}\right]$$

$$= \frac{A}{\beta g'(k^*)}u''(c^*)\{f'(k^*) - g'(k^*)\}.$$

Here we used the equality (◆). From conditions U0 and U2, $u''(c^*) < 0$. The assumption $k^* \in S'$ assures $0 < g'(k^*) < 1/\beta$, which implies $f'(k^*) - g'(k^*) > 0$. Therefore now we have $\tilde{Z}''(k^*) < 0$, a desired result.

Proof of Theorem 4

(i) In EE1 replace $g(y)$ with $\tilde{g}(y,\eta)$, and $\hat{g}(k)$ with $\hat{g}(k,\eta)$, respectively, then we have:

$$-u'(f(k) - y) + \beta u'(f(y) - \tilde{g}(y,\eta))\left[\mu f'(y) - (\mu - 1)\tilde{g}_1(y,\eta)\right] = 0 \qquad \text{(EE3)}$$

where $y = \hat{g}(k,\eta)$ and $\hat{g}(k,0) = \tilde{g}(k,0) = g(k)$.

Differentiating with η,

$$\begin{aligned} & u''(f(k) - y)\hat{g}_2(k,\eta) \\ & +\beta\begin{bmatrix} -u''(f(y) - \tilde{g}(y,\eta))\left[\mu f'(y) - (\mu - 1)\tilde{g}_1(y,\eta)\right]\tilde{g}_2(y,\eta) \\ -u'(f(y) - \tilde{g}(y,\eta))\left[(\mu - 1)\tilde{g}_{12}(y,\eta)\right] \\ +\begin{Bmatrix} u''(f(y) - \tilde{g}(y,\eta))\left[\mu f'(y) - (\mu - 1)\tilde{g}_1(y,\eta)\right]\{f'(y) - \tilde{g}_1(y,\eta)\} \\ +u'(f(y) - \tilde{g}(y,\eta))\left[\mu f''(y) - (\mu - 1)\tilde{g}_{11}(y,\eta)\right] \end{Bmatrix}\hat{g}_2(k,\eta) \end{bmatrix} \\ & = 0. \end{aligned} \qquad (◎)$$

Arranging this equation:

$$\hat{g}_2(k,\eta)\begin{bmatrix} u''(f(k) - y) \\ +\beta\begin{Bmatrix} u''(f(y) - \tilde{g}(y,\eta))\left[\mu f'(y) - (\mu - 1)\tilde{g}_1(y,\eta)\right]\{f'(y) - \tilde{g}_1(y,\eta)\} \\ +u'(f(y) - \tilde{g}(y,\eta))\left[\mu f''(y) - (\mu - 1)\tilde{g}_{11}(y,\eta)\right] \end{Bmatrix} \end{bmatrix}$$

$$= \beta\begin{bmatrix} u''(f(y) - \tilde{g}(y,\eta))\left[\mu f'(y) - (\mu - 1)\tilde{g}_1(y,\eta)\right]\tilde{g}_2(y,\eta) \\ +u'(f(y) - \tilde{g}(y,\eta))\left[(\mu - 1)\tilde{g}_{12}(y,\eta)\right] \end{bmatrix}$$

Evaluating $\eta = 0$ and $k = k^*$ $\;(f(k^*) - k^* = c^*,\; k^* = g(k^*))$, with (★):

$$\hat{g}_2(k^*,0)\begin{bmatrix} u''(c^*)+u''(c^*)\left\{f'(k^*)-g'(k^*)\right\} \\ +\beta u'(c^*)\left[\mu f''(k^*)-(\mu-1)g''(k^*)\right]\end{bmatrix}$$
$$=u''(c^*)\tilde{g}_2(k^*,0)+\beta(\mu-1)u'(c^*)\tilde{g}_{12}(k^*,0)$$

From equality (◆),

$$u''(c^*)+u''(c^*)\left\{f'(k^*)-g'(k^*)\right\}+\beta u'(c^*)\left[\mu f''(k^*)-(\mu-1)g''(k^*)\right]=u''(c^*)\frac{f'(k^*)}{g'(k^*)}.$$

So we obtain

$$\hat{g}_2(k^*,0)=\frac{g'(k^*)}{f'(k^*)}\left\{\tilde{g}_2(k^*,0)+\beta(\mu-1)\frac{u'(c^*)}{u''(c^*)}\tilde{g}_{12}(k^*,0)\right\}.$$

Evaluating at $k=k^*$ and $\eta=0$, with $\tilde{g}(k,\eta)=g(k)+\eta h(k)$, $\tilde{g}_2(k^*,0)=h(k^*)$ and $\tilde{g}_{12}(k^*,0)=h'(k^*)$,

$$\frac{\hat{g}_2(k^*,0)}{\tilde{g}_2(k^*,0)}=\frac{g'(k^*)}{f'(k^*)}\left\{1+\beta(\mu-1)\frac{u'(c^*)}{u''(c^*)}\frac{h'(k^*)}{h(k^*)}\right\}.$$

$\hat{g}_2(k^*,0)$ and $\tilde{g}_2(k^*,0)$ are the slopes of changes of the current and next generation's policy functions in a small change of η, evaluated at $k=k^*$ and $\eta=0$, respectively. So this is a desired result.

(ii) The F.O.C. (Euler eq.) of BE3 is $-Au'(f(k)-y)+\beta\tilde{Z}'(y)=0$. Replacing y with $\hat{g}(k,\eta)$, and $\tilde{Z}(y)$ with $\tilde{Z}(y,\eta)$, then $-Au'(f(k)-\hat{g}(k,\eta))+\beta\tilde{Z}_1(\hat{g}(k,\eta),\eta)=0$ (♯). Differentiating with η,

$$Au''(f(k)-\hat{g}(k,\eta))\hat{g}_2(k,\eta)+\beta\left(\tilde{Z}_{11}(\hat{g}(k,\eta),\eta)\hat{g}_2(k,\eta)+\tilde{Z}_{12}(\hat{g}(k,\eta),\eta)\right)=0.$$

Then we have

$$\hat{g}_2(k,\eta)=\frac{-\beta\tilde{Z}_{12}(\hat{g}(k,\eta),\eta)}{Au''(f(k)-\hat{g}(k,\eta))+\beta\tilde{Z}_{11}(\hat{g}(k,\eta),\eta)}. \qquad (\dagger)$$

The current generation's value function $\hat{Z}(k,\eta)$ is calculated as $\hat{Z}(k,\eta)=Bu(f(k)-\hat{g}(k,\eta))+\beta\tilde{Z}(\hat{g}(k,\eta),\eta)$. Then

$$\begin{aligned}\hat{Z}_2(k,\eta)&=-Bu'(f(k)-\hat{g}(k,\eta))\hat{g}_2(k,\eta)+\beta\left(\tilde{Z}_1(\hat{g}(k,\eta),\eta)\hat{g}_2(k,\eta)+\tilde{Z}_2(\hat{g}(k,\eta),\eta)\right)\\ &=(A-B)u'(f(k)-\hat{g}(k,\eta))\hat{g}_2(k,\eta)+\beta\tilde{Z}_2(\hat{g}(k,\eta),\eta).\end{aligned} \qquad (\ddagger)$$

Here we used the equality (♯). Also $\tilde{Z}_2(\hat{g}(k,\eta),\eta)=h(\hat{g}(k,\eta))$ and $\tilde{Z}_{12}(\hat{g}(k,\eta),\eta)=h'(\hat{g}(k,\eta))$. Then plugging (†) into (‡),

$$\hat{Z}_2(k,\eta)=\beta h(\hat{g}(k,\eta))\left\{1-\frac{h'(\hat{g}(k,\eta))}{h(\hat{g}(k,\eta))}\cdot\frac{(A-B)u'(f(k)-\hat{g}(k,\eta))}{Au''(f(k)-\hat{g}(k,\eta))+\beta\tilde{Z}_{11}(\hat{g}(k,\eta),\eta)}\right\}.$$

Evaluating at $k=k^*$ and $\eta=0$, with $\hat{g}(k^*,0)=g(k^*)=k^*$, $c^*\equiv f(k^*)-k^*$ and

$$\tilde{Z}_{11}(\hat{g}(k^*,0),0)=\tilde{Z}''(k^*)=\frac{A}{\beta g'(k^*)}u''(c^*)\{f'(k^*)-g'(k^*)\},$$

$$\begin{aligned}\frac{\hat{Z}_2(k^*,0)}{\tilde{Z}_2(k^*,0)}&=\beta\left\{1-\frac{h'(k^*)}{h(k^*)}\cdot\frac{(A-B)u'(c^*)}{Au''(c^*)+\frac{A}{g'(k^*)}u''(c^*)\{f'(k^*)-g'(k^*)\}}\right\}\\&=\beta\left\{1-\frac{h'(k^*)}{h(k^*)}\cdot\frac{(1-\mu)u'(c^*)}{u''(c^*)+\frac{1}{g'(k^*)}u''(c^*)\{f'(k^*)-g'(k^*)\}}\right\}\\&=\beta\left\{1+(\mu-1)\frac{h'(k^*)}{h(k^*)}\frac{g'(k^*)}{f'(k^*)}\frac{u'(c^*)}{u''(c^*)}\right\}.\end{aligned}$$

$\hat{Z}_2(k^*,0)$ and $\tilde{Z}_2(k^*,0)(=h(k^*))$ are the slopes of changes of the current and next generation's value functions in a small change of η, evaluated at $k=k^*$ and $\eta=0$, respectively. This is also a desired result.

Proof of Theorem 5

Differentiating (◎) with k,

$$\begin{aligned}&u''(f(k)-y)\hat{g}_{12}(k,\eta)+u'''(f(k)-y)\{f'(k)-\hat{g}_1(k,\eta)\}\hat{g}_2(k,\eta)\\&+\beta\left[\begin{array}{l}\left\{\begin{array}{l}\left(\begin{array}{c}-u'''(f(y)-\tilde{g}(y,\eta))\{f'(k)-\hat{g}_1(k,\eta)\}\left[\mu f'(y)-(\mu-1)\tilde{g}_1(y,\eta)\right]\\-u''(f(y)-\tilde{g}(y,\eta))\left[\mu f''(y)-(\mu-1)\tilde{g}_{11}(y,\eta)\right]\end{array}\right)\tilde{g}_2(y,\eta)\\+\left(\begin{array}{c}-u''(f(y)-\tilde{g}(y,\eta))\left[\mu f'(y)-(\mu-1)\tilde{g}_1(y,\eta)\right]\\-(\mu-1)u''(f(y)-\tilde{g}(y,\eta))\{f'(k)-\hat{g}_1(k,\eta)\}\end{array}\right)\tilde{g}_{12}(y,\eta)\\-u'(f(y)-\tilde{g}(y,\eta))\left[(\mu-1)\tilde{g}_{112}(y,\eta)\right]\end{array}\right\}\hat{g}_1(k,\eta)\\+\left\{\begin{array}{c}u'''(f(y)-\tilde{g}(y,\eta))\left[\mu f'(y)-(\mu-1)\tilde{g}_1(y,\eta)\right]\left\{f'(y)-\tilde{g}_1(y,\eta)\right\}^2\\+2u''(f(y)-\tilde{g}(y,\eta))\left[\mu f''(y)-(\mu-1)\tilde{g}_{11}(y,\eta)\right]\left\{f'(y)-\tilde{g}_1(y,\eta)\right\}\\+u''(f(y)-\tilde{g}(y,\eta))\left[\mu f'(y)-(\mu-1)\tilde{g}_1(y,\eta)\right]\left\{f''(y)-\tilde{g}_{11}(y,\eta)\right\}\\+u'(f(y)-\tilde{g}(y,\eta))\left[\mu f'''(y)-(\mu-1)\tilde{g}_{111}(y,\eta)\right]\end{array}\right\}\hat{g}_1(k,\eta)\hat{g}_2(k,\eta)\\+\left\{\begin{array}{c}u''(f(y)-\tilde{g}(y,\eta))\left[\mu f'(y)-(\mu-1)\tilde{g}_1(y,\eta)\right]\left\{f'(y)-\tilde{g}_1(y,\eta)\right\}\\+u'(f(y)-\tilde{g}(y,\eta))\left[\mu f''(y)-(\mu-1)\tilde{g}_{11}(y,\eta)\right]\end{array}\right\}\hat{g}_{12}(k,\eta)\end{array}\right]=0.\end{aligned}$$

Evaluating $\eta=0$ and $k=k^*$ $(f(k^*)-k^*=c^*,\ k^*=g(k^*))$,

$$u''(c^*)\hat{g}_{12}(k^*,0)+u'''(c^*)\{f'(k^*)-g'(k^*)\}\hat{g}_2(k^*,0)$$
$$+\beta\left[\begin{array}{c}\left\{\begin{array}{c}\left(\begin{array}{c}-u'''(c^*)\{f'(k^*)-g'(k^*)\}\left[\mu f'(k^*)-(\mu-1)g'(k^*)\right]\\ -u''(c^*)\left[\mu f''(k^*)-(\mu-1)g''(k^*)\right]\end{array}\right)\tilde{g}_2(k^*,0)\\ +\left(\begin{array}{c}-u''(c^*)\left[\mu f'(k^*)-(\mu-1)g'(k^*)\right]\\ -(\mu-1)u''(c^*)\{f'(k^*)-g'(k^*)\}\end{array}\right)\tilde{g}_{12}(k^*,0)\\ -u'(c^*)\left[(\mu-1)\tilde{g}_{112}(k^*,0)\right]\end{array}\right\}g'(k^*)\\ +S\cdot g'(k^*)\cdot\hat{g}_2(k^*,0)\\ +T\cdot\hat{g}_{12}(k^*,0)\end{array}\right]=0\ (\bullet),$$

where

$$S=\left\{\begin{array}{c}u'''(c^*)\left[\mu f'(k^*)-(\mu-1)g'(k^*)\right]\left\{f'(k^*)-g'(k^*)\right\}^2\\ +2u''(c^*)\left[\mu f''(k^*)-(\mu-1)g''(k^*)\right]\left\{f'(k^*)-g'(k^*)\right\}\\ +u''(c^*)\left[\mu f'(k^*)-(\mu-1)g'(k^*)\right]\left\{f''(k^*)-g''(k^*)\right\}\\ +u'(c^*)\left[\mu f'''(k^*)-(\mu-1)g'''(k^*)\right]\end{array}\right\},$$

and

$$T=\left\{\begin{array}{c}u''(c^*)\left[\mu f'(k^*)-(\mu-1)g'(k^*))\right]\left\{f'(k^*)-g'(k^*)\right\}\\ +u'(c^*)\left[\mu f''(k^*)-(\mu-1)g''(k^*)\right]\end{array}\right\}.$$

From (▼), (★) and (◆), S and T are respectively calculated as

$$S=\frac{1}{\beta\{g'(k^*)\}^2}\left[u''(c^*)\left\{f''(k^*)-f'(k^*)\frac{g''(k^*)}{g'(k^*)}\right\}+u'''(c^*)\left\{f'(k^*)-g'(k^*)\right\}^2\right],$$

$$T=\frac{1}{\beta g'(k^*)}u''(c^*)\left\{f'(k^*)-g'(k^*)\right\}.$$

By Theorem 4, we already have

$$\hat{g}_2(k^*,0)=\frac{g'(k^*)}{f'(k^*)}\left\{h(k^*)+\beta(\mu-1)\frac{u'(c^*)}{u''(c^*)}h'(k^*)\right\}.$$

Obviously $\tilde{g}_2(k^*,0)=h(k^*)$, $\tilde{g}_{12}(k^*,0)=h'(k^*)$, $\tilde{g}_{112}(k^*,0)=h''(k^*)$. Then plugging all of these into (●), and arranging, again, with (★) and (◆), we derive desired results.

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Approximation Properties For Modified Kantorovich-Type Operators

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Abstract: We introduce a modification of Kantorovich-type operators in polynomial weighted spaces of functions. Then we study some approximation properties of these operators. We give some inequalities for these operators by means of the weighted modulus continuity and also obtain a Voronovskaya-type theorem. Furthermore, in our paper show that the operators give better degree of approximation of functions belonging to weighted spaces than classical Szász- Kantorovich operators.

Keywords: Kantorovich-type operators, modules of continuity, weighted spaces, Voronovskaya theorem.

1. Introduction

In 1930 Kantorovich operators $K_n := L_1[0,1] \to C[0,1]$ were introduced by the following operators:

$$K_n(f)(x) = (n+1)\sum_{k=0}^{\infty}\binom{n}{k}x^k(1-x)^{n-k}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(s)\,ds, n \in N \tag{1}$$

(see [4]) where $f \in L_1[0,1]$ and $x \in [0,1]$. Clearly, Kantorovich operators are linear and positive. Note that Kantorovich operators are extension of classical Bernstein operators in order to study the approximation in the integrable function space $L_1[0,1]$. Inspried by these operators many authors studied Kantorovich extensions of some linear positive operators, some are in [1,6,9,12] and references therein. In the last decade, these kinds of researches have been continuied.

In 1978 Becker [2] studied some approximation problems of Szász-Mirakyan operators.

$$S_n(f)(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), x \in R_0 = [0,\infty) \tag{2}$$

for $f \in C_p$, where C_p with fixed $p \in N_0 := \{0,1,2,...\}$ denotes the polynomial weighted space generated by the weight function

$$w_0(x) := 1, w_p(x) := \left(1+x^p\right)^{-1}, p \geq 1, \tag{3}$$

i.e. C_p is the set of all real-valued functions f continuous on R_0 and such that $w_p f$ is uniformly continuos and bounded on R_0. The norm in C_p is defined by the formula

$$\|f\|_p := \|f(.)\|_p := \sup_{x\in R_0} w_p(x)\left|f(x)\right|. \tag{4}$$

In [2,10], the degree of approximation of $f \in C_p$ by the operators (2) were proved. It was proved that

$$\lim_{n\to\infty} S_n(f)(x) = f(x) \tag{5}$$

for every $f \in C_p, p \in N_0$ and $x \in R_0$. Moreover, the convergence in 5 is uniform on every interval $[x_1, x_2], x_2 > x_1 \geq 0$. Then, Z.Walczak made some works. In [11] Walczak considered the space

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$C_p^1(x) := \{f \in C_p : f' \in C_p\}$ and defined the following modulus of continuity $w_1(f;C_p;t)$ for $f \in C_p$

$$w_1(f;C_p;t) := \sup_{0\le h\le t} \left\|\Delta_h f(.)\right\|_p, \forall t \in R_0 \quad (6)$$

where $\Delta_h f(x) := f(x+h) - f(x)$ for $h, x \in R_0$. Therefore

$$\lim_{t\to 0^+} w_1(f;C_p;t) = 0, f \in C_p.$$

Morever, if $f \in C_p^1$ then there exists a positive constant M such that $w_1(f;C_p;t) \le M_1 t$ for $t \in R_0$. He introduced the following operators: Let $p \in N_0, r \in N$ be fixed numbers. For $f \in C_p$,

$$A_n^*(f;r;x) := \frac{1}{g((nx+1)^2;r)} \sum_{k=0}^{\infty} \frac{(nx+1)^{2k}}{(k+r)!} f\left(\frac{k+r}{n(nx+1)}\right) \quad (7)$$

where

$$g(t;r) = \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!} \quad (8)$$

and

$$g(0;r) = \frac{1}{r!}, g(t;r) = \frac{1}{t^r}\left(e^t - \sum_{j=0}^{r-1} \frac{t^j}{j!}\right), t \in R. \quad (9)$$

Szász-Mirakyan Kantorovich operators is defined as follow;

$$T_n(f)(x) = ne^{-nx} \sum_{k=0}^{\infty} \binom{n}{k} x^k \frac{(nx)^k}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt \quad (10)$$

for $x \in R_0, p \in N_0 := \{1,2,3,...\}, f \in L_1[0,\infty)$. Some modification of the operators may be viewed in [3,4,5,8]. In this work, we consider a Kantorovich-type modification of the operators (7) and obtain the results of Walzack [11] for these operators. Also we study convergence properties of these operators for functions in C_p and C_p^1.

2. Construction of the Operators

Definition 1. Let $p \in N_0$ and $r \in N$ be fixed numbers and (a_n) be a positive sequence such that $\lim_{n\to\infty} a_n = \infty$. For functions $f \in C_p$, we introduce the operators

$$A_n(f;r;x) := \frac{a_n}{g((a_n x+1)^2;r)} \sum_{k=0}^{\infty} \frac{(a_n x+1)^{2k}}{(k+r)!} \int_{\frac{k+r}{a_n}}^{\frac{k+r+1}{a_n}} f\left(\frac{t}{a_n x+1}\right) dt, \quad (11)$$

where (8)-(9) hold.

Linearity and positivity of the operator A_n are clear. Also, we see easily that the following holds;

$$\frac{1}{g(t;r)} \le r!. \quad (12)$$

We shall prove that A_n is an operator from C_p into C_p for every fixed $p \in R_0$. In this paper, we use notation $g_{n,r}(x)$ instead of $g((a_n x+1)^2;r)$. The moments are obtained as follow:

$$A_n(1;r;x) = 1, \quad (13)$$

$$A_n(t;r;x) = \left(x + \frac{1}{a_n}\right)\left(1 + \frac{1}{2(a_n x+1)^2} + \frac{1}{(a_n x+1)^2 (r-1)! g_{n,r}(x)}\right), \quad (14)$$

$$A_n(t^2;r;x) = \left(x + \frac{1}{a_n}\right)^2 \left[1 + \frac{2}{(a_n x+1)^2} + \frac{1}{3(a_n x+1)^4} + \frac{1}{(a_n x+1)^2 (r-1)! g_{n,r}(x)}\left(1 + \frac{r+1}{(a_n x+1)^2}\right)\right], \quad (15)$$

$$A_n\left(t^3;r;x\right)=\left(x+\frac{1}{a_n}\right)^3\left[1+\frac{3}{(a_nx+1)^2}+\frac{5}{3(a_nx+1)^4}+\frac{1}{4(a_nx+1)^6}\right.$$
$$\left.+\frac{1}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\left(1+\frac{2r+7}{2(a_nx+1)^2}+\frac{2r^2+3r+2}{2(a_nx+1)^4}\right)\right] \tag{16}$$

for every fixed numbers $r,n\in N$ and $x\in R_0$.

3. Main Results

We can prove following the Lemmas by using (11)-(16).

Lemma 1. Let $n\in N$ be fixed. Then for all $x\in R_0$, we have

$$A_n\left(\left(t-x\right);r;x\right)=\frac{1}{a_n}+\frac{1}{2a_n(a_nx+1)}+\frac{1}{a_n(a_nx+1)(r-1)!g_{n,r}(x)}, \tag{17}$$

$$A_n\left(\left(t-x\right)^2;r;x\right)=\frac{2}{a_n^2}+\frac{(r-1)!g_{n,r}(x)+3(r+1)+3(a_nx+1)^2}{6a_n^2(a_nx+1)^2(r-1)!g_{n,r}(x)}$$
$$-\frac{x}{2a_n^2(a_nx+1)}+\frac{-x(a_nx+1)^3(r-1)!+a_n^2x^2(a_nx+1)^2}{a_n^2(a_nx+1)^2(r-1)!}+\frac{(a_nx+1)^4(r-1)!}{a_n^2(a_nx+1)^2}, \tag{18}$$

$$A_n\left(\left(t-x\right)^3;r;x\right)=\frac{2}{a_n^3}+\frac{2\left(a_n^2+a_n+1\right)(a_nx+1)^3\left(r-1\right)!g_{n,r}\left(x\right)+2r^2+3r+2}{2a_n^3(a_nx+1)^3\left(r-1\right)!g_{n,r}\left(x\right)}$$
$$+\frac{4(a_nx+1)+(2r+7)(a_nx+1)^2}{2a_n^3(a_nx+1)^3}+\frac{6(a_nx+1)^3+a_n^3(a_nx+1)^2}{2a_n^3(a_nx+1)^3}-\frac{6x(a_nx+1)}{2a_n^3(a_nx+1)^3}$$
$$-\frac{x}{a_n^3(a_nx+1)^2}-\frac{3x(r+1)}{a_n^3(a_nx+1)^2\left(r-1\right)!g_{n,r}\left(x\right)}+\frac{-12a_n^3\left(r-1\right)!g_{n,r}\left(x\right)-3xa_n}{2n^3\left(r-1\right)!g_{n,r}\left(x\right)} \tag{19}$$
$$+\frac{-3x(a_nx+1)+3x^2a_n^2}{a_n^3(a_nx+1)^2\left(r-1\right)!}+\frac{3x(a_nx+1)^2}{(a_nx+1)\left(r-1\right)!g_{n,r}\left(x\right)}-\frac{x}{a_n^3(a_nx+1)^2}+\frac{3x^2}{a_n(a_nx+1)}.$$

Lemma 2. Let $r,s\in N$ be fixed Then there exist positive numbers $\alpha_{s,j}$ depending only j,s,γ_j and $\beta_{s,j}\left(r\right)$ $=r^{j-1}$, depending only on j and s, $1\le j\le s$ such that

$$A_n\left(t^{s+1};r;x\right)=\left(x+\frac{1}{a_n}\right)^{s+1}\left\{\sum_{j=1}^{s+1}\frac{1}{(a_nx+1)^{2(j-1)}}\left(\alpha_{s,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{s,j}\left(r\right)}{(a_nx+1)^2\left(r-1\right)!g_{n,r}\left(x\right)}\right)\right\} \tag{20}$$

for all $n\in N$ and $x\in R_0$ where $\alpha_{s,s}=\alpha_{s,1}=\gamma_1=1,\beta_{s,1}\left(r\right)$ and $\alpha_{s,j},\gamma_j$ are constants.

Proof. We prove this lemma by using the methods and results of Lemma 2 in [11]. From (13) and (16) we see that (20) is obtained for $s=0,1,2$. Let (20) holds for $f_j\left(x\right):=x^j, 1\le j\le s$ with fixed $s\in N$. We shall prove (20) for $f_j(x)=x^{s+1}$. From (7), (11) and (12) it follows that

$$A_n\left(t^{s+1};r;x\right)=\frac{1}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}}+\frac{\sum_{i=1}^{s+1}\binom{s+2}{i}r^{s+1-i}}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}(r-1)!g_{n,r}(x)}$$

$$+\frac{1}{a_n^{s+1}(a_nx+1)}+\frac{\sum_{i=1}^{s}\binom{s+2}{i}}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}}\times\sum_{l=1}^{(s+1)-i}\binom{(s+1)-i}{l}(a_nx+1)^l a_n{}^l A_n^*\left(t^l;r;x\right).$$

Using results of Lemma 2 in [11] and taking the assumption into account, we get that

$$A_n\left(t^{s+1};r;x\right)=\frac{1}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}}+\frac{\sum_{i=1}^{s+1}\binom{s+2}{i}r^{s+1-i}}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}(r-1)!g_{n,r}(x)}$$

$$+\frac{1}{a_n^{s+1}(a_nx+1)}+\frac{\sum_{i=1}^{s}\binom{s+2}{i}}{(s+2)a_n^{s+1}(a_nx+1)^{s+1}}\sum_{l=1}^{(s+1)-i}\binom{(s+1)-i}{l}$$

$$\times(a_nx+1)^{2l}\sum_{j=1}^{l}\frac{1}{(a_nx+1)^{2(j-1)}}\left(\alpha_{l,j}+\frac{\beta_{l,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)$$

$$=\left(x+\frac{1}{a_n}\right)^{s+1}\left\{\frac{1}{(a_nx+1)^{2s}}\left(1+\frac{\sum_{i=1}^{s+1}\binom{s+2}{i}r^{(s+1)-i}}{(s+2)(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)\right.$$

$$+\frac{\sum_{i=1}^{s}\binom{s+2}{i}}{(s+2)(a_nx+1)^{2s}}+\frac{1}{(s+2)(a_nx+1)^{2(s+1)}}+\sum_{j=1}^{s}\sum_{l=2-j+1}^{s}\frac{1}{(a_nx+1)^{2(j-1)}}$$

$$\left.\times\left(\alpha_{l,j}+\frac{\beta_{l,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)\right\}.$$

Hence we have the desired result of (20).

Lemma 3. Let $p\in N_0$ and $r\in N$ be fixed. Then there exists positive constants $M_2=M_2(p,r)$ and $M_3=M_3(p,r)$ depending only on the parameters p and r such that

$$\left\| A_n\left(\frac{1}{w_p(t)};r;.\right)\right\|_p \le M_2, n\in N \tag{21}$$

and for all $f\in C_p$, we have

$$\left\| A_n\left(f;r;.\right)\right\|_p \le M_3\|f\|_p, n\in N. \tag{22}$$

Proof. For $p=0$, we get $A_n(f_0;x,y)=1$. Let $p\in N$ be fixed. From (11)-(16) we have

$$w_p(x)A_n\left(\frac{1}{w_p(t)};f;r;x\right)=w_p(x)\left\{1+A_n\left(t^p;f;r;x\right)\right\}$$

$$=\frac{1}{1+x^p}+\frac{\left(x+\frac{1}{a_n}\right)^p}{1+x^p}\sum_{j=1}^{s}\frac{1}{(a_nx+1)^{2(j-1)}}\left(\alpha_{s,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{s,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)$$

$$\le 1+p\left(\alpha_{s,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{s,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)\le M_2(p;r).$$

Thefore, we obtain

$$\left|w_p(x)A_n\left(\frac{1}{w_p(t)};f;r;x\right)\right|\le\frac{a_n}{w_p(x)g_{n,r}(x)}\sum_{k=0}^{m}\frac{(a_nx+1)^{2k}}{(k+r)!}\int_{\frac{k+r}{n}}^{\frac{k+r+1}{n}}w_p(x)\left|f\left(\frac{t}{(a_nx+1)}\right)\right|dt,$$

which gives the desired result. For (22) we have the following inequalities;

$$A_n\left(\frac{w_p(t)}{w_p(t)}f;r;x\right) = A_n\left(\sup\left(fw_p(t)\right)\frac{1}{w_p(t)};r;x\right)\le\|f\|_pA_n\left(\frac{1}{w_p(t)};r;x\right).$$

Therefore, using (21), we get (22).

Lemma 4. Let $p\in N_0$ and $r\in N$ be fixed. Then there exists a positive constant $M_4=M_4(p,r)$ depending only on the parameters p and r such that

$$\left\| A_n\left(\frac{(t-.)^2}{w_p(t)};r;.\right)\right\|_p \le \frac{M_4}{a_n^2}, n\in N. \tag{23}$$

Proof. For $p=0$ the formulas given in Lemmas 1-3 and (11) imply (23). By (3) and (13)-(16) we have

$A_n\left((t-x)^2/w_p(t);r;x\right)=$

$A_n\left((t-x)^2;r;x\right)+A_n\left(t(t-x)^2;r;x\right)=$

$A_n\left((t-x)^3;r;x\right)+(1+x)A_n\left((t-x)^2;r;x\right)p,$

$n\in N.$

If $p=1$ then

$A_n\left((t-x)^2/w_1(t);r;x\right)=$

$A_n\left((t-x)^2;r;x\right)+A_n\left(t(t-x)^2;r;x\right)=$

$A_n\left((t-x)^3;r;x\right)+(1+x)A_n\left((t-x)^2;r;x\right)$

which by (4) and (12) yield (23) for $p,n\in N$.

Let $p\ge 2$. Applying Lemma 2, we get

$$w_p(x)A_n\left(t^p\left((t-x)^2\right);r;x\right)=w_p(x)\left\{\left(x+\frac{1}{a_n}\right)^{p+2}\sum_{j=1}^{p+2}\frac{1}{(a_nx+1)^{2(j-1)}}\right.$$

$$\times\left(\alpha_{p+2,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{p+2,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)-2x\left(x+\frac{1}{a_n}\right)^{p+1}\sum_{j=1}^{p+1}\frac{1}{(a_nx+1)^{2(j-1)}}$$

$$\times\left(\alpha_{p+1,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{p+1,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)+x^2\left(x+\frac{1}{a_n}\right)^{p}$$

$$\left.\times\sum_{j=1}^{p}\frac{1}{(a_nx+1)^{2(j-1)}}\left(\alpha_{p,j}+\frac{\gamma_j}{(a_nx+1)^2}+\frac{\beta_{p,j}(r)}{(a_nx+1)^2(r-1)!g_{n,r}(x)}\right)\right\}$$

$$=w_p(x)\left\{\left(x+\frac{1}{n}\right)^{p}\frac{(a_nx+1)^2}{n^2}\left[1+\frac{1}{(nx+1)^2}+\frac{1}{(nx+1)^2(r-1)!g_{n,r}(x)}\right]\right.$$

$$+\sum_{j=2}^{p+2}\frac{1}{(nx+1)^{2(j-1)}}\left(\alpha_{p+2,j}+\frac{\gamma_j}{(nx+1)^2}+\frac{\beta_{p+2,j}(r)}{(nx+1)^2(r-1)!g_{n,r}(x)}\right)$$

$$-2x\left(x+\frac{1}{n}\right)^{p}\left[\frac{(nx+1)}{n}\left(1+\frac{1}{(nx+1)^2}+\frac{1}{(nx+1)^2(r-1)!g_{n,r}(x)}\right)\right]$$

$$+x^2\left(x+\frac{1}{n}\right)^{p}\left(1+\frac{1}{(nx+1)^2}+\frac{1}{(nx+1)^2(r-1)!g_{n,r}(x)}\right)$$

$$\left.+\sum_{j=2}^{p}\frac{1}{(nx+1)^{2(j-1)}}\left(\alpha_{p,j}+\frac{\gamma_j}{(nx+1)^2}+\frac{\beta_{p,j}(r)}{(nx+1)^2(r-1)!g_{n,r}(x)}\right)\right\},$$

which by (11) and (4) imply

$$w_p(x)A_n\left(t^p\left((t-x)^2\right);r;x\right)$$

$$\leq w_p(x)\left(x+\frac{1}{a_n}\right)^{p}\frac{1}{a_n^2\left(1+x^p\right)}\left\{2+r+\sum_{j=2}^{p+2}\frac{1}{(a_nx+1)^{2(j-1)}}\right.$$

$$\times\left(\alpha_{p+2,j}+\gamma_{j+}r\beta_{p+2,j}(r)\right)-2x\sum_{j=2}^{p+1}\frac{1}{(a_nx+1)^{2(j-1)}}\left(\alpha_{p+1,j}+\gamma_j+r\beta_{p+1,j}(r)\right)$$

$$\left.+x^2\sum_{j=2}^{p}\frac{1}{(a_nx+1)^{2(j-1)}}\alpha_{p,j}+\gamma_j+r\beta_{p,j}(r)\right\}$$

$$\leq\frac{M_4}{a_n^2}$$

for $x\in R_0$ and $n,r\in N$.

4. Approximation Behaviour of A_n

In this section, we will investigate the approximation behaviour of A_n.

Theorem 1. Let $p \in N_0$ and $r \in N$ be fixed numbers. Then there exists a positive constant $M_5 = M_5(p,r)$ depending only on the parameters p and r such that for every $f \in C_p^1$ and $r \in R_0$ we have

$$\left\| A_n(f;r;.) - f(.) \right\|_p \le \frac{M_5}{a_n} \|f'\|_p, \; n \in N. \quad (24)$$

Proof. Let $x \in R_0$ be a fixed point. Then for $f \in C_p^1$ and $t \in R_0, t \ge x$ we have

$$f(t) - f(x) = \int_x^t f'(u)\,du. \quad (25)$$

By linearity of A_n, (24) and (5) we have

$$\begin{aligned} &A_n\left(f(t);r;x) - f(x)\right) = \\ &A_n\left(\int_x^t f'(u)\,du;r;x\right), \; n \in N. \end{aligned} \quad (26)$$

From (3) and (4) we obtain that

$$\begin{aligned} &\left|\int_x^t f'(u)\,du\right| \le \|f'\|_p \left[\frac{1}{w_p(x)} + \frac{1}{w_p(t)}\right] |t-x|, \\ &t, x \in R_0. \end{aligned} \quad (27)$$

Then, we get

$$\begin{aligned} &w_p(x)\left|A_n\left(f(t);r;x\right) - f(x)\right| = \\ &w_p(x)\left|A_n \int_x^t f'(u)\,du\right| \\ &\le \|f'\|_p \left\{ A_n\left(|t-x|;r;x\right) + A_n\left(\frac{|t-x|}{w_p(x)};r;x\right)\right\} \end{aligned} \quad (28)$$

for $n \in N$. By the Hölders inequality and by Lemmas 1-4 and by (12) it follows that

$$\begin{aligned} &A_n\left(|t-x|;r;x\right) \le \\ &\left\{A_n\left((t-x)^2;r;x\right)\right\}^{\frac{1}{2}} \left\{A_n\left(\frac{1}{w_p(t)};r;x\right)\right\}^{\frac{1}{2}} \\ &\le \frac{M_6(p,r)}{a_n} \\ &w_p(x) A_n\left(\frac{|t-x|}{w_p(x)};r;x\right) \le \\ &w_p(x)\left\{A_n\left(\frac{(t-x)^2}{w_p(t)};r;x\right)\right\}^{\frac{1}{2}} \left\{A_n\left(\frac{1}{w_p(t)};r;x\right)\right\}^{\frac{1}{2}} \\ &\le \frac{M_7(p,r)}{a_n}, \; n \in N. \end{aligned} \quad (29)$$

Hence and by (28) and (29) we obtain (24).

5. Rates of Convergence

In this section, we compute the rate of convergence of $A_n(f;r;.)$ to $f(.)$ by means of the weighted modulus of continuity given by (6).

Theorem 2. Let $p \in N_0$ and $r \in N$ be fixed numbers. Then there exists a positive constant $M_8 = M_8(p,r)$ depending only on the parameters p and r such that for every $f \in C_p^1$ and $n \in N$ we have

$$\left\| A_n(f;r;.) - f(.) \right\|_p \le M_8 w_1\left(f;C_p;\frac{1}{a_n}\right), \; n \in N. \quad (30)$$

Proof. Let $f \in C_p$. We use the Steklov function

$$f_h(x) := \frac{1}{h}\int_x^t f(x+u)\,du, \; x \in R_0, h > 0. \quad (31)$$

From (31) we can write

$$f_h{}'(x) = \frac{1}{h}\Delta_t f(x), \; x \in R_0, h > 0 \quad (32)$$

which imply

$$\|f_h - f\|_p \le w_1\left(f;C_p;h\right), \qquad (33)$$

$$\|f_h'\|_p \le h^{-1} w_1\left(f;C_p;h\right) \qquad (34)$$

for $h>0.$ From this we deduce that $f_h \in C_p^1$ if $f \in C_p$ and $h>0.$ Hence for (32) we can write

$$w_p\left(x\right)\left(A_n(f;r.x)-f(x)\right) \le$$
$$w_p\left(x\right)\left\{\left|A_n\left(f-f_h;x\right)\right|+\left|A_n\left(f_h;x\right)-f_h\left(x\right)\right|\right.$$
$$\left.+\left|f_h\left(x\right)-f\left(x\right)\right|\right\}$$
$$:= L_1\left(x\right)+L_2\left(x\right)+L_3\left(x\right)$$

for $n\in N, h>0$ and $x\in R_0$. For $L_1\left(x\right),$ by using Lemma 3 and (33), we get

$$\|L_1\|_p \le M_1 \|f-f_h\| \le$$
$$M_1 w_1\left(f;C_p;h\right), \|L_3\|_p \le w_1\left(f;C_p;h\right).$$

Similarly, by using Theorem 1 and (34) it follows that

$$\|L_2\|_p \le \frac{M_2}{a_n}\|f_h'\|_p \le \frac{M_2}{a_n h} w_1\left(f;C_p;h\right) h>0, n\in N.$$

Consequently,

$$\left\|A_n\left(f;r;.\right)-f\left(.\right)\right\|_p \le$$
$$\left(1+M_1+\frac{1}{a_n h}M_2\right) w_1\left(f;C_p;h\right).$$

Now, for fixed $n\in N,$ setting $h=\dfrac{1}{a_n}$ in the last equation we obtain

$$\left\|A_n\left(f;r;.\right)-f\left(.\right)\right\|_p \le M_8\left(p,r\right) w_1\left(f;C_p;\frac{1}{a_n}\right).$$

From Theorems 1 and 2 we will give the followings corollaries:

Corollary 1. For every fixed $r\in N$ and $f\in C_p, p\in N_0$ we have

$$\lim_{n\to\infty}\left\|A_n\left(f;r;.\right)-f\left(.\right)\right\|_p = 0. \qquad (35)$$

Corollary 2. For every fixed $r\in N$ and $f\in C_p^1, p\in N_0$ then

$$\left\|A_n\left(f;r;.\right)-f\left(.\right)\right\|_p = o\left(\frac{1}{a_n}\right) \qquad (36)$$

as $n\to\infty$.

Theorem 1 and Corollaries in our paper show that the operator A_n , $n\in N$ give better degree of approximation of functions $f\in C_p, f\in C_p^1$ than classical Szász-Kantorovich operators. Because degree of our operators convergence is $\dfrac{1}{a_n}$ but classical Szász- Kantorovich operators's degree of convergence is $\dfrac{1}{n}$.

Theorem 3. Let $r\in N$ be fixed number. Then for all $f\in C_p^1$ and $r\in N$ we have

$$\lim_{n\to\infty} a_n\left\{A_n\left(f;r;x\right)-f\left(x\right)\right\} = f'\left(x\right) \qquad (37)$$

for every $x>0.$

Proof. Let $x>0$ be a fixed point. Then by Taylor Formula we get

$$f\left(t\right)=f\left(x\right)+f'\left(x\right)\left(t-x\right)+\varepsilon\left(t;x\right)\left(t-x\right)$$

for $t\in R_0$ where $\varepsilon\left(t\right)\equiv\varepsilon\left(t;x\right)$ is a function belonging to C_p and $\varepsilon\left(x\right)=0.$ Hence by (11) and (13)-(16) we have

$$A_n\left(f;r;x\right)=f\left(x\right)+$$
$$f'\left(x\right)A_n\left(\left(t-x\right);r;x\right)+A_n\left(\varepsilon\left(t\right)\left(t-x\right);r;x\right). \qquad (38)$$

By the Hölders inequality and (38) we have

$$A_n\left(\varepsilon\left(t;x\right)\left(t-x\right);r;x\right) \le$$
$$\left\{A_n\left(\varepsilon^2\left(t;x\right);r;x\right)\right\}^{\frac{1}{2}}\left\{A_n\left(\left(t-x\right)^2;r;x\right)\right\}^{\frac{1}{2}}.$$

By Corollary 1 we deduce that

$$\lim_{n\to\infty} A_n\left(\varepsilon^2\left(t\right);r;x\right)=\varepsilon^2\left(x\right)=0.$$

From above equation and Lemma 1 we get

$$\lim_{n\to\infty} a_n A_n\left(\varepsilon(t)(t-x);r;x\right)=0.$$

Theorem 2 show that rate of our operators for pointwise convergence is more fast than classical Szász-Kantorovich operators. Because our operators rate of convergence is $\frac{1}{a_n}$ but classical Szász-Kantorovich operators rate of convergence is $\frac{1}{n}$ pointwisely.

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On Maximal, Discrete, and Area Operators

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Abstract: In this paper, we study the boundednesses of maximal operator g_*, the discrete operator g_d, and the area operator A on Bergman spaces.

Key words: Area operator, Bergman Space, Discrete Maximal operator, Hardy Space, maximal operator

1. Introduction

Let $\mathbb{D}=\{z:|z|<1\}$ denote the unit disk in the complex plane and $\mathbb{T}=\partial\mathbb{D}=\{z:|z|=1\}$ its boundary, the unit circle. Littlewood-Paley operators have been well known, studied by a lot of people, and used to characterize function spaces, such as Hardy spaces, etc. In this paper, we mainly consider the characterizations of classical Bergman spaces by these operators. Here are two Littlewood-Paley operators we have studied:

$$g_*(f)(z)=\left(\int_0^1(1-r)\sup_{\rho<r}|f'(\rho z)|^2\,dr\right)^{1/2}$$

$$g_d(f)(z)=\left(\sum_{n=0}^{\infty}2^{-2n}|f'(r_n z)|^2\,dr\right)^{1/2}$$

where $r_n=1-2^{-n}$ and $z\in\mathbb{D}\cup\mathbb{T}$.

A Luzin area function is defined to be

$$A(f)(z)=\int_{D_z}|f'(w)|^2\,dm(w)$$

where

$D_z=$ the convex hull of $\{|w|<\frac{1}{2}|z|\}\cup\{z\}$.

For $0<p<\infty$, $H^p(\mathbb{D})$ is the usual Hardy space of functions analytic on the unit disk and the norm of f in H^p is as follows

$$\|f\|_{H^p}=\sup_{0\le r<1}\left(\int_0^{2\pi}|f(re^{i\theta}|^p\frac{d\theta}{2\pi}\right)^{1/p}<\infty.$$

It is standard that for $f\in H^p$, its nontangential limit exists almost everywhere on $\mathbb{T}$,

$$\lim_{r\to 1^-}f(re^{i\theta})=f(e^{i\theta})$$

and

$$\|f\|_{H^p}=\|f(e^{i\theta})\|_{L^p(\mathbb{T})}.$$

we refer readers to [11] for Hardy space theory.

For $0<p<\infty$, the Bergman space $A^p(\mathbb{D})$ is defined as the set of all functions f analytic in $\mathbb{D}$ with the following norm

$$\|f\|_{A^p(\mathbb{D})}=\left(\int_D|f(z)|^p\,dm(z)\right)^{1/p}<\infty$$

where $z=x+iy$ and $dm(z)=\frac{dxdy}{\pi}$ in $\mathbb{D}$, the normalized area measure on $\mathbb{D}$. It is well known that $A^p(\mathbb{D})$ is a Bnach space when $1\le p<\infty$ and when $0<p<1$, the space $A^p(\mathbb{D})$ is a quasi-Banach space with p-norm $\|f\|^p_{A^p(\mathbb{D})}$.

The main results of this note are to prove that for f is analytic, f in the Bergman space $\|f\|_{A^p(\mathbb{D})}$ if and only if $g_*(f)$ or $g_d(f)$ is in $L^p(\mathbb{D})$ and the similar results also obtained for the area operator A.

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Throughout this paper, for two nonnegative expressions P and Q, $P \lesssim Q$ means that there exists a positive constant C, not depending on the main factors, such that $P \le CQ$, and $P \approx Q$ means that $P \lesssim Q$ and $Q \lesssim P$.

2. Main Results and Their Proofs

For our convenience and reference, all main results and useful results are stated in this section and proofs are also given with some remarks.

When we prove Theorem 4 and 5 we need the following results on Hardy spaces due to Miroslav Pavlović, see Theorem 1 in [10].

Lemma 1. Let $0 < p < \infty$. For an analytic function, the following conditions are mutually equivalent:

(a) $f \in H^p$;

(b) $g_*(f) \in L^p(\mathbb{T})$;

(c) $g_d(f) \in L^p(\mathbb{T})$.

Furthermore, there are constants C_1, C_2, C_3 independent of f such that

$$\|f\|_{H^p} \le C_1 \|g_*(f)\|_{L^p(\mathbb{D})} \le$$

$$C_2 \|g_d(f)\|_{L^p(\mathbb{D})} \le C_3 \|f\|_{H^p}.$$

In order to prove the second inequality of Theorem 4, we need a Theorem in [11] with $\alpha = 0$ (Theorem 2.30 on page 69). We state it as a lemma here.

Lemma 2. Suppose $0 < p \le 1$. Then there exists a sequence $\{a_k\}$ in $\mathbb{D}$ such that $A^p(\mathbb{D})$ consists of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{1-|a_k|^2}{(1-\overline{a}_k z)^{\frac{2}{p}+1}}, \qquad z \in \mathbb{D}$$

where $\{c_k\}$ belongs to the sequence space l^p and the series converges in the norm topology of $A^p(\mathbb{D})$.

Moreover,

$$\|f\|^p_{A^p(D)} \le \sum_{k=1}^{\infty} |c_k|^p \|f_k\|^p_{A^p(\mathbb{D})},$$

where

$$f_k(z) = \frac{1-|a_k|^2}{(1-\overline{a}_k z)^{\frac{2}{p}+1}}$$

and there is a constant C such that

$$C^{-1}\left(\sum_{k=1}^{\infty} c_k\right)^{1/p} \le \|f\|_{A^p(\mathbb{D})} \le C\left(\sum_{k=1}^{\infty} c_k\right)^{1/p}.$$

A relationship between Littlewood-Paley operators is listed here and we will need it in the proof of Theorem 5.

Lemma 3. For any $f \in A^p(\mathbb{D})$, $g_d(f) \lesssim g_*(f)$.

$$\begin{aligned} g_*(f)^2(z) &= \int_0^1 (1-s) \sup_{\rho<s} |f'(\rho z)|^2 \, ds \\ &= \sum_{n=0}^{\infty} \int_{r_n}^{r_{n+1}} (1-s) \sup_{\rho<s} |f'(\rho z)|^2 \, ds \\ &\ge \sum_{n=0}^{\infty} \sup_{\rho<r_n} |f'(\rho z)|^2 \int_{r_n}^{r_{n+1}} (1-s) ds \\ &\ge \frac{3}{8} g_d(f)^2(z). \end{aligned}$$

Theorem 4. For $0 < p < \infty$ and $f \in A^p(\mathbb{D})$ with $f(0) = 0$, the following are true

$$C_p' \|f\|_{A^p(\mathbb{D})} \le \|g_*(f)\|_{L^p(\mathbb{D})} \le C_p'' \|f\|_{A^p(\mathbb{D})}$$

where C_p' and C_p'' are constants depending only on p.

Moreover

$$\|f\|_{A^p(\mathbb{D})} \approx |f(0)| + \|g_*(f)\|_{L^p(\mathbb{D})}, \qquad 1 \le p < \infty$$

and

$$\|f\|^p_{A^p(\mathbb{D})} \approx |f(0)|^p + \|g_*(f)\|^p_{L^p(\mathbb{D})}, \qquad 0 < p < 1$$

The basic idea of the proof of this theorem is due to Chen and Ouyang in [5] and based on some original results by Littlewood-Paley, Zygmund, and Lemma 1.

Proof of Theorem 4 For $0 < p < \infty$, let $f_r(z) = f(rz)$ for $0 < r < 1$ and $z \in \mathbb{D}$, $z = re^{i\theta}$, $0 \le \theta \le 2\pi$, we have

$$\int_D |f(z)|^p\, dm(z) = \int_0^{2\pi}\int_0^1 r\,|f(re^{i\theta})|^p\, \frac{drd\theta}{\pi}$$

$$= 2\int_0^1 r\int_0^{2\pi} |f_r(e^{i\theta})|^p\, \frac{drd\theta}{2\pi}$$

By (1), it follows that

$$\int_D |f(z)|^p\, dm(z)$$

$$\approx 2\int_0^1 r\int_0^{2\pi} |g_*(f_r)(e^{i\theta})|^p\, \frac{drd\theta}{2\pi}$$

$$\approx \int_0^1 \|g_*(f_r)\|^p\, rdr$$

$$= \int_0^1\int_0^{2\pi}\left[\int_0^1 (1-s)\sup_{\rho<s}|f_r(\rho e^{i\theta})|^2\, ds\right]^{p/2}\frac{d\theta}{2\pi} rdr$$

$$= \int_0^1\int_0^{2\pi}\left[\int_0^1 (1-s)\sup_{\rho<s}|f'(r\rho e^{i\theta})|^2\, r^2 ds\right]^{p/2}\frac{d\theta}{2\pi} rdr$$

$$\lesssim \|g_*(f)\|^p_{L^p(\mathbb{D})}.$$

The first inequality of Theorem 4 is proved.

For $0 < p \le 1$, by Lemma 2, we get

$$f(z) = \sum_{k=1}^{\infty} c_k f_k(z).$$

Standard calculations give us

$$|f_k(\rho z)|^2 =$$

$$(1-|a_k|^2)\left(\frac{2}{p}+1\right)^2 |\bar{a}_k|^2 \frac{1}{|1-\bar{a}_k\rho z|^{2(\frac{2}{p}+2)}}$$

At this point we are in need of a very useful fact which is easy to check. For $0 < t \le 1$, $z \in \mathbb{D}$, we have

$$|1-tz| \le (1-t) + |1-z| \le 3|1-tz|. \qquad (1)$$

Now we have

$$g_*(f_k)(z)$$

$$\lesssim (1-|a_k|^2)\left(\int_0^1 \sup_{\rho<r}\frac{1-r}{[(1-\rho)+|1-\bar{a}_k z|]^{2(\frac{2}{p}+2)}}dr\right)^{1/2}$$

$$\lesssim (1-|a_k|^2)\left(\int_0^1 \sup_{\rho<r}\frac{1}{[(1-\rho)+|1-\bar{a}_k z|]^{2(\frac{2}{p}+1)+1}}dr\right)^{1/2}$$

$$= (1-|a_k|^2)\left(\int_0^1 \frac{1}{[(1-r)+|1-\bar{a}_k z|]^{2(\frac{2}{p}+1)+1}}dr\right)^{1/2}$$

$$\lesssim (1-|a_k|^2)\frac{1}{|1-\bar{a}_k z|^{\frac{2}{p}+1}}.$$

Therefore, for $f = \sum_k c_k f_k$ with $\sum_k |c_k|^p < \infty$, we have

$$\int_D |g_*(f)(z)|^p\, dm(z)$$

$$\le \sum_{k=1}^{\infty}|c_k|^p \int_D |g_*(f_k)(z)|^p\, dm(z)$$

$$\lesssim \sum_{k=1}^{\infty}|c_k|^p\,(1-|a_k|^2)^p\int_D \frac{1}{|1-\bar{a}_k z|^{2+p}}dm(z)$$

$$\lesssim \sum_{k=1}^{\infty}|c_k|^p.$$

Note that

$$\int_D \frac{1}{|1-\bar{a}_k z|^{2+p}}dm(z) \approx \frac{1}{(1-|a_k|^2)^p}.$$

Next for the case of $1 < p < \infty$, if $f \in A^p(\mathbb{D})$, then f has the integral representation [11] (Theorem 2.2 on page 40),

$$f(z) = \int_D \frac{f(w)}{(1-\bar{w}z)^2}dm(w), \qquad \forall z \in \mathbb{D}.$$

So

$$|f'(\rho z)| \le (1-|\rho z|)^{-1/2}\int_D \frac{|f(w)|}{|1-\rho\bar{w}z|^{5/2}}dm(w).$$

Hence we have

$$g_*(f)^2(z) \lesssim$$

$$\int_0^1 \sup_{\rho<r}\frac{1-r}{(1-|\rho z|)}\left[\int_D \frac{|f(w)|}{|1-\rho\bar{w}z|^{5/2}}dm(w)\right]^2 dr.$$

By (1), we have the following estimate

$$3|1-\rho\overline{w}z| \geq \left(1-\frac{\rho}{r}\right)+|1-r\overline{w}z|.$$

Therefore, we obtain

$$g_*(f)^2(z) \lesssim \int_0^1 \left[\int_D \frac{|f(w)|}{|1-r\overline{w}z|^{5/2}} dm(w)\right]^2 dr.$$

By Minkowski's integral inequality, we have

$$g_*(f)^2(z)$$

$$\lesssim \left[\int_D |f(w)| \left(\int_0^1 \frac{1}{|1-r\overline{w}z|^5}\right)^{1/2} dm(w)\right]^2$$

$$\lesssim \left[\int_D |f(w)| \left(\int_0^1 \frac{1}{[(1-r)+|1-\overline{w}z|]^5}\right)^{1/2} dm(w)\right]^2$$

$$\lesssim \left[\int_D \frac{|f(w)|}{|1-\overline{w}z|^2} dm(w)\right]^2.$$

But,

$$f \mapsto \int_D \frac{|f(w)|}{|1-\overline{w}z|^2} dm(w)$$

is bounded on $L^p(\mathbb{D})$ for $1<p<\infty$. The proof of Theorem 4 is completed.

The next theorem is regarding the characterization of analytic function f in Bergman space $A^p(\mathbb{D})$ by the discrete Littlewood-Paley operator $g_d(f)$.

Theorem 5. For $0<p<\infty$ and $f \in A^p(\mathbb{D})$ with $f(0)=0$, there exist constants C_p' and C_p'' depending only on p such that

$$C_p' \|f\|_{A^p(\mathbb{D})} \leq \|g_d(f)\|_{L^p(\mathbb{D})} \leq C_p'' \|f\|_{A^p(\mathbb{D})}.$$

Moreover

$$\|f\|_{A^p(\mathbb{D})} \approx |f(0)| + \|g_d(f)\|_{L^p(\mathbb{D})}, \qquad 1 \leq p < \infty$$

and

$$\|f\|^p_{A^p(\mathbb{D})} \approx |f(0)|^p + \|g_d(f)\|^p_{L^p(\mathbb{D})}, \qquad 0<p<1$$

The proof of the first half of the inequality of Theorem 5 is similar to that of Theorem 4 but we have to be careful with the change from continuous case to discrete case.

Proof of Theorem 5 For $0<p<\infty$ and $f_r(z)=f(rz)$ $0<r<1$ and $z \in \mathbb{D}$, $z=re^{i\theta}$, by (1) and standard calculations, we have

$$\int_D |f(z)|^p \, dm(z) = \int_0^{2\pi}\int_0^1 r\,|f(re^{i\theta})|^p \frac{drd\theta}{\pi}$$

$$= 2\int_0^1 r \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{drd\theta}{2\pi}$$

$$\approx \int_0^1 \|g_d(f_r)\|^p \, rdr$$

$$\lesssim \|g_d(f)\|^p_{L^p(\mathbb{D})}.$$

The last inequality holds because

$$g_d(f_r)(e^{i\theta}) \leq \left(\sum_{n=0}^{\infty} 2^{-2n}\left|f'(rr_n e^{i\theta})\right|^2\right)^{1/2} = g_d(f)(z).$$

The second half of Theorem 5 follows immediately from Theorem 4 and Lemma 3. This completes the proof of Theorem 5.

In the next part, we are going to consider the classical area integral operator, also called the Luzin area operator on Bergman space. To state our results we need the following due to Calderón. In fact it is a special case of Calderón's theorem. See Theorem 1.3 in [10].

Lemma 6. Let $0<p<\infty$. Then $f \in H^p(\mathbb{D})$ if and only if $A(f) \in L^p(\mathbb{D})$.

Theorem 7. For $0<p<\infty$, $f \in A^p(\mathbb{D})$ with $f(0)=0$, there exists a constant C such that

$$\|f\|_{A^p(\mathbb{D})} \leq C\|A(f)\|_{L^p(\mathbb{D})}.$$

Proof For $0<p<\infty$, let $f_r(z)=f(rz)$, $0<r<1$. By Lemma 6, we have

$$\begin{aligned}&\int_D |f(z)|^p\, dm(z)\\&=2\int_0^1\left[\int_0^{2\pi}|f(re^{i\theta})|^p\frac{d\theta}{2\pi}\right]rdr\\&\approx 2\int_0^1\left[\int_0^{2\pi}|A(f_r)(e^{i\theta})|^p\frac{d\theta}{2\pi}\right]rdr\\&\le C\|A(f)\|_{L^p(\mathbb{D})}.\end{aligned}$$

Here in the last step we used the fact that

$$A(f_r)(z)\le A(f)(z).$$

The converse of Theorem 7 is not as good as those of Theorem 4 and 5 when $1\le p<\infty$.

Theorem 8. For $0<p<1$, $f\in A^p(\mathbb{D})$ with $f(0)=0$, there exists a constant C such that

$$\|A(f)\|_{A^p(\mathbb{D})}\le C\|f\|_{L^p(\mathbb{D})}.$$

By Lemma 2, for $0<p<1$,

$$f(z)=\sum_{k=1}^{\infty}c_k f_k(z),$$

where

$$f_k(z)=\frac{1-|a_k|^2}{(1-\bar a_k z)^{\frac{2}{p}+1}}.$$

So we have

$$\begin{aligned}&A(f_k)(z)\\&\lesssim(1-|a_k|^2)\left[\int_{D_z}\frac{1}{|1-\bar a_k w|^{2(\frac{2}{p}+2)}}dm(w)\right]^{1/2}\end{aligned}$$

and

$$\begin{aligned}&\int_D |A(f)(z)|^p\, dm(z)\le\\&\sum_{k=1}^{\infty}|c_k|^p(1-|a_k|^2)^p\left[\int_{D_z}\frac{1}{|1-\bar a_k w|^{2(\frac{2}{p}+2)}}dm(w)\right]^{p/2}.\end{aligned}$$

To get through our proof, we need to estimate the inner integral in the last line by variable changes.

$$\begin{aligned}&\int_{D_z}\frac{1}{|1-\bar a_k w|^{2(\frac{2}{p}+2)}}dm(w)\\&\le\int_{|w|\le|z|}\frac{1}{|1-\bar a_k w|^{2(\frac{2}{p}+2)}}dm(w)\\&=\int_D\frac{1}{|1-\bar a_k|z|w|^{2(\frac{2}{p}+2)}}\frac{1}{|z|^2}dm(w)\\&\approx\frac{1}{|z|^2}\frac{1}{(1-|a_k z|^2)^{2(\frac{2}{p}+1)}}.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\int_D |A(f)(z)|^p\, dm(z)\\&\le\int_D\frac{1}{|z|^p}\frac{1}{(1-|a_k z|^2)^{2+p}}dm(z)\\&\le\frac{1}{(1-|a_k|^2)^2}\int_D\frac{1}{|z|^p}\frac{1}{(1-|a_k z|^2)^p}dm(z)\\&\le\frac{1}{(1-|a_k|^2)^2}\int_D\frac{1}{|z|^p}\frac{1}{(1-|z|^2)^p}dm(z)\\&\lesssim\frac{1}{(1-|a_k|^2)^2}\int_0^{2\pi}\int_0^1 r^{1-p}(1-r)^{-p}drd\theta\\&\lesssim\frac{1}{(1-|a_k|^2)^2}B(2-p,1-p).\end{aligned}$$

where $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the Beta function. Hence

$$\int_D |A(f)(z)|^p\, dm(z)\lesssim\sum_k |c_k|^p.$$

We are done with the proof of Theorem 8.

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Volterra Integral Equations and Some Nonlinear Integral Equations with Variable Limit of Integration as Generalized Moment Problems

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Abstract: In this paper we will see that, under certain conditions, the techniques of generalized moment problem will apply to numerically solve an Volterra integral equation of first kind or second kind. Volterra integral equation is transformed into a one-dimensional generalized moment problem, and shall apply the moment problem techniques to find a numerical approximation of the solution. Specifically you will see that solving the Volterra integral equation of first kind $f(t) = \int_a^t K(t,s)x(s)ds \;\; a \le t \le b$ or solve the Volterra integral equation of the second kind $x(t) = f(t) + \int_a^t K(t,s)x(s)ds \;\; a \le t \le b$ is equivalent to solving a generalized moment problem of the form $\mu_n = \int_a^b g_n(s)x(s)ds \;\; n = 0,1,2,\ldots$. This shall apply for to find the solution of an integrodifferential equation of the form $x'(t) = f(t) + \int_a^t K(t,s)x(s)ds$ for $a \le t \le b$ and $x(a) = a_0$ Also considering the nonlinear integral equation: $f(x) = \int_a^x y(x-t)y(t)dt$ This integral equation is transformed a two-dimensional generalized moment problem. In all cases, we will find an approximated solution and bounds for the error of the estimated solution using the techniques ofgeneralized moment problem.

Key words: Generalized moment problems, solution stability, Volterra integral equations, nonlinear integral equations.

1. Introduction

An equation of the form

$$x(t) = f(t) + \lambda \int_a^t K(t,s)x(s)ds \;\; a \le t \le b$$

where $f(t)$ *y* $K(t,s)$ are known functions, λ is a numerical parameter and $x(t)$ is a unknown function, is a Volterra integral equation of second kind. The function $K(t,s)$ is the kernel of the Volterra integral equation. If $f(t) = 0$ then the integral equation is said homogeneous.

The equation

$$f(t) = \int_a^t K(t,s)x(s)ds \;\; a \le t \le b$$

where $x(t)$ is the unknown function, is a Volterra integral equation of first kind. In many scientific and engineering problems of Volterra integral equations are present and have attracted much attention to find analytical and numerical methods for their solution. Some applications of Volterra integral equations are: population dynamics, spread of epidemics, semiconductor devices, inverse problems, etc.

One of the fundamental methods of solving Volterra integral equations of second kind is the method of resolvents [1], [9] where the solution is given by

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$$x(t) = f(t) + \lambda \int_a^t R(t,s,\lambda) f(s) ds$$

The $R(t,s,\lambda)$ is the resolvent function of the integral equation and is defined as the sum of the series

$$R(t,s,\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(t,s)$$

wherein the cores iterated $K_{n+1}(t,s)$ satisfy a recurrence relation.

Another fundamental method it is the method of successive approximations [1], where the solution is determined as the limit of the sequence $\{x_n(t)\}_n$ $n = 0,1,2,\ldots$ whose general term is found by the recurrence formula

$$x_n(t) = f(t) + \lambda \int_a^t K(t,s) x_{n-1}(s) ds$$

Other methods of resolution [10], [11], [12], involving the Laplace transform, are used to solve Volterra integral equations of convolution

$$x(t) = f(t) + \lambda \int_a^t K(t-s) x(s) ds$$

Volterra integral equations of first kind, under certain conditions, can be reduced to a Volterra integral equation of second kind.

2. The Generalized Moment Problem

The generalized moment problem [2], [6], [7], [8] is defined as finding the function $\boldsymbol{f(x)}$ on a domain $\boldsymbol{\Omega} \subset \boldsymbol{R^d}$ satisfying the equations

$$\boldsymbol{\mu_n = \int_\Omega g_n(x) f(x) dx \; n \in N} \quad (1)$$

where $(\boldsymbol{g_n})$ is a given sequence of functions in $\boldsymbol{L^2(\Omega)}$ linearly independent. The moment problem is an ill-conditioned problem. There are several methods for constructing regularized solutions. One is the method of truncated expansion.

The truncated expansion method consists in approximating (1) by finite moment problem

$$\mu_i = \int_\Omega g_i(x) f(x) dx \; i = 1,2,\ldots,n \quad (2)$$

In the subspace generated by $g_1, g_2, \ldots, g_n$ the solution is stable. In the case where the data $\mu_1, \mu_2, \ldots, \mu_n$ are inexact convergence theorems and error estimates for the regularized solutions must be applied.

It can be proved that [8] a necessary and sufficient condition for the existence of a solution of (1) is that

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} C_{ij} \mu_j \right)^2 < \infty$$

where C_{ij} are given by (13).

3. Volterra Linear Integral Equation of Second Kind

We want to find a function $\boldsymbol{x(t)} \in \boldsymbol{L^2(a,b)}$ such that

$$\boldsymbol{x(t) = f(t) + \int_a^t K(t,s) x(s) ds \;\; a \le t \le b} \quad (3)$$

where

$\boldsymbol{f(t)} \in \boldsymbol{L^2(a,b)}$ and $\boldsymbol{K(t,s)} \in \boldsymbol{L^2(R)}$

$\boldsymbol{R = (a,b) \times (a,b)}$ are known functions.

Theorem 1: *If* $\boldsymbol{f(t)} \in \boldsymbol{L^2(a,b)}$ *and* $\boldsymbol{K(t,s)} \in \boldsymbol{L^2(R)}$ $\boldsymbol{R = (a,b) \times (a,b)}$ *then* (3) *has a unique solution in* $\boldsymbol{L^2(a,b)}$ [1].

To write (3) as a moment problems:

$$\boldsymbol{-f(t) = -x(t) + \int_a^t K(t,s) x(s) ds} \quad (4)$$

We take a basis $\{\boldsymbol{\psi_n(t)}\}_{\boldsymbol{n}}$ in $\boldsymbol{L^2(a,b)}$ and both sides of (4) are multiplied by $\boldsymbol{\psi_n(t)}$ and integrated between *a* and *b*

$$\int_a^b \boldsymbol{-f(t)\psi_n(t)\, dt}$$

$$= \int_a^b \boldsymbol{-x(t)\psi_n(t) dt}$$

$$+ \int_a^b \int_a^t \boldsymbol{K(t,s) x(s) \psi_n(t)\, ds\, dt}$$

We call $\boldsymbol{\int_a^b -f(t)\psi_n(t)\, dt = \mu_n}$

In addition

$$\int_a^b \int_a^t \boldsymbol{K(t,s)x(s)\psi_n(t)\,ds\,dt}$$

$$= \int_a^b \boldsymbol{x(s)} \int_s^b \boldsymbol{K(t,s)\psi_n(t)\,dt\,ds}$$

$$= \int_a^b \boldsymbol{x(s)g_n^*(s)ds}$$

Hence

$$\mu_n = \int_a^b -x(t)\psi_n(t)dt + \int_a^b x(s)g_n^*(s)ds$$

$$= \int_a^b -x(t)\psi_n(t)\,dt$$

$$+ \int_a^b x(t)g_n^*(t)dt$$

$$= \int_a^b x(t)[-\psi_n(t) + g_n^*(t)]\,dt$$

$$= \int_a^b x(t)G_n^*(t)dt$$

Consequently

$$\boldsymbol{\mu_n = \int_a^b x(t)G_n^*(t)dt \; n \in N} \tag{5}$$

If $\{\boldsymbol{G_n^*(t)}\}_{\boldsymbol{n}}$ are linearly independent is solved (5) as a generalized moment problem.

Let us see under what conditions $\{G_n^*(t)\}_n$ are linearly independent.

We have $g_n^*(s) = \int_s^b K(t,s)\psi_n(t)\,dt$.

Further $-\psi_n(s) + g_n^*(s) = G_n^*(s)$
We consider the operator

$$L(\varphi) = -\varphi + \int_s^b K(t,s)\varphi(t)\,dt$$

Then $L(\varphi)$ is linear. If L is nonsingular, that is $L(\varphi) = 0 \implies \varphi = 0$, then L preserves the linear independence.

In this case $\{L(\psi_n)\}_n = \{G_n^*\}_n$ would be linearly independent.

But $L(\varphi) = 0$ can be viewed as a Volterra integral equation of second kind with $f(s) = 0$

$$\boldsymbol{0 = -\varphi(s) + \int_s^b K(t,s)\varphi(t)\,dt \implies}$$

$$\boldsymbol{\varphi(s) = \int_s^b K(t,s)\varphi(t)\,dt}$$

$$\boldsymbol{\varphi(s) = \int_b^s (-K(t,s))\varphi(t)\,dt} \tag{6}$$

If we assume that

$$\boldsymbol{K(t,s) \in L^2(R)\; R = (a,b) \times (a,b)}$$

then as $\boldsymbol{\varphi(s) = 0}$ is solution of (6) by the previous theorem is the only solution of (6) in $\boldsymbol{L^2(a,b)}$.

Consequently L is nonsingular.

4. Volterra Linear Integral Equation of First Kind.

We want to find a function $\boldsymbol{x(t) \in L^2(a,b)}$ such that

$$\boldsymbol{f(t) = \int_a^t K(t,s)x(s)ds \; a \le t \le b} \tag{7}$$

with $\boldsymbol{f(t) \in L^2(a,b)}$ and $\boldsymbol{K(t,s) \in L^2(R)\; R = (a,b) \times (a,b)}$ known functions.

The following result can be seen in [1]:

Given a Volterra integral equation of first kind, it can be written as a Volterra integral equation of second kind by applying derivation in (7) with respect to t

$$\boldsymbol{f'(t) = \int_a^t K_t(t,s)x(s)ds + K(t,t)x(t)} \tag{8}$$

we write $\boldsymbol{K^*(t,s) = \frac{K_t(t,s)}{K(t,t)}}$ and $\boldsymbol{f^*(t) = \frac{f'(t)}{K(t,t)}}$ then

$$f^*(t) = \int_a^t K^*(t,s)x(s)ds + x(t) \tag{9}$$

from here (9) is a Volterra integral equation of second kind.

In the case of $K(t,t)=0$ then remains (8) an equation of first kind. It derives again until that $K_t^{(n)}(t,t)\neq 0$.

We must have $K(t,s)$ and $x(t)$ continuous functions in their respective domains, $K(t,s)$ and $f(t)$ differentiables functions with respect to *t*, and it must also be continuous $K_t(t,s)$.

Thus if $K(t,t)\neq 0$ on *(a,b)* and taking into account that it must be $f(a)=0$, (7) is equivalent to (9).

If $K^*(t,s)\in L^2(R)$ and $f^*(t)\in L^2(a,b)$ then (9) (and (7)) has a unique solution in $L^2(a,b)$.

If we have

$$f(t)=\int_a^t K(t,s)g(x(s))ds \quad a\le t\le b \qquad (10)$$

so with the above arguments we arrive at

$$f^*(t)=\int_a^t K^*(t,s)g(x(s))ds+g(x(t)) \qquad (11)$$

In this case (11) is analogous to the generalized moment problem

$$\mu_n=\int_a^b g(x(t))G_n^*(t)dt \quad n\in N \qquad (12)$$

To solve numerically (5) as a generalized moments problem, truncated expansion method detailed in [3] and generalized in [5] is applied for the corresponding finite problem with $i=0,1,2,\dots,N$. We write $p_N(t)$ to approximate $x(t)$.

Is taken a base $\varphi_i(t)$ $i=0,1,2,\dots$ of $L^2(a,b)$ obtained from the sequence $G_i^*(t)$ $i=0,1,2,\dots,N$ by Gram-Schmidt method and necessary functions are added in order to have an orthonormal basis..

We then approximate the solution *x(t)* with [5]:

$$p_N(t)=\sum_{i=0}^{N}\lambda_i\,\varphi_i(t)$$

$$\text{where } \lambda_i=\sum_{j=0}^{i}C_{ij}\;\mu_j \quad i=0,1,\dots,N$$

And the coefficients C_{ij} verifies

$$C_{ij}=\left(\sum_{k=j}^{i-1}(-1)\frac{\langle G_i^*(t)\mid\varphi_k(t)\rangle}{\|\varphi_k(t)\|^2}C_{kj}\right)\cdot\|\varphi_i(t)\|^{-1} \qquad 1<i\le N \quad ; \quad 1\le j<i$$

$$C_{ii}=\|\varphi_i(t)\|^{-1} \qquad i=0,1,\dots,N \qquad (13)$$

It can be seen in [5] the following theorem

Theorem 2: *Let* $\{\mu_k\}_{k=0}^{N}$ *be a set of real numbers and suppose that x(t)* $\in L^2(a,b)$ *verifies for some N* ε *and* E *(two positive numbers)*

$$\sum_{k=0}^{N}\left|\int_a^b G_k^*(t)x(t)dt-\mu_k\right|^2\le\varepsilon^2 \quad and$$

$$\int_a^b\left|x'(t)\right|^2dt\le E^2 \quad then$$

$$\int_a^b|p_N(t)-x(t)|^2dt\le\|C^TC\|\varepsilon^2+\frac{(b-a)^2}{4(N+1)^2}E^2$$

where C is the matrix with coefficients given by (13)

If we apply the truncated expansion method to solve the equation (11) would obtain an approximation $p_N(t)$ for $g(x(t))$.

Thus if g^{-1} is continuous, then $g^{-1}(p_N(t))$ is an estimate of $x(t)$

And if g^{-1} is Lipschitz in a domain D that includes the image of $x(t)$, ie if

$$\|g^{-1}(x)-g^{-1}(y)\|\le\lambda\|x-y\|$$

for some λ and $\forall x,y\in D$ then

$$\int_a^b\left|g^{-1}(p_N(t))-x(t)\right|^2dx$$

$$\le\lambda\left(\|C^TC\|\varepsilon^2+\frac{(b-a)^2}{4(N+1)^2}E^2\right)$$

5. Application

Suppose the integrodifferential equation

$$x'(t)=f(t)+\int_a^t K(t,s)x(s)ds \quad a\le t\le b \quad (14)$$

with initial condition $x(a)=a_0$.

We integrate from *a* to *t*

$$\int_a^t x'(t)dt=\int_a^t f(t)dt+\int_a^t\int_a^t K(t,s)x(s)ds\,dt$$

Thus

$$x(t) - x(a) = F(t) - F(a) + \int_a^t \int_s^t K(t,s)x(s)dtds$$

where $F(t)$ is the primitive of $f(t)$.

If we write $K^p(t,s)$ the primitive of $K(t,s)$ with respect to t, then

$$x(t) - x(a) = F(t) - F(a) + \int_a^t [K^p(t,s) - K^p(s,s)]x(s)ds$$

If we replace $F(t) - F(a) + x(a) = G(t)$ and $K^p(t,s) - K^p(s,s) = K^*(t,s)$ then

$$x(t) = G(t) + \int_a^t K^*(t,s)x(s)ds \qquad (15)$$

That is to say leads to a Volterra integral equation of second kind. Therefore resolver (14) is equivalent to solving (15).

6. Nonlinear Integral Equation with Variable Limit of Integration

Suppose we want to find $\boldsymbol{y(t) \epsilon L^2(a,b)}$ such that

$$\int_a^x \boldsymbol{y(t)y(x-t)dt = f(x)\ \ a \le x \le b} \qquad (16)$$

with $\boldsymbol{f(x)} \in \boldsymbol{L^2(a,b)}$ known.

We take a basis of $\boldsymbol{L^2(a,b)}$. We multiply both sides of (16) and integrate between *a* and *b*:

$$\boldsymbol{\mu_n} = \int_a^b \boldsymbol{f(x)\psi_n(x)dx}$$

$$= \int_a^b \int_a^x \boldsymbol{y(t)y(x-t)\psi_n(x)dtdx}$$

Then

$$\int_a^b \int_a^x \boldsymbol{y(t)y(x-t)\psi_n(x)dtdx}$$

$$= \int_a^b \int_a^{b-t} \boldsymbol{y(t)y(s)\,\psi_n(t+s)dsdt}$$

Consequently

$$\int_a^b \int_a^{b-t} \boldsymbol{y(t)y(s)\,\psi_n(t+s)dsdt = \mu_n\ n\epsilon N} \qquad (17)$$

It can be considered (17) as a two-dimensional generalized moments problem over a region

$$\boldsymbol{\Omega = \{(t,s);\ a \le s \le b - t\,;\ a \le t \le b\}}$$

with $g_n(t,s) = \psi_n(t+s)$ and the unknown function is $x(t,s) = y(t)y(s)$

It is chosen $\{\psi_n(t)\}_n$ such that $\{\psi_n(t+s)\}_n$ are linearly independent.

By solving the corresponding finite problem

$$\int_a^b \int_a^{b-t} \boldsymbol{y(t)y(s)\,\psi_i(t+s)dsdt = \mu_i\ i = 0,1,\ldots,N}$$

applying the truncated expansion method we find the approximation $p_N(t,s)$ for $y(t)y(s)$.

Thus $p_N(t,t)$ will be an approximation of $y^2(t)$. Consequently $\sqrt{p_N(t,t)}$ will be an estimate of $y(t)$

Theorem 2 can be adapted to the case of a two-dimensional moments problem [4] considering a rectangular region *R* such that $\Omega \subset R$.

Note that if $f(x) \in L^2(a,b)$ then (17) , and therefore (16), has a solution in $L^2(a,b)$ because

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i} C_{ij}\,\mu_j\right)^2 = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{i} C_{ij}\int_a^b f(x)\psi_j(x)\right)^2 = \sum_{i=1}^{\infty}\left(\int_a^b f(x)\sum_{j=1}^{i} C_{ij}\,\psi_j(x)\right)^2 = \|f(x)\|^2 < \infty$$

7. Numerical Examples

7.1 We Consider the Volterra Integral Equation of Second Kind

$$\boldsymbol{x(t) = 1 - t - \frac{3}{2}t^2 + \frac{t^3}{3} + \int_0^t \left(\frac{1+t}{1+s}\right)x(s)ds\ \ 0} < t < 1$$

The solution is $x(t) = 1 - t^2$. Was taken $N = 6$ and

$$\psi_n(t) = t^n\ n = 0,1,2,\ldots,N$$

The approximate solution is

$$p_6(t) = \frac{1}{1+t}(0.9999147621585023 + 1.0049081485367966t - 1.0672475817558449t^2 - 0.6253280251875469t^3 - 1.0184797250168018t^4 + 1.4149009386296558t^5 - 0.9279894745774144t^6 + 0.17611185238095847t^7 + 0.04332685368242366t^8)$$

Theorem 2 provides an estimate of the "accuracy" of the approximate solution. Is calculated for the example given

$$\|p_6(t) - x(t)\| = 0.0000182523$$

7.2 We Consider the Volterra Integral Equation of First Kind

$$\boldsymbol{t^2} = \int_{\boldsymbol{0}}^{\boldsymbol{t}} \boldsymbol{e^{t+s}\, x(s) ds}\ \boldsymbol{0} < t < 1$$

The solution is $x(t) = e^{-2t}(2t - t^2)$.

Was taken $N = 6$ and

$$\psi_n(t) = t^n\ n = 0,1,2,\ldots,N$$

The approximate solution is

$$p_6(t) = -83.95739831942318 + 83.95743097212348e^t - 81.95933679986479t - 46.95163623263498t^2 - 8.153902721602144t^3 - 7.676928797912827t^4 + 1.1473834615675322t^5 - 0.5328318171899457t^6$$

and $\|p_6(t) - x(t)\| = 7.74288 \times 10^{-6}$

7.3 We Consider the Integrodifferential Equation

$$\boldsymbol{x'(t) = 2t - \frac{t^5}{4} - t^3 + \int_{0}^{t} st\, x(s) ds\ 0} < t < 1$$

The solution is $\boldsymbol{(t) = t^2 + 2}$.

Was taken $N = 6$ and $\boldsymbol{\psi_n(t) = t^n}\ n = 0,1,2,\ldots,N$

The approximate solution is

$$p_6(t) = 2.000011350675333 - 0.00060008055269661\,35t + 1.0074722441931296t^2 - 0.036727180247674074t^3 + 0.08179727671799264t^4 - 0.07332085167364494t^5 - 0.008280528273741879t^6 + 0.05157218926604704t^7 - 0.02138479219837439t^8 + 0.0003789028271436808t^9 - 0.0009346712381873\,03t^{10}$$

and $\|p_6(t) - x(t)\| = 3.57206 \times 10^{-6}$

7.4 We Consider the Integral Equation

$$\frac{1}{2}sen(x) = \int_0^x y(t)y(x-t)dt\ 0 < x < 1$$

The solution is $\boldsymbol{y(x)} = \sqrt{\frac{1}{2}}\,\boldsymbol{J_0(x)}$ where

$$J_0(x) = BesselJ(0,x)$$

Was taken $\boldsymbol{N = 4}$ and

$$\psi_n(t) = t^n\ n = 0,1,2,\ldots,N$$

The approximate solution is

$$p_4(x) = \left[0.4596976941318603 - 0.09533610866670236\left(-\frac{2}{3} + 2x\right) - 0.07464930125266528\left(-\frac{1}{2} + 4x^2 - \frac{6}{5}\left(-\frac{2}{3} + 2x\right)\right) + 0.008845907552301498\left(-\frac{2}{5} + 8x^3 - \frac{6}{5}\left(-\frac{2}{3} + 2x\right) - \frac{12}{7}\left(-\frac{1}{2} + 4x^2 - \frac{6}{5}\left(-\frac{2}{3} + 2x\right)\right)\right)\right]^{\frac{1}{2}}$$

and $\|\boldsymbol{p_4(x) - y(x)}\| = \boldsymbol{0.000959458}$

8. Conclusions

Given a Volterra integral equation of second kind of the form

$$\boldsymbol{x(t) = f(t) + \int_a^t K(t,s)x(s)ds \quad a \le t \le b}$$

with $\boldsymbol{f(t)} \in \boldsymbol{L^2(a,b)}, \boldsymbol{K(t,s)} \in \boldsymbol{L^2(R)}$ $\boldsymbol{R = (a,b) \times (a,b)}$ known functions, it can be written as a one-dimensional generalized moments problem, and can apply the techniques of moments problem to find a numerical approximation of the solution.

The Volterra integral equation of first kind

$$\boldsymbol{f(t) = \int_a^t K(t,s)x(s)ds \quad a \le t \le b}$$

where $f(t) \in L^2(a,b)$, $K(t,s) \in L^2(R)$ $R = (a,b) \times (a,b)$ known functions, can be written as a Volterra integral equation of second kind if $K(t,t) \neq 0$ on (a,b) , $f(a) = 0$, $K_t(t,s)$ and $f'(t)$ continuous functions in their respective domains.

The nonlinear integral equation

$$\int_a^x y(t)y(x-t)dt = f(x) \quad a \le x \le b$$

with $f(x) \in L^2(a,b)$ known function, can be written as a two-dimensional generalized moments problem.

In all cases the moments problem techniques can be applied to find a numerical approximation of the solution.

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Differential Groupoids

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Abstract: The basic properties and some examples of the differential groupoids are studied.

Key words: Differential space, groupoid.

1. Introduction

In 1981 the notion of *differential group* and the notion of *group differential structure* (based on the notion of Sikorski's differential space – see [9]) was introduced and investigated by the second author in his PhD thesis [4]. Independently, in the same time, an analogous notions was investigated by P. Multarzyński in his PhD thesis (prepared in the Jagiellonian University in Krakov). Some results of this works have been published in [5], [6], [7], and [3] however most of them have never been presented in in the form of an article. Meanwhile, during last ten years, an interest in the theory of differential groups and groupoids appeared, concerned in applications of them in general relativity and cosmology (see references in [8]). This article is the first of the series of papers concerning differential groupoids and describing main results and many details of the theory of differential groups.

Section 2 of the paper contains basic definitions concerning theory of groupoids and theory of differential spaces. Basic definition and facts concerning groupoids can be find in [10] and [11] whereas foundations of theory of differential spaces can be find in [9]. In Section 3 we give the definition of a differential groupoid which is illustrated by an elementary example. Section 4 contains two another examples of topological and differential groupoids.

Without any other explanation we use the following symbols: $\mathbb{N}$-the set of natural numbers; $\mathbb{Z}$-the set of integers; $\mathbb{R}$-the set of reals.

2. Preliminaries

Definition 1. The sequence $(G, X, \alpha, \beta, m, \varepsilon, \tau)$ is called a *groupoid* G *over the base* X if G and X are arbitrary nonempty sets and: (i) the map $\alpha: G \to X$ called a *target* and the map $\beta: G \to X$ called a *source* are surjections; (ii) the map $m: G^{(2)} \to G$, where

$$G^{(2)} := \{(g, h) \in G \times G : \beta(g) = \alpha(h)\},$$

called a *multiplication* satisfies the following conditions:

- $(gh)k = g(hk)$ - *associativity*,
- $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h)$

for each $g, h, k \in G$ (instead of $m(g, h)$ we write gh); (iii) the embedding $\varepsilon: X \to G$ called *the identity section* is such that:

$$\varepsilon(\alpha(g))g = g = g\varepsilon(\beta(g)),$$
$$\alpha(\varepsilon(x)) = x = \beta(\varepsilon(x))$$

for each $g \in G$ and $x \in X$;

(iv) the map $\tau: G \to G$ (denote by $g^{-1} = \tau(g)$) called the *inverce map* is such that

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$$g\tau(g) = \varepsilon(\alpha(g)) \text{ and } \tau(g)g = \varepsilon(\beta(g))$$

for each $g \in G$.

For the definition, basic properties and applications of groupoids see [10] or [11].

Definition 2. A *subgroupoid* of the groupoid $(G, X, \alpha, \beta, m, \varepsilon, \tau)$ is a sequence $(H, \alpha(H), \alpha_{|H}, \beta_{|H}, m_{|H^{(2)}}, \varepsilon_{\alpha(H)}, \tau_{|H})$, where H is nonempty subset of G which is closed under the multiplication and the inverse i. e. (i) if $g, h \in H$ and $(g, h) \in G^{(2)}$, then $gh \in H$; (ii) if $h \in H$, then $h^{-1} \in H$.

Definition 3. The groupoid $(G, X, \alpha, \beta, m, \varepsilon, \tau)$ over the set X is called a *topological groupoid*, if G and X are topological spaces, X is a Hausdorff space and the mappings $\alpha, \beta, m, \varepsilon$ and τ are continuous. Then τ is a homeomorphism.

We recall now the definition of a (Sikorski's) differential space. Let M be a nonempty set and let $\mathcal{C}$ be a family of real valued functions on M. Denote by $\tau_{\mathcal{C}}$ the weakest topology on M with respect to which all functions of $\mathcal{C}$ are continuous. A subbase of the topology $\tau_{\mathcal{C}}$ consists of sets of the form

$$\{p: f(p) < a\} \text{ and } \{p: f(p) > a\},$$

where $a \in \mathbb{R}$ and $f \in \mathcal{C}$. A function $f: M \to \mathbb{R}$ is called *a local $\mathcal{C}$-function on M* if for every $m \in M$ there is a neighborhood V of m and $\alpha \in \mathcal{C}$ such that $f_{|V} = \alpha_{|V}$. The set of all local $\mathcal{C}$-functions on M is denoted by $\mathcal{C}_M$. Note that any function $f \in \mathcal{C}_M$ is continuous with respect to the topology $\tau_{\mathcal{C}}$. Then $\tau_{\mathcal{C}_M} = \tau_{\mathcal{C}}$ (see [1], [2]).

A function $f: M \to \mathbb{R}$ is called *$\mathcal{C}$-smooth function on M* if there exist $n \in \mathbb{N}, \omega \in C^\infty(\mathbb{R}^n)$ and $\alpha_1, \dots, \alpha_n \in \mathcal{C}$ such that

$$f = \omega \circ (\alpha_1, \dots, \alpha_n).$$

The set of all $\mathcal{C}$-smooth functions on M is denoted by $sc\mathcal{C}$. Since $\mathcal{C} \subset sc\mathcal{C}$ and any superposition $\omega \circ (\alpha_1, \dots, \alpha_n)$ is continuous with respect to $\tau_{\mathcal{C}}$ we obtain $\tau_{sc\mathcal{C}} = \tau_{\mathcal{C}}$ (see [1], [2]).

Definition 4. A set $\mathcal{C}$ of real functions on M is said to be a (*Sikorski's*) *differential structure* if: (i) $\mathcal{C}$ is *closed with respect to localization* i.e. $\mathcal{C} = \mathcal{C}_M$; (ii) $\mathcal{C}$ is closed with respect to superposition with smooth functions i.e. $\mathcal{C} = sc\mathcal{C}$.

In this case a pair $(M, \mathcal{C})$ is said to be a *(Sikorski's) differential space* (see [9]). Any element of $\mathcal{C}$ is called a *smooth function on M* (with respect to $\mathcal{C}$).

It is easy to prove that the intersection of any family of differential structures defined on a set $M \neq \emptyset$ is a differential structure on M (see [1], [2], Proposition 2.1).

Let $\mathcal{F}$ be a set of real functions on M. Then the intersection $\mathcal{C}$ of all differential structures on M containing $\mathcal{F}$ is a differential structure on M. It is the smallest differential structure on M containing $\mathcal{F}$. One can easy prove that $\mathcal{C} = (\mathrm{sc}\mathcal{F})_{\mathrm{M}}$. This structure is called *the differential structure generated by* $\mathcal{F}$ and is denoted by $gen(\mathcal{F})$. Functions of $\mathcal{F}$ are called *generators* of the differential structure $\mathcal{C}$. We have also $\tau_{(sc\mathcal{F})_M} = \tau_{sc\mathcal{F}} = \tau_{\mathcal{F}}$.

Let $(M, \mathcal{C})$ and $(N, \mathcal{D})$ be differential spaces. A map $F: M \to N$ is said to be *smooth* if for any $\beta \in \mathcal{D}$ the superposition $\beta \circ F \in \mathcal{C}$. We will denote the fact that $\mathcal{F}$ is smooth writing

$$F: (M, \mathcal{C}) \to (N, \mathcal{D}).$$

If $F: (M, \mathcal{C}) \to (N, \mathcal{D})$ is a bijection and $F^{-1}: (N, \mathcal{D}) \to (M, \mathcal{C})$ then F is called a *diffeomorphism*.

If A is a nonempty subset of M and $\mathcal{C}$ is a differential structure on M then $\mathcal{C}_A$ denotes the differential structure on A generated by the family of restrictions $\{\alpha_{|A}: \alpha \in \mathcal{C}\}$. The differential space $(A, \mathcal{C}_A)$ is called *a differential subspace* of $(M, \mathcal{C})$. One can easy prove that if $(M, \mathcal{C})$ and $(N, \mathcal{D})$ are differential spaces and $F: M \to N$ then

$F: (M, \mathcal{C}) \to (N, \mathcal{D})$ iff $F: (M, \mathcal{C}) \to (F(M), F(M)_{\mathcal{D}})$.

If the map $F: (M, \mathcal{C}) \to (F(M), F(M)_{\mathcal{D}})$ is a diffeomorphism then we say that $F: M \to N$ is *a diffeomorphism onto its range* (in $(N, \mathcal{D})$). In particular the natural embedding

$$A \ni m \mapsto i(m) := m \in M$$

is a diffeomorphism of $(A, \mathcal{C}_A)$ onto its range in $(M, \mathcal{C})$.

If $\{(M_i, \mathcal{C}_i)\}_{i\in I}$ is an arbitrary family of differential spaces then we consider the Cartesian product

$$\prod_{i\in I} M_i$$

as a differential space with the differential structure

$$\widehat{\bigotimes_{i\in I}} \mathcal{C}_i$$

generated by the family of functions

$$\mathcal{F} := \{\alpha_i \circ pr_i : i \in I, \alpha_i \in \mathcal{C}_i\},$$

where

$$\prod_{i\in I} M_i \ni (m_i) \mapsto pr_j\big((m_i)\big) =: m_j \in M_j$$

for any $j \in I$. The topology

$$\tau \widehat{\bigotimes_{i\in I}} \mathcal{C}_i$$

coincides with the standard product topology on

$$\prod_{i\in I} M_i.$$

We will denote the differential structure

$$\widehat{\bigotimes_{i\in I}} \mathcal{C}^\infty(\mathbb{R})$$

on $\mathbb{R}^I$ by $\mathcal{C}^\infty(\mathbb{R}^I)$. In the case when I is an n-element finite set the differential structure $\mathcal{C}^\infty(\mathbb{R}^I)$ coincides with the ordinary differential structure $\mathcal{C}^\infty(\mathbb{R}^n)$ of all real-valued functions on $\mathbb{R}^n$ which posses partial derivatives of any order (see [9]). In any case a function $\alpha: \mathbb{R}^I \to \mathbb{R}$ is an element of $\mathcal{C}^\infty(\mathbb{R}^I)$ iff for any $a = (a_i) \in \mathbb{R}^I$ there are $n \in \mathbb{N}$, elements $i_1, i_2, \cdots, i_n \in I$, a set U open in $\mathbb{R}^n$and a function $\omega \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$a \in U[i_1, i_2, \cdots, i_n] := \{(x_i) \in \mathbb{R}^I : (x_{i_1}, x_{i_1}, \cdots, x_{i_n}) \in U\}$$

and for any $x = (x_i) \in U[i_1, i_2, \cdots, i_n]$ we have

$$\alpha(x) = \omega(x_{i_1}, x_{i_1}, \cdots, x_{i_n}).$$

Let $\mathcal{F}$ be a family of generators of a differential structure $\mathcal{C}$ on a set M. *The generator embedding* of the differential space $(M, \mathcal{C})$ into the Cartesian space defined by $\mathcal{F}$ is a mapping

$$\phi_{\mathcal{F}}: (M, \mathcal{C}) \to \left(\mathbb{R}^{\mathcal{F}}, C^\infty(\mathbb{R}^{\mathcal{F}})\right)$$

given by the formula

$$\phi_{\mathcal{F}}(m) = \big(\alpha(m)\big)_{\alpha\in\mathcal{F}}$$

(for example if $\mathcal{F} = \{\alpha_1, \alpha_2, \alpha_3\}$ then $\phi_{\mathcal{F}}(m) = \big(\alpha_1(m), \alpha_2(m), \alpha_3(m)\big) \in \mathbb{R}^3 \cong \mathbb{R}^{\mathcal{F}}$). If $\mathcal{F}$ separates points of M the generator embedding is a diffeomorphism onto its image. On that image we consider a differential structure of a subspace of $\left(\mathbb{R}^{\mathcal{F}}, C^\infty(\mathbb{R}^{\mathcal{F}})\right)$ (see [2], Proposition 2.3).

3. Basic Properties of Differential Groupoids

Definition 5. Let $(G, X, \alpha, \beta, m, \varepsilon, \tau)$ be a groupoid. A differential structure $\mathcal{C}$ on G is called *a groupoid differential structure*, if the following conditions are satisfied: (i) the multiplication map $m: G^{(2)} \to G$ is smooth with respect to the differential structure of the differential subspace on $G^{(2)} \subset G \times G$; (ii) the inverse map $\tau: G \to G$ and the mappings $\varepsilon \circ \alpha: G \to G$ and $\epsilon \circ \beta: G \to G$ are smooth..

A groupoid G equipped with a groupoid differential structure $\mathcal{C}$ is called *a differential groupoid.*

On $G \times G$ we consider natural differential structure of the Cartesian product which we denote by $\mathcal{C}\widehat{\otimes}\mathcal{C}$.

Example 1. Let $(X, \mathcal{D})$ be a differential space. Then the groupoid of pairs ($G = X \times X$ – see [11]) with the differential structure $\mathcal{C} := \mathcal{D} \mathbin{\widehat{\otimes}} \mathcal{D}$ is a differential groupoid.

Let $\mathcal{C}$ be a groupoid differential structure on a groupoid $(G, X, \alpha, \beta, m, \varepsilon, \tau)$. We know that $\varepsilon(X) \subset G$. On the set $\varepsilon(X)$ there exists the structure of differential subspace of G, i. e. $\mathcal{C}_{\varepsilon(X)}$. Then we will consider X as a support of the differential space $(X,\mathcal{D})$, where the differential structure $\mathcal{D}=\{f\circ\varepsilon : f\in\mathcal{C}_{\varepsilon(X)}\}$ is said to be *induced from the differential structure* $\mathcal{C}_{\varepsilon(X)}$ (or $\mathcal{C}$) *by the map* ε. One can easy show that the identity section ε, the target map α and the source map β are smooth with respect to $\mathcal{D}$ i. e. $\varepsilon:(X,\mathcal{D})\to(G,\mathcal{C})$ and $\alpha,\beta:(G,\mathcal{C})\to(X,\mathcal{D})$.

Let H be a subgroupoid of a groupoid G endowed with a groupoid differential structure $\mathcal{C}$. It is easy to show that the set $\mathcal{C}_H$ is a groupoid differential structure on H. Then the pair $(H,\mathcal{C}_H)$ is called *a differential subgroupoid* of the differential groupoid $(G,\mathcal{C})$. We will write shortly then H is a differential subgroupoid of a differential groupoid G.

4. Examples of Topological and Differential Groupoids

Example 2. Let G be a set of all diffemorphisms between compact subsets of $\mathbb{R}^n$. For arbitrary element g of the set G we have: $g:\left(K_1,\mathcal{C}^\infty(\mathbb{R}^n)_{K_1}\right)\to\left(K_2,\mathcal{C}^\infty(\mathbb{R}^n)_{K_2}\right)$, where K_1 i K_2 are compact subsets in $\mathbb{R}^n$ or shortly $g:K_1\to K_2$. We denote by X_n the family of all non-empty compact subsets in $\mathbb{R}^n$. Let the value of the $\alpha:G\to X_n$ at the element $g\in G$ be the image of the map $g:K_1\to K_2$, i. e. $\alpha(g)=K_2$, and the value of the map $\beta:G\to X_n$ at g be the domain of the diffeomorphism g, i. e. $\beta(g)=K_1$. The value of the embedding $\varepsilon:X_n\to G$ for each compact set $K\in X_n$ is the identity map i. e. $\varepsilon(K)=id_K$. The value of the map $\tau:G\to G$ at $g\in G$ is equal to the inverse map i. e. $\tau(g)=g^{-1}$. As before we put $G^{(2)}=\{(g_1,g_2)\in G^2:\beta(g_1)=\alpha(g_2)\}$. The multiplication m: $G^{(2)}\to G$ is defined by equation: $m(g_1,g_2)=g_1\circ g_2$, where $\circ$ is an ordinary mappings composition. Then the sequence $(G,X_n,\alpha,\beta,m,\varepsilon,\tau)$ is a groupoid.

Let for any two compact sets $K_1,K_2\in X_n$ and any two diffeomorphisms $g_1,g_2\in G$

$$d_n(K_1,K_2)=\sup_{x\in K_1}\left(\inf_{y\in K_2}\|x-y\|\right)$$
$$=\sup_{y\in K_2}\left(\inf_{x\in K_1}\|x-y\|\right)$$

and

$$\tilde{d}_n(g_1,g_2)=d_{2n}(graf\ g_1,graf\ g_2).$$

Then d_n and $\tilde{d}_n$ are metrics on X_n and G, respectively. Mappings $\alpha,\beta,m,\varepsilon$ and τ are continuous with respect to the topology τ_{d_n} and $\tau_{\tilde{d}_n}$ given by metrics d_n and $\tilde{d}_n$ respectively. Hence $(G,X_n,\alpha,\beta,m,\varepsilon,\tau)$ is a topological groupoid. We will denote this groupoid by $DCS(\mathbb{R}^n)$.

Example 3. Let G be such as in Example 2 and let $G_0\subset G$ contains all diffeomorphisms $g\in G$ for which the domain D_g is the cloasure of its interior i.e.

$$D_g=cl\left(int(D_g)\right).$$

Let us consider the set

$$\tilde{G}=\{(g,a)\in G_0\times\mathbb{R}^n : a\in D_g\},$$

where $n\in\mathbb{N}$ is constant.

As the base of the groupoid $G=\left(\tilde{G},X,\alpha,\beta,m,\varepsilon,\tau\right)$ we take the set X composed of all pairs (K,a), where K is a compact subset of $\mathbb{R}^n$ and $a\in K$.

The source and target maps are defined in the following way:

$$\alpha(g,a)=\left(R_g,g(a)\right)\text{ oraz }\beta(g,a)=(D_g,a),$$

where D_g is the domain and R_g is the image of the diffeomorphism g.

Groupoid action m on pairs (g,a) and (h,b) is done, if $D_h=R_g$ and $b=g(a)$. Then $m\left((h,b),(g,a)\right)=(h\circ g,a)$. The identity secion ε we define by:

- $\varepsilon(K,a)=(id_K,a)$ for any $(K,a)\in X$ and the inverse map τ has the value
- $\tau(g,a)=\left(g^{-1},g(a)\right)$ for any $(g,a)\in\tilde{G}$. On the set $\tilde{G}$ we consider the family of
- functions $\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3$, where $\mathcal{F}_1$ is a

family of functions of the form

- $f_\eta(g,a) = \eta\big(g(a)\big)$ for $(g,a) \in \tilde{G}$ and $\eta \in C^\infty(\mathbb{R}^n)$,

$\mathcal{F}_2$ is a family of functions δ_i, where for any multiindex $i \in (\mathbb{N} \cup \{0\})^n$we have

$$\delta_i(g,a) = \frac{\partial^{|i|} g}{\partial x^i}(a) \text{ for } (g,a) \in \tilde{G}$$

(all partial derivatives exists because $D_g = cl\left(int\big(D_g\big)\right)$) and $\mathcal{F}_3$ is a family of functions p_η defined by

$$p_\eta\ (g,a) = \eta(a) \text{ for } (g,a) \in \tilde{G},$$

where $\eta \in C^\infty(\mathbb{R}^n)$.

The family of functions $\mathcal{F}$ generates the differential structure $\mathcal{C}$ on $\tilde{G}$ ($\mathcal{C} = gen\,\mathcal{F}$).

We will prove that $\mathcal{C}$ is a groupoid differential structure on $\tilde{G}$. For it is enough to show that each compositions of functions from families $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ with mappings $m, \tau, \varepsilon \circ \alpha$ and $\varepsilon \circ \beta$ are smooth.

Let us take an arbitrary function $f_\eta \in \mathcal{F}_1$. Then we have

$$f_\eta\left(m\big((h,b),(g,a)\big)\right) = f_\eta(h \circ g, a)$$

$$= \eta\big((h \circ g)(a)\big) = \eta\left(h\big(g(a)\big)\right)$$

$$= \eta\big(h(b)\big) = f_\eta(h,b)$$

which means that $\big(f_\eta \circ m\big)(\xi,\sigma) = f_\eta(\xi)$ for all $(\xi,\sigma) \in \tilde{G}^2$. Hence $f_\eta \circ m$ is an element of the differential structure $\mathcal{C}\widehat{\otimes}\mathcal{C}$ on the space $\tilde{G}^2$. We have also

$$f_\eta\big(\tau(g,a)\big) = f_\eta\big(g^{-1}, g(a)\big) = \eta\left(g^{-1}\big(g(a)\big)\right)$$

$$= \eta(a) = p_\eta(g,a)$$

for each element $(g,a) \in \tilde{G}$ which means that $f_\eta \circ \tau = p_\eta \in \mathcal{F}_3 \subset \mathcal{C}$.

Subsequently, for each element $(g,a) \in \tilde{G}$

$$f_\eta\big((\varepsilon \circ \alpha)(g,a)\big) = f_\eta\left(\varepsilon\big(\alpha(g,a)\big)\right) =$$

$$f_\eta\left(\varepsilon\left(R_g, g(a)\right)\right) = f_\eta\left(id_{R_g}, g(a)\right) =$$

$$\eta\left(id_{R_g}\big(g(a)\big)\right) = \eta\big(g(a)\big) = f_\eta(g,a).$$

Then $f_\eta \circ (\varepsilon \circ \alpha) = f_\eta \in \mathcal{F}_1 \subset \mathcal{C}$. Similarly, for each element $(g,a) \in \tilde{G}$

$$f_\eta\big((\varepsilon \circ \beta)(g,a)\big) = f_\eta\left(\varepsilon\big(\beta(g,a)\big)\right) = f_\eta\left(\varepsilon\big(D_g, a\big)\right)$$

$$= f_\eta\left(id_{D_g}, a\right) = \eta\left(id_{D_g}(a)\right)$$

$$= \eta(a) = p_\eta(g,a)$$

which means that

$$f_\eta \circ (\varepsilon \circ \beta) = p_\eta \in \mathcal{F}_3 \subset \mathcal{C}.$$

Now we can make similar considerations for an arbitrary function

$$\delta_i\left(m\big((h,b),(g,a)\big)\right) = \delta_i(h \circ g, a) =$$

$$= \frac{\partial^{|i|}}{\partial x^i}(h \circ g)(a) =$$

$$= \sum_{\substack{1 \le |j| \le |i| \\ |k_1|+\cdots+|k_n| = |j| \\ |s_1|+\cdots+|s_n| = |i|}} c^{k_1 \cdots k_n}_{j, s_1, \cdots, s_n} \frac{\partial^{|j|}}{\partial y^j}\Big(h\big(g(a)\big)\Big)\left(\frac{\partial^{|s_1|}}{\partial x^{s_1}} g(a)\right)^{k_1} \cdots \left(\frac{\partial^{|s_n|}}{\partial x^{s_n}} g(a)\right)^{k_n}$$

$$= \sum_{\substack{1\le|j|\le|i| \\ |k_1|+\cdots+|k_n|=|j| \\ |s_1|+\cdots+|s_n|=|i|}} c_{j,s_1,\cdots,s_n}^{k_1\cdots k_n}\, \delta_j(h,b)\delta_{s_1}^{k_1}(g,a)\cdots\delta_{s_n}^{k_n}(g,a)$$

where $c_{j,s_1,\cdots,s_n}^{k_1\cdots k_n} \in \mathbb{Z}$. Then $\delta_i \circ m$ is a polynomial function of several variables composed with elements of $\mathcal{F}_2$ and because of that it is an element of $\mathcal{C}$.

Let us consider the superposition $\delta_i \circ \tau$. We have

$$\delta_i(\tau(g,a)) = \delta_i(g^{-1}, g(a)) =$$

$$= \frac{\partial^{|i|}}{\partial x^i} g^{-1}(g(a)).$$

It is know from the course of calculus that the derivative $\frac{\partial^{|i|}}{\partial x^i} g^{-1}(b)$ is a rational function of partial derivatives of the map g taken at the point $g^{-1}(b)$. Then $\frac{\partial^{|i|}}{\partial x^i}(g^{-1})(g(a))$ is a rational function of partial derivatives of the map g taken at the point a (which are elements of $\mathcal{F}_2$). Hence it belongs to $\mathcal{C}$.

Subsequently we consider the superposition $\delta_i((\varepsilon \circ \alpha))$.

$$\delta_i((\varepsilon\circ\alpha)(g,a)) = \delta_i\left(\varepsilon(\alpha(g,a))\right)$$

$$= \delta_i\left(\varepsilon\left(R_g, g(a)\right)\right)$$

$$= \delta_i\left(id_{R_g}, g(a)\right) = \frac{\partial^{|i|}}{\partial x^i} id_{R_g}(g(a))$$

$$= \frac{\partial^{|i|}}{\partial x^i} g(a) = \delta_i(g,a)$$

for each element $(g,a) \in \tilde{G}$. Then $\delta_i \circ (\varepsilon\circ\alpha) = \delta_i \in \mathcal{F}_2 \subset \mathcal{C}$. Similarly

$$\delta_i((\varepsilon\circ\beta)(g,a)) = \delta_i\left(\varepsilon(\beta(g,a))\right) =$$

$$\delta_i\left(\varepsilon(D_g, a)\right) = \delta_i\left(id_{D_g}, a\right) = \frac{\partial^{|i|}}{\partial x^i} id_{D_g}(a) =$$

$$constant \text{ (0 or 1)}.$$

Since $\delta_i((\varepsilon\circ\beta))$ is a constant function it belongs to $\mathcal{C}$.

Let's take any function $p_\eta \in \mathcal{F}_3$. Then we have

$$p_\eta\left(m((h,b),(g,a))\right) = p_\eta(h\circ g, a) = \eta(a) =$$

$$p_\eta(g,a).$$

Then $(p_\eta \circ m)(\xi,\sigma) = p_\eta(\xi)$ for all $(\xi,\sigma) \in \tilde{G}^{(2)}$. It means that the superposition $p_\eta \circ m$ is a element of the differential structure $\mathcal{C}\widehat{\otimes}\mathcal{C}$ on the space $\tilde{G}^{(2)}$.

We have

$p_\eta(\tau(g,a)) = p_\eta(g^{-1}, g(a)) = \eta(g(a)) = f_\eta(g,a)$ for each element $(g,a) \in \tilde{G}$. Hence $p_\eta \circ \tau = f_\eta \in \mathcal{F}_1 \subset \mathcal{C}$. Since

$$p_\eta((\varepsilon\circ\alpha)(g,a)) = p_\eta\left(\varepsilon(\alpha(g,a))\right)$$

$$= p_\eta\left(\varepsilon\left(R_g, g(a)\right)\right)$$

$$= p_\eta\left(id_{R_g}, g(a)\right) = \eta(g(a))$$

$$= f_\eta(g,a)$$

for each element $(g,a) \in \tilde{G}$ we obtain that $p_\eta \circ (\varepsilon\circ\alpha) = f_\eta \in \mathcal{F}_1 \subset \mathcal{C}$. Similarly

$$p_\eta((\varepsilon\circ\beta)(g,a)) = p_\eta\left(\varepsilon(\beta(g,a))\right)$$

$$= p_\eta\left(\varepsilon(D_g, a)\right) = p_\eta\left(id_{D_g}, a\right)$$

$$= \eta(a) = p_\eta(g,a)$$

for each element $(g,a) \in \tilde{G}$. It means that $p_\eta \circ (\varepsilon\circ\beta) = p_\eta \in \mathcal{F}_3 \subset \mathcal{C}$.

Finally we see that $\mathcal{C}$ is a groupoid differential structure on $\tilde{G}$ i. e. $(G,\mathcal{C})$ is a differential groupoid.

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Determining Bookkeeping Cash Maximum of Serbian Army Units by Using Multicriteria Optimization

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Abstract: Normal practice of financial management in the defense system is crucial for the performance of assigned tasks. Payment transactions in cash, in addition to non-cash payment system are very important if we take into account the specificity of the defense system. With limited financial resources optimization level of bookkeeping cash limit should provide continuous funding of units and institutions of the defense system. The aim of this paper is to show that using the method of analytic hierarchy process (AHP) we can help optimize the allocation of cash financial funds within the defense system.

Key words: Decision making, bookkeeping, cash transactions, criteria, alternatives

1. Introduction

One of the main challenges with which people are faced in everyday situations is how to make the right decision for the given problem. One way is the use of multi-criteria optimization, which offers a range of representative methods for making the right decisions. Methods of analytic hierarchy process (AHP) is one of the most highly used methods for multi-criteria decision making, where a decision is made based on a number of criteria and multiple time periods. It is this method that is going to be used in the paper to determine the relative weights of the criteria and the optimal solution to the problem, i.e. determining the cash maximum for the Serbian Army (SA) units.

The aim of this paper is to based on rational and scientific approach, show a way of solving the problem of decision-making, using multi-criteria optimization in determining the amount of cash maximum in SA units. It is the applicative aspect of the paper that should arise from the elaborated example which is its fundamental contribution. Multiple criteria decision-making plays a key role in many real-life problems. This has been confirmed in practice, whether it is applied to state authorities, managers of companies or any other businesses, because they all face situations where they choose in a range of alternatives the best one, based on the existing criteria. This paper provides empirical analysis of a multi-criteria problem with which managers in the defense system are faced, with a recommendation for the creation and implementation of a model that will improve the decision-making process.

2. Organization of Cash Operations in the Serbian Army

Area of financial operations in SA is regulated with a number of normative acts, each of which in its area regulates the performance of certain actions and procedures. One of the most important regulations is the Regulation on financial operations in the Ministry of Defense and Serbian Army [1] in which, among other things, the performance of bookkeeping cash operations is regulated.

Serbian Army, as a direct budget beneficiary can use its provided funds for the following purposes [2]:

- acquisition of assets, works and services;

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- payment of personnel costs and
- specific purposes.

State and purpose of use of the approved funds must constantly be monitored and recorded in an appropriate manner. Especially significant are the funds that must be provided daily for lesser payments in units, in order to ensure normal functioning. For this reason each units has its own cash operation, and the realization is done through cashiers that were established by formation.

Cash operations include downloading, storing and issuing or trade of cash. It should be taken into account that the regulations limit cash payments up to a certain specified amount. To prevent cash buildup in the cashier's desk, cash maximum is determined, which represents the largest amount of cash that can be kept in the cashier's desk. Height of cash maximum is determined by a decision of the Head of budget Administration of the Ministry of Defense. Article 39 of the Regulations on financial operations of the Ministry of Defense and Serbian Army (OMG 17/2011) regulates that the supervisor of the beneficiary authorizes the person that takes over the cash in the manner and procedure prescribed by the regulation on budget execution system.

One of the problems present in the work of financial services authority, which directly make decisions in the budget Administration, is how to determine the maximum cash for each unit. It is the aim of this paper to help solve the aforementioned problem, using the method of multi-criteria optimization.

3. Methods of Multicriteria Analysis

There are numerous methods to solve multi-criteria decision making models that can be divided based on several criteria, and the best in today's time are:

- ELECTRE method;
- PROMETHEE method;
- AHP (analytic hierarchy proccess) method;
- TOPSIS method;
- SAW method and other.

Special attention in this paper will be devoted to the AHP method, which is a method of multi-criteria decision making, created to assist decision makers in solving complex decision problems involving a large number of decision makers, a number of criteria in multiple periods. Methodologically speaking AHP is based on the decomposition of a complex problem into a hierarchy. The goal is at the top of the hierarchy, while the criteria, sub-criteria and alternatives are at lower levels. AHP holds all the parts of the hierarchy in relationship, so it's easy to see how a change in one affects the other criteria.

The process of solving the decision-making problem is often very complex due to the presence of conflicting objectives among the available criteria or alternatives. The problem is to choose the alternative that will best meet the set goals. Field of application of this method is multi-criteria decision making, where based on a defined set of criteria and attribute values for each alternative the most appropriate is selected. In order to easily facilitate the application of this method a software tool in decision support systems has been developed named Expert Choice.

The process of realization of the AHP method includes four main phases [3]:

- Structuring the problem, which consists of decomposing any complex decision making problem into a series of hierarchies, where each level represents a smaller number of manageable attributes. They are then decomposed in a second set of elements which correspond to the next level. This way of hierarchical structuring of any decision making problem is an effective way of dealing with the complexity of real life problems and identifying significant attributes in order to achieve the overall goal of the problem.
- Data collection is the beginning of the second phase of the AHP method. The decision maker assigns relative scores in pairs of attributes of one hierarchical level and does this at all levels of the entire hierarchy. The best known scale used to assign weights is Saaty's nine-point scale.

- Rating the relative weights implies that the comparison matrix in pairs, is translated into problems of determining their own values, to obtain the normalized and unique own vectors, with weights for all attributes at each level of the hierarchy.
- Determining the solution to the problem is the last stage that involves finding the so-called composite normalized vector. Once the vector of sequence of criterion values in a model is determined, in the next round it is necessary to determine the order of importance of alternatives in the model with respect to the same procedure, within each of the monitored criteria.

3.1 Formulating the Mathematical Model of Multi-Criteria Decision Making

Model of the multi-criteria decision making has the following mathematical formulation [3]:

$$max \left[f_1(x), f_2(x), ..., f_p(x)\right], \ p \geq 2 \quad (1)$$

with limitations:

$$g_i(x) \leq 0, \ i = \overline{1,m}$$

$$x_j \geq 0, \ j = \overline{1,n}$$

where:

n – number of variables;

p – number of criterion functions;

m – number of limitations;

X – n- dimensional vector of variables xj, $j = \overline{1,n}$

fk - function (goal) of the criteria, $k = \overline{1,p}$

gi(x) – set of constraints, $i = \overline{1,m}$

It should be noted that the maximization of the function vector is carried out with the given constraints, since the minimization criteria can be converted into maximization criteria, and:

$$\max \mathrm{f}_r(x) = -\min\left[-\mathrm{f}_r(x)\right], r \in (1,p) \quad (2)$$

By solving the model above a set of permissible solutions is obtained, vector X which belongs to the set of positive integers $\mathrm{X} \in R^n$, for which applies:

$$X = \left[x \middle| g_i(x) \leq 0, i = \overline{1,m}, x_j \geq 0, j = \overline{1,n}\right] \quad (3)$$

Thus resulting set of solutions X, to which corresponds a set of values of the function criteria, or the vector f(x), so that the set of permissible solutions X can be mapped into a criterion set S:

$$f(x) = \left[f_1(x), f_2(x), ..., f_p(x)\right] \quad (4)$$

$$S = \left[f(x) \middle| x \in X\right]$$

3.2 Defining the Terms in the Decision-Making Problem

Defining the criteria occupies an important place in the process of deciding on cash maximum amount which is determined for the units. Criterion as a term refers to the attributes that are related to alternatives between which we select. They can be divided into qualitative and quantitative criteria depending on the degree of measurability. Quantitative criteria are those that can be accurately measured and are expressed in different units of measurement. Qualitative criteria are those that cannot be expressed numerically. They can be divided into two subgroups: the attributes whose values cannot be accurately measured, but can be ranked by the "intensity" and attribute basis of which no quantitative comparison of alternatives can be done. There are plenty of ways to translate qualitative criterion values in quantitative. The most commonly used scales are: in-line scale, interval scale and ratio scale. The second criterion, which is also used for the distribution of decision making criteria is the direction of correlation of their values and utility providers. According to the stacking direction there are [4]:

- Revenue criteria;
- Expenditure Criteria and
- Non-monotonic criteria.

In the process of the observed choices there are great number of criteria available, which are more or less important and precisely defined at the beginning, and in our case they are: distance of the unit from the Accounting Centre, unit level, unit size and unit type.

Alternatives are the solutions which are emerging as a choice between which we select the best one. For simplicity of presentation three SA units are taken into consideration between which we select to whom to assign the highest cash maximum. They have characteristics that match the criteria that have been defined.

4. Application of the AHP Method in Multicriteria Optimization of Determining the Height of Bookkeeping Cash Maximum

The process of determining the maximum cash amount in SA units which are the executors of the approved plan of financial resource expenditure represents a problem which we will try to realize using the method of multi-criteria optimization. Mitigating factor during usage of any method of multi-criteria decision-making is the fact that they are all software supported, and the mentioned software in our case can be found at the internet address: www.odlucivanje.fon.rs. However in this paper the emphasis is not directly placed on the application of this software but the logical-mathematical setting of the problem.

Justification of this paper can be ontologically substantiated by facts of appropriateness of the optimization of determining the height of cash maximum from the competent authority within the defense system that is, to show that in practice the use of these methods may lead to an optimal solution. Also an important requirement that is going to be satisfied in this way is the scientific foundation of the procedure of decision-making.

The assumption in this problem is determining cash maximum amount for accounting purposes in order to maintain permanent liquidity of the financial assets in the SA units. In order to find the optimal solution for given assumptions four criteria are used in relation with three possible alternatives which will be considered.

Criteria in this problem are:

- K1 – Distance of the unit from the Accounting Centre (AC) is one of the criteria to be taken for determining the cash maximum amount, which is the criterion of maximization. It is necessary to determine the distance of the unit from the AC because of the need to determine the time interval in which the documents are submitted to the AC. For example some units due to the physical distance only deliver their accounting documents twice a week making it difficult to justify the consumed cash as a condition to receive newly approved.
- K2– Unit level in accordance with the Decision on authorization for management and replacement of movable property and the procurement of works and services to the MoD and SA [5]. By this Decision the commanders of beneficiaries among which are cash funds have the authority delegated by the Defense Minster concerning managing funds. In this regard, depending on the degree of autonomy to manage the funds greater or smaller amount of cash is required, the units will be observed as the commander 1 (the lowest level of authority), commander 2 (medium degree of authority) and commander 3 (the highest level of authority).
- K3 – Unit size is the maximization criterion, and refers to the number of people within the unit which significantly affects the level of cash maximum due to an increase in personnel expenses. Personnel expenditures conditionally progressively increase due to the increase of personnel during peacetime and war formations.
- K4 – Unit type is determined depending on the composition and use of a particular unit or institution. Within this criterion in determining the amount of cash maximum it is significant to determine the degree of significance of the quantity of cash as an instrument of maintaining continuous liquidity with a goal to support permanent combat readiness of SA units. As a condition of determining the maximum cash amount it is significant to differentiate between infantry, artillery, armor, special units, logistics and others.

The decision making matrix in this case is shown in table 1:

Quantifying this matrix, using Saaty nine-point scale for assigning weights the following matrix is obtained:

4.1 Evaluation of the Relative Weights of Criteria

At the beginning of processing the problem it is necessary to start by determining the relative weights of the criteria that is, significance of the criteria. To estimate the relative weights of the criteria we will use Saaty's scale [6].

Based on the data obtained from the evaluation of relative weights of the criteria, using the same procedure alternatives should be observed as well. Comparison of alternatives will also be done by using Saaty's scale. After forming the tables of comparing the weights in pairs for each alternative, we will calculate own vectors.

Table 1 Decision making matrix

Alternatives	Criteria			
	K_1	K_2	K_3	K_4
Unit 1	360	Commander 2	2300	infantry
Unit 2	150	Commander 1	500	armor
Unit 3	90	Commander 3	80	logistic

Table 2 Quantified input data

Alternatives	Criteria			
	K_1	K_2	K_3	K_4
Unit 1	360	5	2300	5
Unit 2	150	3	500	7
Unit 3	90	9	80	9

Table 3 Evaluation of relative weights of the criteria

	K_1	K_2	K_3	K_4
K_1	1	(5)	(4)	(7)
K_2	5	1	4	(5)
K_3	4	(4)	1	(6)
K_4	7	5	6	1
Σ	17	6,45	11,25	1,51

Table 4 Calculation of own vectors with corresponding own values

	K_1	K_2	K_3	K_4	Σ	W(Σ/4)
K_1	0,059	0,031	0,022	0,092	0,204	0,051
K_2	0,294	0,155	0,355	0,132	0,936	0,234
K_3	0,235	0,039	0,089	0,113	0,476	0,119
K_4	0,412	0,775	0,533	0,662	2,382	0,595

Table 5 Calculation of own vectors corresponding to own values (Distance of units from the AC)

	Unit 1	Unit 2	Unit 3	Σ	W(Σ/3)
Unit 1	1	9	7	2,329	0,776
Unit 2	(9)	1	(3)	0,205	0,068
Unit 3	(7)	3	1	0,465	0,155

Table 6 Calculation of own vectors corresponding to own values (Unit level)

	Unit 1	Unit 2	Unit 3	Σ	W(Σ/3)
Unit 1	1	5	(7)	0,569	0,189
Unit 2	(5)	1	(9)	0,180	0,060
Unit 3	7	9	1	2,251	0,750

Table 7 Calculation of own vectors corresponding to own values (Unit size)

	Unit 1	Unit 2	Unit 3	Σ	W(Σ/3)
Unit 1	1	7	9	2,251	0,750
Unit 2	(7)	1	5	0,569	0,189
Unit 3	(9)	(5)	1	0,180	0,060

Table 8 Calculation of own vectors corresponding to own values (Unit type)

	Unit 1	Unit 2	Unit 3	Σ	W(Σ/3)
Unit 1	1	(5)	(7)	0,215	0,072
Unit 2	5	1	(5)	0,695	0,232
Unit 3	7	5	1	2,089	0,696

Table 9 Determining the amount of cash maximum

	K_1	K_2	K_3	K_4	Total priorities of alternatives
	0,051	0,234	0,119	0,595	
Unit 1	0,776	0,189	0,750	0,072	0,216
Unit 2	0,068	0,060	0,189	0,232	0,178
Unit 3	0,155	0,750	0,060	0,696	0,604

4.2 Determining the Solution to the Problem

After assessing relative weights of alternatives with respect to each criterion we approach to determining the maximum cash amount of the observed units. The choice of units is made based on the received own vectors of alternatives and previously obtained own vectors of criteria. Total priorities of alternatives are obtained by multiplying each alternative by its weight within the observed criterion in order and finally adding up the results.

From Table 9 it can be seen that after implementing the procedure of AHP method for the given example, the order of alternatives would be as follows: "Unit 1" (22%), "Unit 2" (18%), "Unit 3" (60%), which shows that the best decision would be to award the highest cash maximum to "Unit 3".

5. Conclusions

Everyone - individuals, politicians, professionals, business men daily consider and make small and big decisions - decisions that affect individuals, families, business systems or social communities - of regions, countries and even the world as a whole. In most cases that is, problems solved there are several solutions. But the question that arises is which solution to choose? One that considers and decides takes into account several aspects of the problem being solved: some reasons speak in favor of deciding in one way, but other reasons say that such decisions are often reviewed and often amended.

Thus the practice of problem solving in the defense system shows that they can be resolved in different ways, taking into account the relevant criteria. The possibility of using a number of representative methods that are available when deciding on the amount of maximum cash makes the work even easier and raises the level of quality of the decision to a higher level. It is an example of using the

AHP method in choosing which unit should be assigned the highest maximum cash is shown in a rather simple way how with a precise procedure a decision can be made, and while doing so recognizing all the set criteria on which the selection is made. It is also in this way shown that there are significant arguments for this method to be based on scientific grounds.

In the specific problem (formulated criteria, assumed input data) people who decide in the defense system, that need to decide on the maximum cash limit will not make a mistake if the decision relates to the choice of alternative "Unit" 3. This decision stemmed from conducted methodological procedure of applying the AHP method, where in a scientific-friendly and reliable manner the solution of multi-criteria problem was got.

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