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Bootstrapping Students' Emergent Ideas: Case Study of Students' Guided (Re)Invention of Abstract Math Concepts

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As a collective, the undergraduate mathematics teaching faculty resist abandoning the dominant mode of lecturing despite a widely held belief that it is ineffective to learning. This design-based research problematizes the homogeneous lecture-style teaching and advocates for student-centered inquiry based learning in undergraduate mathematics classes. Grounded on Realistic Mathematics Education, Teaching for Robust Understanding framework, and Variation Theory of Learning, I propose a set of design principles to support students' guided generalization of abstract concepts through a sequence of well-articulated examples. Results show that student-generated vocabularies can bootstrap their collaborative inquiry, and I operationalized five steps that characterize the bootstrapping process. This study suggests that facilitating collaborative inquiry demands teachers to proactively look for ways to connect students' intuitive, informal grounds of understanding with the formal, to-be-learned ideas as well as capitalizing on student-generated resources as bearing powerful potential for students to selectively modify and reuse in concert with their emerging goals and mathematical tools.

Keywords: mathematics education, inquiry based learning, design based research

Introduction

It is well documented that lecturing is the dominant instructional mode in undergraduate mathematics classes (Davis & Hersh, 1981; Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016; Lutzer, Rodi, Kirkman, & Maxwell, 2007; Speer, Smith III, & Horvath, 2010), despite a widely held belief among educators and researchers that it is ineffective and it does not reflect the nature of how people learn or do mathematics (Dreyfus, 1991; Kyle, 1997; National Academy of Science et al., 2007; National Research Council, 1996; National Science Foundation, 1996). Many scholars have criticized that lecturing minimizes student participation (Artemeva & Fox, 2011; Lew et al., 2016) and that the predominance of definitions, theorems, and proofs in undergraduate lectures (Davis & Hersh, 1981; Dreyfus, 1991) leaves little room for engaging students' informal ways of thinking about and understanding mathematical concepts (Fukawa-Connelly, Lew, Mejia-Ramos, & Weber, 2014). These criticisms are grounded in evidence from educational research that (a) active student participation results in more robust learning outcomes as compared to passive listening (Chi & Wylie, 2014; Freeman et al., 2014), specifically in mathematics (Braun et al., 2019; Kogan & Laursen, 2014) and for underrepresented groups of students (Kogan & Laursen, 2014); (b) explicitly building on students' informal, intuitive understanding of to-be-learned concepts facilitates deep learning (Abrahamson, Dutton, & Bakker, 2021; Dewey, 1944; Gravemeijer, 1999; Kamii &

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Dominick, 1998; Karmiloff-Smith, 1988); and (c) fostering student agency and engagement in classroom discourse contributes to their more productive disciplinary identity and habits of mind (Gresalfi, Martin, Hand, & Greeno, 2009; Kober, 2015).

Yet as a collective, the undergraduate mathematics teaching faculty have little interest in changing the status quo (Johnson, Keller, & Fukawa-Connelly, 2018; Woods & Weber, 2020). Recent results from Johnson et al. (2018) show that more than 82% of 131 mathematicians across institutions believe "lecture is the best way to teach"; the study also suggests that "what [the] instructors think is best for student learning (students doing mathematical work in class) is not happening with any frequency" (2018, p. 283). With a widespread mismatch between instructors' beliefs about student learning and students' actual learning experiences in undergraduate mathematics classrooms, it is imperative that the field of mathematics education develops effective means to both penetrate through the resisting minds and support them with research-based alternatives that take into account students' prior knowledge, center on students' emergent ideas, and demand that students actively grapple with course materials in the classroom. This design-based research is an attempt to contribute to the collective effort of scholars in problematizing the homogeneous lecture-style teaching and the lack of active sense-making therein (e.g., Viirman, 2015; Zarkh & Geisler, 2021) and in advocating for student-centered pedagogical alternatives such as inquiry based learning (Hayward, Kogan, & Laursen, 2016; Laursen & Rasmussen, 2019) in undergraduate mathematics classrooms.

Design Problem

Abstract algebra, an upper-division course required of mathematics majors across countries, is typically believed to be one of the most troublesome undergraduate courses (Dubinsky Dautermann, Leron, & Zazkis, 1994; Hart, 1994; Selden & Selden, 1987). The axiomatic, abstract, and complex nature of the content accounts for most of the difficulties students encounter in this course (e.g., Melhuish, 2015). Prior to taking abstract algebra, many students have little experience dealing with axiomatic systems (Dubinsky et al., 1994), adding additional layers of complexity in learning as axioms together with definitions (Edwards & Ward, 2004) and proofs (Hart, 1994; Selden & Selden, 1987) are the key focuses of the course. Objects in abstract algebra are primarily (if not exclusively) discussed and argued metaphysically (Hazzan, 1999; Leron, Hazzan, & Zazkis, 1995), and understanding even a basic concept requires a complex coordination of a number of complicated ideas (Leron et al., 1995). With a high level of abstraction and a well-documented reputation for difficulty, abstract algebra is thus an extraordinarily suitable content area for investigating how students make sense of and generalize abstract math concepts.

As an accomplished novice, I have finished the honors sequence of abstract algebra (a total of three courses) at a major public university and was concurrently taking the graduate level abstract algebra course while engaging in this design-based research. When I was first introduced to abstract algebra, it took me four weeks before I was able to see any connections between the course materials and my other mathematical knowledge and experiences—it took me even longer to be able to leave the formal plane and reason about course objects intuitively. And it was not just me who experienced an unanticipated difficulty in understanding abstract algebra; among the 38 students who enrolled in the honors abstract algebra course with me in Fall 2019, 25 dropped out before the final exam. It is important to note that honors courses are not required for graduation; students who choose to take honors courses typically plan to apply for PhD programs in mathematics—and yet we shut the door for many aspiring mathematicians not because they cannot learn abstract algebra well, but because of the traditional approach of lecturing abstractions, superficially going over examples, and expecting students to be able to identify and prove abstract properties by way of yet another abstraction without engaging students' intuitive understanding fails them. It is thus critical that we advance our collective understanding of how students make sense of abstract concepts in abstract algebra and ground our pedagogical innovations on students' agentic sense-making processes.

For the purpose of this study, I will focus on how students develop an intuitive background knowledge around and (re)invent the generalization of the concept of a group. A group is typically the first concept students learn in abstract algebra, and the way it is introduced in textbooks often triggers students' feeling of inadequacy and intimidation. Figure 1 shows a textbook definition of a group by Dummit and Foote (2004, pp. 16-17). It is my hypothesis that by leveraging students' prior knowledge and ways of thinking as well as providing adequate background and motivation for the concept, students will experience the learning of a group as a sense-making endeavor and come to internalize productive self imagines as agentic, powerful mathematical thinkers who can (re)invent meaningful concepts just like mathematicians.

Definition. A group is an ordered pair $(G, \bigstar)$ where G is a set and $\bigstar$ is a binary operation on G satisfying the following axioms:

1. There exists an element e in G, called the identity of G, such that for all $a \in G$ we have $a \bigstar e = e \bigstar a = a$.
2. For each $a \in G$ there is an element a^{-1} of G, called an inverse of a, such that $a \bigstar a^{-1} = a^{-1} \bigstar a = e$.
3. $(a \bigstar b) \bigstar c = a \bigstar (b \bigstar c)$ for all $a, b, c \in G$. In other words, $\bigstar$ is associative.

Figure 1. Textbook definition of a group.

Previous Solutions

In this section, I will present some seminal research on the teaching and learning of abstract algebra from the "New Math" era as well as more recent literature. The goal is to identify productive aspects of existing studies while critically examining their limitations. Melhuish (2015) characterizes the majority of the literature as falling into two categories: investigating student misconceptions and exploring instructional innovations. Examples of the first strand include studies that examine common learning difficulties with proof (Hart, 1994; Selden & Selden, 1987; Weber, 2001), abstraction (Hazzan, 1999), concept complexity (Leron et al., 1995), and process-object duality (Dubinsky et al., 1994). In particular, Hart (1994) finds that "the amount of academic experience with abstract algebra does not necessarily reflect the level of understanding" (p. 56).

Scholars from the "New Math" movement (Branca, 1974; Dienes & Jeeves, 1965; Lant, 1980) have explored a number of instructional experiments in which students are presented with concrete instantiations of the concept of a group. However, Brown DeVries, Dubinsky, and Thomas (1997) point out that these studies typically offer information about younger students' (elementary/middle grades) abilities to "deal with the specific examples, but do not address the learning of the underlying conceptual mathematics" (p. 190). Despite the criticism, Steiner's (1966, cited in Veith & Bitzenbauer, 2022, pp. 4-5) heuristics for establishing algebraic structures while making the concept engaging are worth mentioning: (1) how many models in the direct mathematical periphery tap the algebraic structure, and (2) which of those algebraic structures allow for an adequate (in relation to school mathematics), deductive approach as well as exploratory approach? Building on similar principles, Griesel (1965, cited in Veith & Bitzenbauer, 2022, p. 5) proposes that the following groups are suitable for introduction: dihedral groups; frieze groups; cyclic groups; additive groups of integers, rational numbers, real numbers; multiplicative groups of positive rational numbers; orthogonal groups.

Freudenthal (1973a) problematizes the traditional (and still widespread!) instructional approach of starting with abstract definitions and superficially going over examples in undergraduate mathematics classrooms; he believes that the educational process should resemble the actual invention process and move from the particular to the general, not vice versa. Freudenthal advocates for introducing groups as systems of automorphisms (roughly, compositions of structures that map objects onto oneself) so that group axioms could be verified conceptually rather than algorithmically. Similarly, Burn (1996) argues for starting an abstract algebra course with an informal investigation of permutations and symmetries, an approach that has been espoused by several recent researchers (Larsen, 2009; Veith & Bitzenbauer, 2022). In a similar spirit, Wilfried (1977, cited in Veith & Bitzenbauer, 2022, p. 3) puts forward three fundamental rules for introducing abstract algebra:

1. Algebraic structures must not be introduced in a formal way or independent of tangible models. The axiomatic nature can be discussed if an opportunity presents itself, but it is of secondary importance.

2. Mathematical terms may only be introduced if they can be demonstrated or illustrated on the basis of already existing terms.

3. The goal of abstraction is only possible if the underlying concepts have been studied thoroughly.

Dubinsky and colleagues pioneer investigations into how students develop their understanding of major group-theory concepts using their Action, Process, Object, Schema (APOS) framework (Dubinsky 1997; Dubinsky et al., 1994). Building on this analysis, they explore an instructional approach that invites students to code group axioms and use computer programming as a tool to learn and determine whether a given system forms a group (Leron & Dubinsky, 1995). While one may argue that programming language is similar to the formal definition in that both require given systems to satisfy certain axioms, Dubinsky and colleagues' instructional approach introduces additional prerequisites for learning the concept of a group that may not be necessary. In addition, the authors analyze exam results and interviews without taking into account students' thinking in the programming process; nor did they document significant differences between their instruction and other approaches (Brown et al., 1997). Notwithstanding, it is important to keep Dubinsky, Dautermann, Leron, and Zazkis's (1997) caution in mind: although the concept of a group can be seen in specific examples, examples themselves do not address the challenge of abstracting a generic group notion or learning to perform calculations therein.

With respect to students' learning experiences, Freudenthal (1973a) proposes measuring their progress in understanding algebraic properties using three milestones: (1) an algebraic property emerges implicitly from students' intuitive activity with an example structure, (2) the property appears explicitly in students' general description, and (3) students repurpose the property as a lens to classify other structures (cited in Veith & Bitzenbauer, 2022, p. 9). In a similar vein, Simpson and Stehlikova (2006) characterize five shifts in attention as students move from concrete examples to more abstract reasoning: (1) view the elements of a set as objects to be operated on, (2) attend to the interrelationships between set elements, (3) focus on names and symbols that define the underlying abstract structure, (4) perceive other examples of the general structure, and (5) move to a formal system and derive consequences. Simpson and Stehlikova (2006) note that the first shift involves reification (Sfard, 1991), the second signifies a shift of attention from particular objects to the interrelationships between objects, and the last three "involve relating the definitional properties of the (teacher-defined) abstract algebraic structure with the interrelationships noted in the initial example (and in subsequent instances of the structure)" (p. 352).

More recently, Larsen and colleagues initiate a group-theory curriculum where students build on their informal knowledge and reinvent formal concepts (Larsen, Johnson, & Bartlo, 2013). Underlying the curriculum is Larsen's (2009) local instructional theory for the guided reinvention of the group and isomorphism concepts using the Realistic Mathematics Education framework, particularly the heuristic of guided reinvention and emergent models (Gravemeijer, 1994; 1999). Larsen's local theory takes as its starting point an exploratory activity in identifying and symbolizing the set of symmetries of an equilateral triangle. As students begin to analyze combinations of pairs of symmetries, the group structure emerges as a model of the students' mathematical activity. The students then move from manipulating geometric figures to algebraically analyzing combinations using their set of rules, which may include axioms featured in the definition of a group as well as relations specific to the symmetries of an equilateral triangle. The students are then guided to simplify their rules to include only essential ones and to ensure its completeness by including non-obvious rules (e.g., the existence of inverse). In the last phase, the students analyze other systems and try to formalize a group concept using the shared properties of different systems.

In the same spirit, Veith and Bitzenbauer (2022) espouse Brown et al.'s (1997) suggestion of using an experimental, hands-on approach to teach the concept of a group. Very similar to how Larsen (2009) starts his instructional sequence, students participating in Veith and Bitzenbauer's learning activity begin by physically manipulating an equilateral, acrylic-glass triangle and exploring ways to map the triangle onto itself. Veith and Bitzenbauer scaffold the first step of abstraction by numbering all vertices of the triangle for the students and the authors specifically choose transparent materials so that students can see the numbers even when they flip the triangle. Once the students find all possible mappings, they are invited to describe the mappings geometrically and symbolically, after which they are encouraged to record their descriptions in a tabular. To highlight to the students that geometric transformations are not commutative in general, Veith and Bitzenbauer ask students to compute and compare different combinations of mappings. If time permits, the students are challenged to express each mapping using only two elements: rotating $120°$ counterclockwise and flipping the triangle along a fixed axis. This last activity provides students with an opportunity to explore the group structure and discover two fundamental "atoms" (or elements) of the system.

My concern about starting with symmetries of a geometric figure and spending much time analyzing emergent rules in this context (which is very demanding!), is that students may find themselves busy sorting out the nitty-gritty details of the basic calculations of the unfamiliar system and run the risk of losing a sense of the forest. While it's a good mathematical practice to try and characterize essential features, we have to admit that most of students' common observations may be specific to the triangle example, as they don't yet have other examples to help them focus their attention. So when guiding students to simplify and complete their set of rules using the triangle example, it may be difficult—in the eyes of students—to distinguish rules contingent on the specific example and rules that are generalizable. It is important to note that if at this point teachers pitch in and impose their normative view without fully justifying why, for example, the non-obvious rule of inverse is necessary, then they run the risk of giving away the core of students' (re)invention. Moreover, although prescribing symbols to label vertices can facilitate discussion across different groups of students and foreshadow the standard way of writing symmetries of geometric figures, I am concerned that it leaves little room for Veith and Bitzenbauer (2022) to see students' intuitive, informal ways of making sense of and representing the situation. With respect to Larsen's (2009) theoretical perspectives, while the Realistic Mathematics Education framework offers many insights into bridging students' informal ideas and the to-be-learned knowledge and guiding students' reinvention process, I suspect that it does not provide adequate explanations for *how* students transition from having models of situated activity to conceptualizing models for more general activity.

Cognitive Domain Analysis

So what exactly is difficult in learning the concept of a group? As noted by Melhuish (2015), the compactness of formal definitions in abstract algebra often obscures the complexity in learning new concepts that "students have yet to develop the necessary mental constructs" (p. 25). Indeed, the concept of a group is complex in that it is made up of some abstract sub-constructs: sets, binary operations, identities, inverses, and associativity. Understanding all the sub-constructs is not a trivial task; but in order to understand the concept of a group, students necessarily need to possess a robust understanding of all the sub-constructs (Dubinsky et al., 1994). Second, since a group is typically the first mathematical concept students encounter that is defined solely by abstract properties (Hazzan, 1999), the level of abstraction and the unfamiliarity with axioms contribute to students' difficulty in envisioning a mathematical object from the definition. Third, coordinating different sub-constructs, especially ones that students have not yet seen being discussed together, raises a significant challenge for many students (Leron et al., 1995). As noted by Dubinsky et al. (1994), students have a temptation to treat groups as primarily sets of discrete elements; it is only after they encounter situations where their initial conceptualization is not sufficient that they may start to coordinate both operations and the group elements into their mental image of a group. Fourth, students may have difficulty building intuitions around the concept in order to complement the formal definition and internalize it as a tool for future problem solving. Results from Weber and Alcock (2004) suggest that many undergraduate mathematics students equate the formal definition as equivalent to their intuition and struggle to leave the formal space. In contrast to the undergraduates who over-rely on symbol-pushing strategies and are often unable to provide valid proofs or correct evaluations of mathematical statements, the graduate counterparts are found to be able to navigate flexibly between abstract and concrete representation using their intuitions (Weber & Alcock, 2004). Last but not least, students may experience difficulty conceptualizing groups as agents that can act on objects and as subjects upon which actions can be applied (by other groups), perhaps due to a lack of experience seeing mathematical objects having their own agency. Since group action is a deep topic that involves other group-theory concepts, it does not fit with the current study whose participants just start to learn about a group. So I will leave the last aspect of difficulty to future studies and only address the first four aspects.

Theoretical Frameworks

In this section, I will present some theoretical frameworks that have significantly informed this study: Realistic Mathematics Education, Teaching for Robust Understanding framework, and the Variation Theory of Learning. They are grounded on the Constructionist (Papert & Harel, 1991) assumption that students are active creators and transformers (as opposed to passive recipients) of knowledge and that students construct new knowledge from what they already know (Blumenfeld, Marx, Patrick, Krajcik, & Soloway, 1997; Bransford

Brown, & Cocking, 2000). All taken together, they provide focus and direction to the design of my proposed learning activity that will be discussed in detail in the Design Solution section.

Realistic Mathematics Education (RME) developed at the Freudenthal Institute in the Netherlands is rooted in Freudenthal's (1971; 1973b) idea of mathematicization and his interpretation of mathematics as a human activity. Freudenthal believes that mathematics education should support students to reinvent mathematics through organizing and mathematizing situations or mathematical relations that have substance to them (Cobb, Zhao, & Visnovska, 2008). He emphasizes that an instructional sequence should be *experientially real* to students so that they can engage in meaningful mathematics immediately (Gravemeijer, 1990; Streefland, 1991). It is in this sense that the approach is called Realistic Mathematics Education. Mathematization is a key process in RME for several reasons: (a) it is a major form of professional practice, (b) it provides a powerful tool for students to make sense of the world, and (c) it fosters student agency and facilitates students' reinvention of mathematical ideas starting from their informal, intuitive understandings of given situations. RME places students' active sense-making of mathematics at the center of the design process and it frames the fundamental design challenge as that of supporting the emergence of mathematical meaning in students (Cobb et al., 2008). A major goal of mathematics education in the eyes of Freudenthal and other RME scholars is to support students' mathematical learning as a process of guided reinvention (Cobb et al., 2008; Gravemeijer, 1994; 1999), which Gravemeijer and Doorman succinctly characterize as to "allow learners to come to regard the knowledge that they acquire as their own private knowledge, knowledge for which they themselves are responsible" (1999, p. 116).

An implicit assumption underlying the RME framework is that instructions should explicitly tap into students' prior knowledge and experiences as well as their intuitive, emerging ways of understanding the world. According to Cobb et al. (2008), the immediate goal for RME designers is that students' interpretations "lead to the development of informal ways of speaking, symbolizing, and reasoning across a range of initial instructional activities" (p. 108). National Research Council (2001) refers to the alignment between instructional activities and students' existing intuitions as "horizontal coherence", and a common RME strategy is to invite students to create and elaborate symbolic models of their informal mathematical activity, which may involve drawing diagrams, making tables, developing informal notations, and many more (Cobb et al., 2008). With appropriate scaffolding, students' *models of* their informal mathematical activity can be transformed into *models for* more general mathematical activity in other situations (Gravemeijer, 1999; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). Cobb et al. (2008) note that the approach of building up from students' intuitive, informal ideas to normative representations typically demands teachers to judiciously plan instructional activities and when appropriate, revoice students' ideas and introduce semiotic tools to support their development of more sophisticated reasoning. The introduction of new representations can highlight students' resources that are useful for achieving their current mathematical goals and can serve as powerful tools for future communication and problem solving (Gravemeijer, 1994).

Van den Heuvel-Panhuizen and Drijvers (2020) characterize for RME an *activity principle* that "mathematics is best learned by doing mathematics" (pp. 522-523). This fits with the Constructivist stance that advocates for active learning and it also aligns with Enactivism (Hutto, 2019; Hutto, Kirchhoff, & Abrahamson, 2015; Varela, 1999) that theorizes knowing and learning as situated doing (Gallagher, 2005; Abrahamson et al., 2021). There are a number of RME scholars (e.g., Eyrikh, Fishman, Bazhenov, Markova, & Pitsuk, 2019; Larsen, 2009; Veith & Bitzenbauer, 2022) who have espoused a hands-on, experimental approach to teaching mathematics. These lines of work are grounded in educational research that exploring and physically interacting with mathematical objects can (a) elicit students' practical epistemologies (Sandoval, 2005) about mathematics and their influences on their learning, (b) help students learn and experience mathematical ideas in new ways (Abrahamson, 2019), (c) promote students' (intrinsic) motivation and engagement in learning relevant ideas and contribute to better school achievement (Schiefele, Krapp, & Winteler, 1992; Wigfield, Eccles, Schiefele, Roeser, & Kean, 2006), and (d) support an experiential basis for learning and foster a deeper understanding of the contents (Veith & Bitzenbauer, 2022; Weber & Larsen, 2008). The RME approach of doing mathematics also relates to Problem Based Learning (Hmelo-Silver, 2004; Savery, 2015) where students learn through facilitated problem solving that involves identifying useful information, making connections between various ideas, applying knowledge to unfamiliar situations, and reflecting on the effectiveness of chosen strategies. Wilson (1996) delineates a set of pedagogical heuristics to facilitate active learning through problem solving: (1) anchor all learning activities to a larger task, (2) support students to develop ownership for the content and the problem solving process, (3) design authentic tasks to match with students' cognitive abilities, (4) design learning environments to support and challenge student thinking, (5) encourage testing ideas against different views and alternative contexts, and (6) provide opportunities for and support reflections on the contents learned as well as the learning processes (pp. 137-140).

Similar to RME, the Teaching for Robust Understanding (TRU) framework (Schoenfeld, 2013; 2014; 2017; 2019) positions teachers as master craftsmen (as opposed to authorities who teach "at" students) with a goal of creating powerful learning environments where agentic, knowledgeable, and flexible mathematical thinkers and problem solvers emerge naturally. Essentially, TRU suggests that the quality of a learning environment depends on the extent to which students have opportunities along five dimensions: (1) the richness of mathematical core ideas and practices, (2) the level of cognitive demand and students' sense-making opportunities, (3) meaningful and equitable access for all students to central concepts and practices, (4) opportunities to build positive disciplinary identities through presenting, discussing, and refining ideas, and (5) the responsiveness of the environment to student thinking (Schoenfeld et al., 2023). A combination of knowledge (including concepts and tools), practices, disciplinary orientations, and habits of mind forms the basis of Dimension 1, The Discipline. Dimension 2, Cognitive Demand, highlights the role of productive struggle in student learning (Stein & Smith, 1998; Hess, 2006) and suggests that students be provided with opportunities to stretch their current understandings. Recommended instructional strategies for supporting this dimension include providing clarifications and scaffolding (through, for example, heuristic advice, raising issues, suggesting approaches, etc.) without telling students what to do (Schoenfeld, 2019). Dimension 3, Equitable Access to Content, is built upon research findings that suggest effective classrooms necessarily entail meaningful participation by all students (Boaler, 2008; Cohen & Lotan, 1997; Schoenfeld, 2002). Dimension 4, Agency, Ownership, and Identity focuses on the extent to which students have opportunities to generate and share ideas as well as the extent to which student contributions are recognized and supported as part of classroom routines. The core question for this dimension is, "What opportunities do all students have to see themselves and others as proficient disciplinary thinkers, to grapple with challenges and construct new understandings, to build on others' ideas, and demonstrate their understandings?" (Schoenfeld & the Teaching for Robust Understanding Project, 2016). Dimension 5, Formative Assessment, involves orchestrating classroom activities to reveal students' emerging ideas and using the information gathered about student understanding to inform subsequent classroom activities (Black, Harrison, & Lee, 2003; Shepard, 2000).

Originating from the phenomenography tradition, the Variation Theory of Learning (Marton, 2015; Marton & Booth, 1997) takes as its departure the question of how to "help learners handle novel situations in powerful ways" (Marton & Pang, 2006, cited in Kullberg, Runesson Kempe, & Marton, 2017). It posits that learning implies differentiation rather than accumulation (cf. J. J. Gibson & E. J. Gibson, 1955) and that variation is a necessary component of learning. An underlying assumption of variation theory is that what students "attend to or discern are of decisive significance for how [they] understand or experience the object of learning" (Kullberg et al., 2017, p. 560). A number of studies have suggested that variation can enhance student learning (Bartolini Bussi, Sun, & Ramploud, 2013; Marton, 2015; Marton & Pang, 2013; Watson & Mason, 2006), partly because systematics variation helps make target features noticeable (Watson & Mason, 2006) and partly because students are provided with more opportunities to discern defining aspects and their relations and thus are better positioned to make sense of novel contexts (Kullberg et al., 2017). In mathematics education specifically, scholars (Al-Murani, 2007; Goldenberg & Mason, 2008; Gu, Huang, & Marton, 2004; Guo, Pang, Yang, & Ding, 2012; Huang & Li, 2017; Rowland, 2008; Sun, 2011) have explored variation in sets of instructional examples as well as how variation and invariance within and between examples help students learn mathematical structures. Several researchers have proposed that to help students understand or generalize a new concept, teachers may point to examples that share the aimed-at features but differ otherwise so that defining aspects of the concept naturally distinguish itself from the background (Marton & Pang, 2013; Kullberg et al., 2017). According to the theory, fusion of learning is made possible when dimensions of variation corresponding to multiple critical aspects are opened up at the same time (Kullberg et al., 2017). Marton (2015) puts forward a prototype for sequencing patterns of variation and invariance to facilitate learning: start with the undivided object of learning (typically a problem aimed at getting the students acquainted with the situation), then provide opportunities for comparison and contrast, followed by opportunities for generalization, and finally opportunities for fusion (p. 263).

Design Principles

Inspired by the aforementioned theoretical lenses and Van den Akker's (1999, p. 9; 2013, p. 63) approach to describing design principles (Bakker, 2018) and arguments (Easterday, Rees Lewis, & Gerber, 2016), I devise a set of principles that both inform and are informed by my four iterations of design-research cycles. While the following principles are operationalized from my work of students' guided generalization of the concept of a group, I believe they can be applied with minor modification to guide students' (re)invention of any abstract concept across grade levels and disciplines. They are as follows.

If you want to design an intervention in which multiple examples are used to support students' guided generalization of abstract concepts, you are advised to:

- Start with simple, classic exemplary cases that afford intuitive comparison, because this allows students to spontaneously conceptualize similarities (Gravemeijer, 1990; Marton & Pang, 2013).
- Add complexity by introducing examples that at first glance may seem to validate students' simplistic conceptions but upon some careful examination, some nuance begins to surface and students come to see a need to specify and refine their earlier conceptions in order to work with the similar yet distinct phenomena. These examples can highlight specific areas for the students to discern more carefully (Marton & Booth, 1997; Marton & Pang, 2013) and engage them in the professional practice of critically reflecting upon and revising conjectures.
- Challenge students to build and make sense of a complex, unfamiliar example (Larsen, 2009; Veith & Bitzenbauer, 2022) using their revised conceptualization. This will push students to apply their understanding

and in the event that they find their current understanding insufficient, they may fold back to their conceptualization in simpler examples and seek deeper understanding of the underlying structure so as to help them make sense of the idea in a more complex context (Pirie & Kieren, 1989; 1991; 1994).

- Only then encourage students to generalize common properties and formalize an abstract concept of which all considered examples are special cases. This is because at this point students will have solid grounding of the shared properties and can recognize them even in disguised forms; thus they are ready for abstraction and generalization (Dubinsky, 1997; Marton, 2015; Schoenfeld, 2014; Sfard, 1991; Wilfried, 1977).

Design Solution

In the context of guiding students to (re)invent the concept of a group, I propose to start by inviting students to identify and describe similarities between two familiar systems: the set of integers under addition and the set of rational numbers under multiplication. This is grounded on research findings that students appear to prefer working with familiar number systems when presented with objects of abstract algebra (Selden & Selden, 1987; Weber & Larsen, 2008) and that in Pre K-12 mathematics classrooms, students have ample experience working with integers and rational numbers even though the systems are often treated as sets without any structure (Veith & Bitzenbauer, 2022). Students' familiarity with the two systems makes the first task experientially real (Gravemeijer, 1990; Streefland, 1991), creating a low floor for students to start exploring binary operation (and the concept of a group) without much need for abstraction (yet). Once students have some initial conceptualization of the similarities between the two infinite systems, I introduce the modulo-5 system under addition to add some complexity to their domain of exploration and to provide them with more opportunities to discern defining features. The choice of the third system is informed by the variation theory of learning (Marton, 2015; Marton & Booth, 1997); the goal is to familiarize the students with finite systems and highlight to them that not all number systems need have an infinite number of elements.

In accordance with the Constructionist viewpoint that instructors are facilitators and not authorities of learning (Rhodes & Bellamy, 1999), I explicitly position students as experts of their own thinking and let them know that I will not lecture but will support them from the back. Informed by the TRU framework (Schoenfeld, 2014; 2019), I strive to create a safe, non-judgmental learning environment for students to freely experiment and venture not necessarily complete ideas; I also incorporate formative assessment with Socratic questioning techniques (Padesky, 1993; Paul & Elder, 2019) to elicit students' emerging understanding and probe them to think deeper about their assumptions and the properties that they use or do not use in the process. In the language of TRU, I make it explicit that all participants are expected to contribute to the exploration process and that it is going to be challenging—the cognitive demand resides in the task itself and in my constant probing to push students to think critically about their own reasoning. In the event that the students are stuck with finding more similarities from the three systems, I scaffold by asking, for example, "I am curious, can we find an element in each of the three systems such that it basically does nothing under the operation?" This question highlights to students that systems with different appearance can have very similar underlying structures and that we should not just focus on surface properties; it also scaffolds an enlightening moment when students see—after some thinking and discussion—that 0 plays the same (identity) role in addition as 1 plays in multiplication, offering them a new way to think about the familiar number systems.

After students establish some basic understanding of the similarities of the three systems, I challenge them to build and make sense of a complex, unfamiliar system involving the symmetries of an equilateral triangle. This is grounded on the observation that congruent mappings and transformations are part of secondary school mathematics curriculum (so part of what students already know) and Burn's (1996) argument that permutations and symmetries are fundamental concepts in abstract algebra. This reduces the need to construct new mathematics while providing a means to explicitly connect students' prior knowledge with the abstract concept to be (re)invented. Espousing Van den Heuvel-Panhuizen's (2020) and others' idea that students need to get their hands dirty and *do* mathematics in the learning process, I provide a physical triangle and a square cardboard with a hole in the middle that fits perfectly with the triangle (see Figure 2). This enables hands-on interaction with the mathematical content, affording students to manually manipulate the rotation and reflection of the triangle so as to perform and understand abstract group operations in this unfamiliar system. By engaging students in an embodied (Nguyen & Larson, 2015; Shapiro & Stolz, 2019) inquiry into the mathematics, my approach aligns well with the literature that a hands-on, experiential approach to abstract algebra is fruitful for learning the concept of a group (Brown et al., 1997; Veith & Bitzenbauer, 2022).

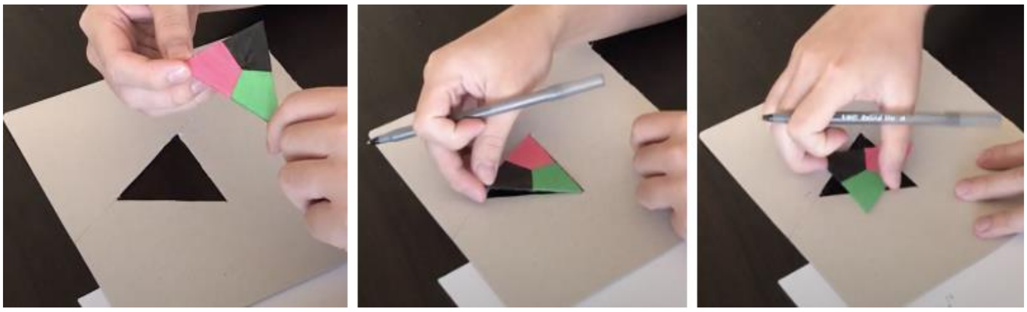

Figure 2. The triangle and the corresponding square cardboard.

In contrast to Veith and Bitzenbauer's (2022) strategy of prescribing labels of all vertices and leading students to adopt the standard representation, I try to leverage students' intuitive ways of seeing and representing the situation; thus I un-math the activity through coloring the triangle. I intentionally partition the triangle into three congruent parts and color them using pink, green, and black as shown in Figure 2. These three colors signify that the three vertices of the triangle are distinct and that when the students rotate or flip the triangle, they should keep track of the position of each vertex. I also color the back face of the triangle to ensure that the triangle looks exactly the same when it is flipped. The provided materials invite students to mathematize—identify, describe, and symbolize—elements in this system using their own repertoires of language, offering opportunities for them to actively construct their mathematical reference and hence develop their sense of agency and ownership over the exploration process. This connects back to the dimension of Agency, Ownership, and Identity in the TRU framework.

Bearing in mind the RME heuristic of guided reinvention (Gravemeijer, 1994; 1999), I invite the students to come up with their own ways of defining a group operation for the triangle system as opposed to just telling them what the operation looks like. Knowing that this is a challenging task, I scaffold by first inviting students to reflect on the previous examples and highlighting to them that all considered operations take two elements as inputs and result in a one-element output. This sets the general structure of their to-be-defined operation. Then I suggest reviewing the operation of the also-finite modulo system using a group table as it may place the students in a better position to define an operation in the new example. The introduction of the table offers a semiotic tool for the students to learn and define a group operation concretely by showing the results of all possible operation outcomes; it also foreshadows the professional practice of using a Cayley group table (Cayley, 1854) to visually represent the structure of a finite group. The left table in Figure 3 shows a to-be-completed modulo-5 group table I use to stimulate a discussion around and invite students' appropriation of the tool. Students' engagement with the modulo-5 group table reveals much of their understandings about binary operations; the patterns they notice about the modulo-5 group table (e.g., each element appears exactly once in each row and column) also serve as enabling constraints (Gelman & Williams, 1998) for the students to reason about and check their emerging definition of operations in the triangle system. In the language of Pirie and Kieren (1989; 1991), I explicitly invite students to fold back to their initial understanding of binary operations using simpler examples in order to elicit their conceptualizations and provide an opportunity for them to examine and deepen their understandings.

+	0	1	2	3	4
0					
1					
2					
3					
4					

Figure 3. To-be-completed Cayley group tables for the modulo-5 system (left) and the triangle system (right).

With some observed patterns from the modulo-5 group table, students are challenged to complete a similar table for the triangle system using their self-generated representations of each element and of the operation symbol. I offer a structure as seen in the right table of Figure 3 to scaffold students' use of the tool in the new context. The demanding nature of the table-filling task (since, among others, it is challenging to keep track of the position of each vertex after some combination of rotation and reflection) invites students to physically interact with the triangle and constantly look for emerging rules to help them simplify the calculation. When students write down a result without explicitly manipulating the triangle, I probe to elicit their underlying reasoning and follow up by either affirming their insights or challenging their assumptions. An important moment in the discussion arises when students feel a need to distinguish different types of order—they don't yet know the terminology of associativity and commutativity, but the results from the triangle group table seem to challenge their earlier observation that "order doesn't matter". This opens the door for a rich discussion about different types of orders that they just begin to see and articulate, and I probe students to think more deeply about what exactly those types of order are and how they differ. Since the operation in the triangle system is not commutative, from the variation theory point of view (Watson & Mason, 2006), the triangle example provides a ground to challenge students' common assumption that associativity implies commutativity, affording students opportunities to deepen their understanding of these two notions. The distinction between associativity and commutativity is essential because (a) the Pre K-12 mathematics curriculum seldom, if at all, question why the properties hold in our number systems or discuss how the two properties are different, resulting in many students' temptation to conflate them (Melhuish & Fagan, 2018; Larsen, 2010), and (b) associativity is a defining property of groups whereas commutativity is only a common but non-defining property.

After the students work through the group table for the triangle system, I invite them to make sense of what each similar property they derive earlier looks like in this new example. This application task elicits students' understanding of important notions such as identity, inverse, and the different types of order they conceptualize earlier; it also provides opportunities for students to examine and refine their sets of defining features. The learning activity climaxes with an invitation to generalize a mathematical concept of which all considered examples are special cases, a task that students are now ready for because through the in-depth, intimate interaction with the different systems, they have developed a solid grounding upon which they can build their generalized concept. Following Freudenthal's (1973a) advice, the learning activity I propose starts with a sequence of concrete examples and only later challenges students to formalize abstractions—it resembles the actual, messy process of constructing mathematical definitions as opposed to merely presenting the final, clean results to students. At the end of the learning activity, I present a copy of a textbook definition of a group (which we see in Figure 1) and invite students to compare their definitions and the formal definition used by the current mathematical community. This is a point where students see how the whole learning activity manifests itself in the formal concept, how their intuitive representations connect to the standard representations—and more powerfully, how their constructed definition is essentially a reinvention of important mathematics.

Research Questions

Since students are given much freedom expressing their intuitive ways of thinking and representing situations, I am interested in how they make sense of the emerging mathematics and what role their self-generated vocabularies play in their collaborative inquiry. More specifically, my research question is: *Can student-generated vocabulary bootstrap their collaborative inquiry, and, if so, what is the process of this phenomenon?*

Methods

I used design-based research (Anderson & Shattuck, 2012; Bakker, 2018; Kelly, Lesh, & Baek, 2014) along with semi-structured task-based interviews (Ginsburg, 1997; Goldin, 2000) to conduct this study. According to Anderson and Shattuck (2012, pp. 16-17), design-based research is characterized by the following: (1) focusing on the design and testing of a significant intervention, (2) involving multiple iterations, (3) situating in a real educational context, and (4) resulting in practical design principles or grounded theorization, among others. In the spirit of design-based research, I iteratively conducted four design-enactment-reflection-refinement cycles (Lewis et al., 2020; Wong, Boticki, Sun, & Looi, 2011), where I developed, tested, and revised conjectures and used unanticipated phenomena to inform subsequent cycles of investigation. For each enactment, I video- and audio-recorded all participation and I collected and scanned all participants' written works after they completed the study. I approached data analysis in several phases in the reflection process: (1) I reviewed the recordings multiple times to have an overview of my data with a focus on catching unsuccessful or unanticipated moments, (2) I developed analytic memos to record my critical examination along with the corresponding time and the reasons I hypothesized to be contributing factors; I also identified emerging themes and brainstormed some potential research questions in this phase, (3) I developed fine-grain transcripts and operationalized a preliminary data analysis in an attempt to answer my emerging research questions, and (4) I took another holistic review of the recordings to evaluate whether my analysis provided a full story and if not, what were the missing parts. In total, there were six students participating in this research, none of whom had taken Abstract Algebra or had known what a group was prior to their participation. Two focal participants of this paper, J and E, were first-year undergraduates intended to major in Computer Science at UC Berkeley. They participated in this study in pairs for two hours, and the author had been a Graduate Student Instructor for their Calculus course for 13 weeks prior to their participation in the study. (Note: they did not receive any privilege in the Calculus course for participation).

Results

I reviewed my video data and transcripts multiple times from which I operationalized five key steps that characterize the process of students bootstrapping collaborative inquiry using their generated vocabularies. They are: (1) generate informal vocabulary to describe observed phenomena, (2) attempt to predict similar phenomena, (3) attempt to explain similar phenomena, (4) attempt to address conflicting ideas, and (5) notice conflation and revise initial conception. I will illustrate the five steps using the case of "order" J and E generated in the learning activity. As shown in Figure 4, J and E noticed that "order doesn't matter" across three different examples; as will become clear later, this conception provided a shared anchor for J and E to analyze and rethink what they really meant by "order". In the words of Goodwin (2013, p. 11), the student-generated vocabulary presents a "semiotic landscape with quite diverse resources" for them to decompose, reuse, and transform their initial understanding of observed phenomena into more complete, sophisticated conceptions.

Systems \ Properties	"PRIME" element that will NOT affect Results	Undoing #'s	Common order
$(\mathbb{Z}, +)$	0	(-)	order doesn't matter for these operations
$(\mathbb{Q}, \times)$	1	$\frac{1}{x}$	
(mod 5 ,+)	0	There's always a # that you add, you can use to undo it	

Figure 4. The vocabulary of "order".

In the second step, E used their earlier computation of 2♡6=5 to challenge J's result of 2♡6=4 in the triangle system and predicted that 6♡2 should equal 5 instead. After some careful checking of the results, they accepted that 2♡6 was indeed 4. I (abbreviated as S) scaffolded a reconciliation by suggesting that order does matter in this triangle system and probed:

S: What about parentheses? Do you remember when you are adding numbers…

E: I think it *does* matter, just because order matters, so the order we do it in will matter.

In the above exchange, E used her reasoning to explain what she thought were similar phenomena—that both order of numbers and order of operations deal with "order" and thus should not be logically independent. This signifies the third step of the bootstrapping process, namely attempting to explain similar phenomena using the generated vocabulary. To highlight to the students that E's reasoning was not valid (though plausible), I invited them to try out some numbers. After some computation, they found that 4♡(3♡2)=(4♡3)♡2, which appeared to contradict E's reasoning that "because order matters, so the order we do it in will matter". This unexpected contradiction surprised the students and motivated them to test out more cases.

J: Cool~

E: Emmm…

//**S:** Is that a coincidence? Or is it always true?

E: Hmm, [straightens her back] let's try a different one.

//**J:** Ok yeah, try a different one, try like three. Three, err, parenthesis…

As will become apparent soon, E and J were attempting to find examples to confirm E's claim that "because order matters, so the order we do it in will matter". After some effort, they finally thought that they found one such example (see Figure 5). "Even though these two [pointing to 4♡(3♡5)=3 and (4♡3)♡5=3] gave the same answer," claimed E, "this one [pointing to 4♡(5♡3)=1] was different." This conclusion marks the fourth step in the bootstrapping process, namely attempting to address conflicting ideas using (although not-entirely correct) examples.

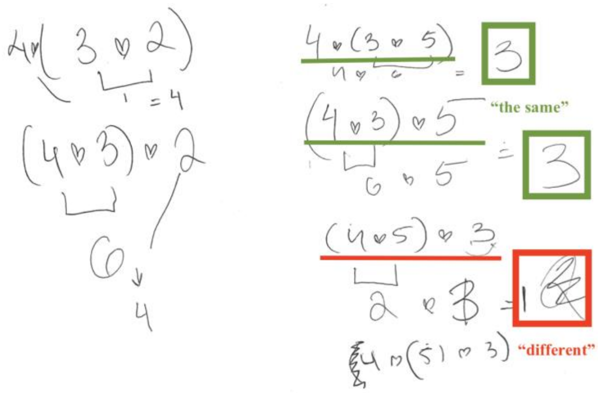

Figure 5. "Same" vs. "Different".

Sensing that this is a good opportunity to highlight some nuances underlying their generated vocabulary of "order", I scaffolded and guided them to distinguish different types of "order".

S: I see. Well, but this one you didn't just add the parentheses [pointing to (4♡3)♡5=3 and 4♡(5♡3)=1]. You also reorder the numbers [moving two parallel-and-apart hands so that they cross each other and switch the location].

E: Oh, so order matters.

//S: There're two types of orders.

//E: Yeah. Err…

S: So maybe we should fix that. That's a very good observation. So when we say order matters, we mean two things. One is like [pointing to the "Common order" column of the similarity table in Figure 5]...

E: Order matters as in the order of numbers, matter. But the order of parentheses does not matter.

E's success in distinguishing two different types of order they conflated earlier denotes a milestone in their bootstrapping process from informally describing phenomena to grasping a challenging, formal disentanglement between commutativity and associativity. This concludes the five steps of the bootstrapping process; it also signifies a major milestone in the students' sense-making of the triangle example in comparison with other discussed-upon examples.

Concluding Remarks

The focal students' notion of "order" and their critical application and revisit to their vocabulary bootstrapped their collaborative inquiry into rich mathematics. While much research is needed to examine whether the proposed bootstrapping process is valid, reliable, or comprehensive, there is reason to believe that this is a step forward into understanding students' sense-making more deeply from the bottom up. In another dimension, the results appear to confirm that my proposed learning activity facilitated the students' active sense-making about the concept of a group, shedding light on a potentially productive way of helping students generalize abstract mathematical ideas and learn traditionally-intimidating concepts better. Based on the feedback I received from the six participants, the learning activity helped them experience the enlightening, aesthetic aspect of doing mathematics because they were guided to discover important mathematics with much freedom and agency as opposed to passively taking in results from authorities. The mathematics they engaged in were meaningful to them; they were stretched with support and they felt that they owned the mathematics they (re)invented. Furthermore, the results suggest that my design provides an existence proof of using inquiry-based learning to systematically mine students' intuitions, capitalize on their emergent ideas, and grant students' authority and ownership over their intellectual process in undergraduate mathematics classrooms. One implication from this study is that when helping students generalize or (re)invent mathematical ideas, teachers may proactively look for ways to connect students' intuitive, informal grounds of understanding with the formal, to-be-learned ideas. Building on Goodwin's (2013) notion of co-operative transformation zones, the study also suggests that teachers capitalize on student-generated resources as bearing powerful potential for students to selectively modify and reuse in concert with their emerging goals and mathematical tools.

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