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N-Body Problem Extension Hypothesis

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Abstract: A system of mutually interacting point-mass objects is considered. All objects act on each other with mutual forces. A consistent relativity of motion based on the elimination of the observer is introduced. The center of inertia of the entire system is a common reference point. Central and mutual quantities are defined and the relationship between them is derived. A simple method of numerical approximation of the evolution of the central motion is presented. The scale invariance of the classical n-body problem is used to challenge the physical correctness of the problem formulation. Subsequently, a hypothesis is expressed about the extension of Coulomb's law as well as Newton's law of gravity. The consequences of the hypothesis are illustrated by the simulation of the helium-2 atom and the simulation of the motion of the stars S2, S4716 in the vicinity of the black hole in the core of our galaxy.

Keywords: Barycenter, partial and resultant mutual forces, central and mutual inertia.

1. Introduction

The general n-body problem provides one of the richest unsolved dynamical problems [1]. The only possible way to reliably solve an n-body problem is through numerical integration of the equations of motion. The aim of the paper is to present a slightly modified approach to the formulation of the n-body problem based on the elimination of the reference point of the initial positions and velocities. An observer is replaced by the common center of inertia. Consequently central, and mutual quantities are introduced and the relation between them is defined. An ordinary relativity of both the central and the mutual vector quantities is defined. A simple numerical approximation of the global solution in central quantities is presented. For the classical inverse square force law interaction, an existing scale invariance challenging the classical n-body problem is pointed out. The existing scale invariance initiates the need to modify the mutual force interaction between material elements at high speeds. The consequence of this modification is the attraction of like charges, the repulsion of unlike charges, and the gravitational repulsion at high speeds comparable to the speed of light propagation in a vacuum.

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2. Mutual Inertia Definition

Let the initial positions and velocities of the point-mass model elements m_i in respect of an observer be $\mathbf{r}_{io}$, $\mathbf{v}_{io}$, and let the total inertia of the system is

$$m = \sum m_i \tag{1}$$

We replace an observer by the common center of inertia:

$$\mathbf{r}_{co} = m^{-1}\sum m_i\mathbf{r}_{io}, \qquad \mathbf{r}_{ic} = \mathbf{r}_{io} - \mathbf{r}_{co} \tag{2}$$

$$\mathbf{v}_{co} = m^{-1}\sum m_i\mathbf{v}_{io}, \qquad \mathbf{v}_{ic} = \mathbf{v}_{io} - \mathbf{v}_{co} \tag{3}$$

Having the set of point-mass objects m_i with the central positions and velocities r_{ic}, v_{ic} in respect of the common center of inertia, we can write

$$\sum m_i\mathbf{r}_{ic} = 0, \ \sum m_i\mathbf{v}_{ic} = 0 \tag{4}$$

Mutual vectors are defined by the equalities

$$\mathbf{r}_{ij} = \mathbf{r}_{ic} - \mathbf{r}_{jc}, \mathbf{v}_{ij} = \mathbf{v}_{ic} - \mathbf{v}_{jc} \tag{5}$$

From Eq. (4, 5) we have for positions (as well as for velocities):

$$\begin{aligned}
&\mathbf{r}_{ic} = \mathbf{r}_{jc} + \mathbf{r}_{ij} \\
&=> \sum_j m_j\mathbf{r}_{ic} = \sum_j m_j (\mathbf{r}_{jc} + \mathbf{r}_{ij}) \\
&=> m\ \mathbf{r}_{ic} = \sum_j m_j\mathbf{r}_{jc} + \sum_j m_j\mathbf{r}_{ij} \\
&=> \mathbf{r}_{ic} = 0 + m^{-1}\sum_j m_j\mathbf{r}_{ij} \\
&=> m_i\mathbf{r}_{ic} = m^{-1}\sum_j m_i m_j\mathbf{r}_{ij} \\
&=> m_i\mathbf{r}_{ic} = \sum_j m_{ij}\mathbf{r}_{ij},
\end{aligned} \tag{6}$$

where we have defined the scalar quantity

$$m_{ij} = m^{-1} m_i m_j \quad (7)$$

to be the mutual inertia of the point mass objects m_i, m_j within the system of total inertia m.

The term reference frame will not be used, since it consists of two independent terms, namely a reference point and a reference vector base. The only reference vector base used in this paper is the inertial (non-rotating) vector base. Nevertheless, many reference points will be used, namely local centers of inertia. From the practical point of view, there is a single global reference point, namely the center of inertia of all point-mass objects under consideration.

3. Central and Mutual Forces

Let $^{\#}\mathbf{f}_{ij}$ be a partial mutual force exerted by the object j on the object i in the hypothetical absence of the remaining objects under consideration. In accordance with the law of action and reaction, $^{\#}\mathbf{f}_{ij} + {}^{\#}\mathbf{f}_{ji} = 0$. Let us define the central forces to be the sum of all partial mutual forces

$$\mathbf{f}_{ic} = \sum_j {}^{\#}\mathbf{f}_{ij} \quad (8)$$

Let the central force (exerting on the object m_i in respect of the common center of inertia) be $\mathbf{f}_{ic} = m_i \mathbf{a}_{ic}$. We define the resultant mutual forces

$$\mathbf{f}_{ij} = m_{ij} \mathbf{a}_{ij} = m_{ij} (\mathbf{f}_{ic} / m_i - \mathbf{f}_{jc} / m_j) \quad (9)$$

The relation between the resultant mutual forces and the partial mutual forces is

$$\sum_j \mathbf{f}_{ij} = \sum_j {}^{\#}\mathbf{f}_{ij} \quad (10)$$

Proof:

$$\begin{aligned}
\sum_j \mathbf{f}_{ij} &= \sum_j m_{ij} (\mathbf{a}_{ic} - \mathbf{a}_{jc}) = \sum_j m_{ij} (\mathbf{f}_{ic}/m_i - \mathbf{f}_{jc}/m_j) \\
&= m^{-1} \sum_j (m_j \sum_k {}^{\#}\mathbf{f}_{ik} - m_i \sum_k {}^{\#}\mathbf{f}_{jk}) \\
&= m^{-1} \sum_k \sum_j (m_j {}^{\#}\mathbf{f}_{ik} - m_i {}^{\#}\mathbf{f}_{jk}) \\
&= m^{-1} \sum_k (\sum_j m_j {}^{\#}\mathbf{f}_{ik} - m_i \sum_j {}^{\#}\mathbf{f}_{jk}) \\
&= m^{-1} \sum_k (m\, {}^{\#}\mathbf{f}_{ik} + m_i \sum_j {}^{\#}\mathbf{f}_{kj}) \\
&= m^{-1} (m \sum_k {}^{\#}\mathbf{f}_{ik} + m_i \sum_j \sum_k {}^{\#}\mathbf{f}_{kj}) \\
&= \sum_k {}^{\#}\mathbf{f}_{ik}, \text{ since } \sum_j \sum_k {}^{\#}\mathbf{f}_{kj} = 0 \ \blacksquare
\end{aligned}$$

The second subscript c for central quantities will be simply omitted by writing $\mathbf{r}_i$, $\mathbf{v}_i$, $\mathbf{a}_i$, $\mathbf{f}_i$ instead of $\mathbf{r}_{ic}$, $\mathbf{v}_{ic}$, $\mathbf{a}_{ic}$, $\mathbf{f}_{ic}$. Central and mutual forces, as well as position, velocity, and acceleration are relative vector quantities that always refer to two points or point objects, even if the second subscript c is omitted.

4. Motion Development

Having defined the central forces Eq. (8), the new central positions and velocities of all point-mass objects under consideration will be approximated simply by

$$\mathbf{r}_i(t) = \mathbf{r}_i + \mathbf{v}_i t + \mathbf{a}_i t^2/2 \quad (11)$$

$$\mathbf{v}_i(t) = \mathbf{v}_i + \mathbf{a}_i t \quad (12)$$

where $\mathbf{a}_i = \mathbf{f}_i / m_i$ are central accelerations and t is the sufficiently small step of the motion development approximation with the desired precision. It is recommended to repeat the centralization procedure Eqs. (2), (3) in order to suppress possible numerical noise.

5. Central and Mutual Quantities Relation

Having introduced the above notation, the following equalities may be easily verified

$$m_i = \sum_j m_{ij}, \qquad m = \sum_i \sum_j m_{ij} \quad (13)$$

$$m_i \mathbf{r}_i = \sum_j m_{ij} \mathbf{r}_{ij}, \sum_i m_i \mathbf{r}_i = \sum_i \sum_j m_{ij} \mathbf{r}_{ij} = 0 \quad (14)$$

Analogously for velocities and accelerations. The following equalities hold for the relations between central and mutual vector quantities

$$2 \sum_i m_i \mathbf{p}_i . \mathbf{q}_i = \sum_i \sum_j m_{ij} \mathbf{p}_{ij} . \mathbf{q}_{ij} \quad (15)$$

$$2 \sum_i m_i \mathbf{p}_i \times \mathbf{q}_i = \sum_i \sum_j m_{ij} \mathbf{p}_{ij} \times \mathbf{q}_{ij} \quad (16)$$

In Eqs. (15), (16) $\mathbf{p}_i$, $\mathbf{q}_i$ are central vectors and $\mathbf{p}_{ij}$, $\mathbf{q}_{ij}$ mutual vectors of position, velocity, or acceleration, i.e., 18 relations altogether. For example

$$2 \sum_i m_i \mathbf{v}_i . \mathbf{v}_i = \sum_i \sum_j m_{ij} \mathbf{v}_{ij} . \mathbf{v}_{ij} \quad (17)$$

is an equality of the central and the mutual kinetic energy, and

$$2 \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i = \sum_i \sum_j m_{ij} \mathbf{r}_{ij} \times \mathbf{v}_{ij} \quad (18)$$

is an equality of the central and the mutual angular momentum. Another example is an equality of the central and the mutual potential energy

$$2 \sum_i m_i \mathbf{a}_i . \mathbf{r}_i = \sum_i \sum_j m_{ij} \mathbf{a}_{ij} . \mathbf{r}_{ij} \quad (19)$$

which may be written in the form

$$2 \sum_i \mathbf{f}_i . \mathbf{r}_i = \sum_i \sum_j \mathbf{f}_{ij} . \mathbf{r}_{ij} \quad (20)$$

6. Decomposition

The motion development is being calculated in central quantities with respect to the common barycenter, but each locality may be depicted with respect to its barycenter, i.e., local point of reference. Let the set of all objects under consideration consists of disjunct subsets a, b, … with objects creating the localities. We express the sum through all indexes $i \in x$ by the symbol $\sum_{i \in x}$, and define the following local quantities for localities $x \in \{a, b, …\}$:

$$m_x = \sum_{i \in x} m_i \tag{21}$$

$$\mathbf{r}_x = m_x^{-1} \sum_{i \in x} m_i \mathbf{r}_i \tag{22}$$

$$\mathbf{r}_{ix} = \mathbf{r}_i - \mathbf{r}_x \tag{23}$$

and the following mutual quantities for pairs of localities $x, y \in \{a, b, …\}$:

$$m_{xy} = m_x\, m_y / m \tag{24}$$

$$\mathbf{r}_{xy} = \mathbf{r}_x - \mathbf{r}_y \tag{25}$$

Analogously for velocity and acceleration vectors. The same Eqs. (15), (16) hold for the relations between central and mutual vector quantities inside each locality as well as for central and mutual vector quantities concerned with localities.

7. Inverse Square Force Law

In principle, partial mutual forces are arbitrary and may be given by the superposition of physical laws. Nevertheless, let us suppose the partial mutual forces to be given by the inverse square law in the form

$$^{\#}\mathbf{f}_{ij} = k_{ij}\, r_{ij}^{-3}\, \mathbf{r}_{ij}, \tag{26}$$

where $k_{ij} = -G\, m_i\, m_j$ in the case of the Newton's law of gravity or $k_{ij} = (4\pi\varepsilon_0)^{-1} q_i\, q_j$ in the case of the Coulomb's law. In accordance with [2] the equations of motion may be written in the form

$$m_i \mathbf{r}_i'' = \sum_j {}^{\#}\mathbf{f}_{ij} \tag{27}$$

where $\mathbf{r}_i$ are the central position vectors of the i-th object relative to the barycenter of all objects under consideration.

The known energy integral for the equations of motion Eq. (27) can be written in central or mutual quantities with the same integration constant q:

$$(1/2) \sum_i m_i\, \mathbf{v}_i . \mathbf{v}_i + \sum_i \mathbf{f}_i . \mathbf{r}_i = q \tag{28}$$

$$(1/4) \sum_i \sum_j m_{ij}\, \mathbf{v}_{ij} . \mathbf{v}_{ij} + (1/2) \sum_i \sum_j \mathbf{f}_{ij} . \mathbf{r}_{ij} = q \tag{29}$$

There is a scaling invariance of initial conditions and the corresponding solution of Eq. (27):

$$r_i(0) \rightarrow s^{-2}\, r_i(0) \tag{30}$$

$$v_i(0) \rightarrow s\, v_i(0) \tag{31}$$

$$t \rightarrow s^{-3}\, t \tag{32}$$

If the initial conditions and the time step of the approximation are changed according to the parameter $s > 0$, we will formally see the same evolution of the motion, except for the change in scale. For example, when the parameter $s = 10$, the new distances are 1/100 of the original, the new velocities are 10 times higher, and the time step is 1/1000 of the original.

8. Force Law Extension Hypothesis

The inverse square force law in the classical n-body problem formulation should be somehow modified so as not to provide the above-mentioned scaling invariance without bound. Consider two point-like charges q_1, q_2 moving relative to each other. The interaction between them is expressed by Lorentz relation

$$\mathbf{f}_{12} = q_1\, (\mathbf{E} + \mathbf{v}_{12} \times \mathbf{B}), \tag{33}$$

$$\mathbf{E} = (4\pi\varepsilon_0)^{-1}\, q_2\, r_{12}^{-3}\, \mathbf{r}_{12} \tag{34}$$

$$\mathbf{B} = (\mu_0/4\pi)\, q_2\, r_{12}^{-3}\, \mathbf{v}_{12} \times \mathbf{r}_{12}. \tag{35}$$

Eliminating **E, B** from Eq. (33), a hypothesis of the Coulomb's law extension may be written in the form

$$\mathbf{f}_{12} = k_{12}\, r_{12}^{-3}\, [\mathbf{r}_{12} + c^{-2}\, \mathbf{v}_{12} \times (\mathbf{v}_{12} \times \mathbf{r}_{12})], \tag{36}$$

where c is the speed of light and $k_{12} = (4\pi\varepsilon_0)^{-1} q_1\, q_2$. If we assume that the gravitational force is a residual force acting between electrically neutral objects, a hypothesis of the extension of Newton's law of gravity has the form of Eq. (36) with $k_{12} = -G\, m_1\, m_2$.

The position vector $\mathbf{r}_{12}$ of the two-body problem in [2] changes according to the vector differential equation

$$m_{12}\, \mathbf{r}_{12}'' = \mathbf{f}_{12}, \tag{37}$$

where $m_{12} = m_1 m_2/(m_1 + m_2)$, but, in accordance with the above hypothesis, $\mathbf{f}_{12}$ is given by Eq. (36). The first-order equations of motion can be written in the form

$$\mathbf{r}_{12}' = \mathbf{v}_{12} \quad (38)$$

$$m_{12}\,\mathbf{v}_{12}' = k_{12}\, r_{12}^{-3}\, [\mathbf{r}_{12} + c^{-2}\,\mathbf{v}_{12} \times (\mathbf{v}_{12} \times \mathbf{r}_{12})] \quad (39)$$

Since $\mathbf{v}_{12}.\ [\,\mathbf{v}_{12} \times (\mathbf{v}_{12} \times \mathbf{r}_{12})] = 0$, Eqs. (38), (39) yield the same well-known energy integral

$$m_{12}\,\mathbf{v}_{12}.\ \mathbf{v}_{12} / 2 + k_{12}\, r_{12}^{-1} = q, \quad (40)$$

where q is an integration constant given by initial conditions. Obviously, the same energy integral does not mean the same evolution of the kinetic and the potential energy. Only the sum of the energies remains the same. The slightly modified angular momentum integral is

$$m_{12}\,\mathbf{r}_{12} \times \mathbf{v}_{12} = \mathbf{p}\,\exp\,[-\,k_{12}\, m_{12}^{-1}\, c^{-2}\, r_{12}^{-1}], \quad (41)$$

where $\mathbf{p}$ is a constant vector given by initial conditions, but the magnitude of the angular momentum is not constant.

9. Extended N-Body Problem Energy Integral

The extended partial mutual forces may be written in the form

$$^{\#}\mathbf{f}_{ij} = {}^{\$\#}\mathbf{f}_{ij} + {}^{@\#}\mathbf{f}_{ij}, \quad (42)$$

$$^{\$\#}\mathbf{f}_{ij} = k_{ij}\, r_{ij}^{-3}\, \mathbf{r}_{ij}, \quad (43)$$

$$^{@\#}\mathbf{f}_{ij} = k_{ij}\, r_{ij}^{-3}\, c^{-2}\, \mathbf{v}_{ij} \times (\mathbf{v}_{ij} \times \mathbf{r}_{ij}), \quad (44)$$

The equations of motion formally have the same form Eq. (27). Since $\mathbf{v}_{ij}.\ [\,\mathbf{v}_{ij} \times (\mathbf{v}_{ij} \times \mathbf{r}_{ij})] = 0$, the energy integral in central or mutual quantities has the forms

$$(1/2)\,\textstyle\sum_i m_i\,\mathbf{v}_i.\ \mathbf{v}_i + \sum_i {}^{\$}\mathbf{f}_i.\ \mathbf{r}_i = q \quad (45)$$

$$(1/4)\,\textstyle\sum_i \sum_j m_{ij}\,\mathbf{v}_{ij}.\ \mathbf{v}_{ij} + (1/2)\sum_i \sum_j {}^{\$}\mathbf{f}_{ij}.\ \mathbf{r}_{ij} = q, \quad (46)$$

$$^{\$}\mathbf{f}_i = \textstyle\sum_j {}^{\$\#}\mathbf{f}_{ij} \quad (47)$$

$$^{\$}\mathbf{f}_{ij} = m_{ij}\,({}^{\$}\mathbf{f}_i / m_i - {}^{\$}\mathbf{f}_j / m_j) \quad (48)$$

By comparing Eqs. (47), (48) with Eqs. (8), (9) the inverse square force law extension hypothesis suppresses at high speeds the scaling invariance given by Eqs. (30)-(32), but leaves the sum of the central and mutual kinetic and potential energies constant. Moreover, Coulomb's law does not offer any stable solutions for equally charged elements. However, this is not true for the force law extension hypothesis given by Eq. (36). At high mutual velocities comparable to the speed of light, both repulsion and attraction occur.

10. Helium-2

The simulation of two protons [3] orbiting around a common barycenter, and two electrons symmetrically dancing back and forth, suggests that a hypothesis of the Coulomb's law extension may not be futile. The initial conditions of the 4-body motion simulation in SI units:

m_i	q_i	r_x	r_y	r_z	v_x	v_y	v_z
m_p	q_p	- 1.535e-19	0	0	0	1.5e8	0
m_p	q_p	1.535e-19	0	0	0	- 1.5e8	0
m_e	q_e	- 1.782e-18	0	0	0	- 1.5e8	0
m_e	q_e	1.782e-18	0	0	0	1.5e8	0

$m_p = 1.6726\text{e-}27$ kg,

$m_e = 9.1095\text{e-}31$ kg,

$q_p = -\,q_e = 1.60219\text{e-}19$ C

11. Rosette Shape

The European Southern Observatory's huge telescope located in Atacama Desert in Chile, from its 27 years of effective observation on the star named S2, orbiting a massive black hole in our galaxy, gave awesome data that we never had before [4]. What makes it so important is its orbital specialty around Sagittarius A*. Data from the ESO telescope was analyzed to finally find that S2 orbits Sagittarius A* in

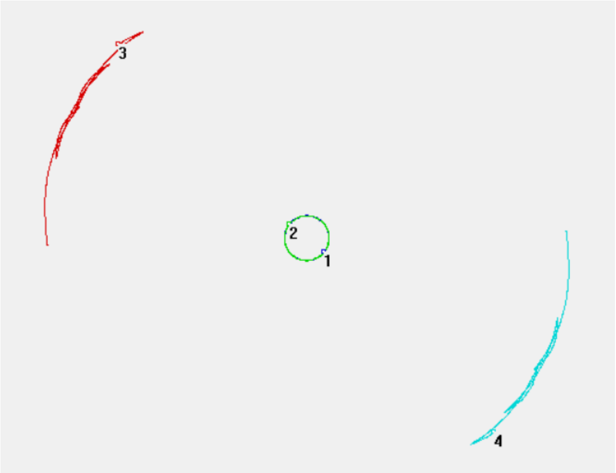

Fig. 1 Simulated motion of 2 protons and 2 electrons.

16.05 years, and follows a motion from orbit to orbit, resulting in a rosette-shaped motion around the black hole. As of July 2022, the orbital period of the star S4716 was the shortest known of any star in the Milky Way galaxy. It orbits the Sagittarius A* in 4.0 years, on an elliptical orbit with an eccentricity of 0.75 [5]. The parameters of the simulation of the motion of stars S2, S4716:

Sagittarius A*	m_1 = 8e36 kg
Star S2	m_2 = 2e31 kg
Star S4716	m_3 = 2e31 kg
minimal distance	$r_{12}(0)$ = 1.8e13 m
minimal distance	$r_{13}(0)$ = 1.496e13 m
maximal speed	$v_{12}(0)$ = 7.47e6 m/s
maximal speed	$v_{13}(0)$ = 7.9041e6 m/s
step	t = 315.576 s

The simulation does not give a nice-looking rosette shape, because the angular shift from orbit to orbit is about 0.2 and 0,16 deg/orbit, Fig. 2.

The scaling given by Eq. (30)-(32) for s = 10 highlights the tendency to angular shift of pericenters, as seen in Fig. 3. The initial conditions for scaling:

minimal distance	$r_{12}(0)$ = 1.8e11 m
minimal distance	$r_{13}(0)$ = 1.496e11 m
maximal speed	$v_{12}(0)$ = 7.47e7 m/s
maximal speed	$v_{13}(0)$ = 7.9041e7 m/s
approximation time step	t = 0.315576 s

General Relativity predicts that bound orbits of one object around another are not closed, as in Newtonian Gravity, but precess forwards in the plane of motion [6].

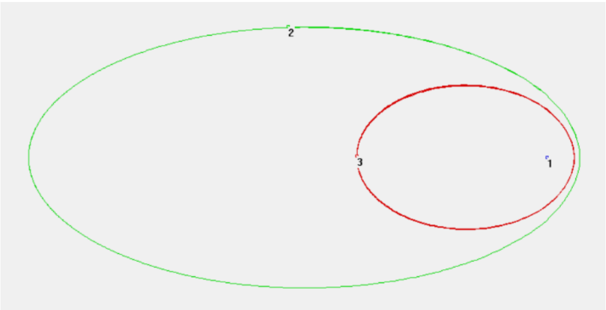

Fig. 2 Trajectory simulation by S2, S4716 stars data.

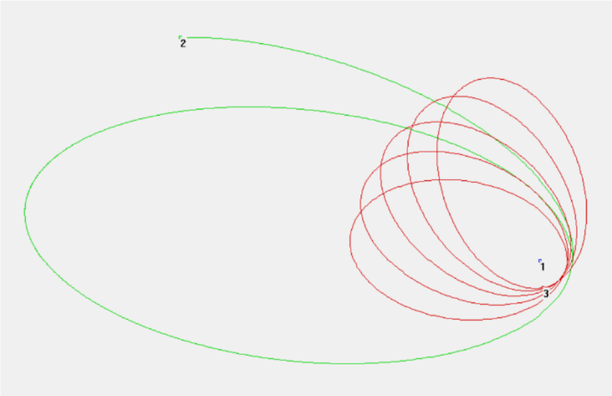

Fig. 3 Rosette simulation by S2, S4716 stars data scaling.

The above rosette shape simulations yield the opposite sense of angular shift with respect to orbital motion.

12. Conclusion

The generalization of the classical n-body problem formulation has been introduced. Mutual force between any pair of point-mass objects has been defined hypothetically as partial with no regard on other pairs. The relation between partial and resultant forces has been derived through central forces with respect to the common barycenter of all point-mass objects under consideration. In principle, everything is related to everything and it is only a question of what is already negligible and what is not. Decomposition into localities suggests a possible disintegration of globality into relatively independent localities. Experiments suggest that two identical solar systems several light-years apart can be considered relatively independent. The mutual inertia and the partial versus resultant mutual forces concept have been introduced. The dynamic model of the mutually interacting localities has been developed on the very general basis admitting any kind of the partial mutual force interaction in accordance with the law of action and reaction. Thus, any kind of motion dynamics from the micro up to the macro systems may be generated as well as simulated and animated by the presented point-mass model.

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