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Quaternion Realization of Mobile Weapon Aiming Stabilization

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Abstract: Quaternion realization of aiming stabilization of the mobile weapon is considered. Gyro compassing combined with an inclinometer is used for the measurement of the initial chassis orientation in respect of the geographical north and the horizontal plane. The same 3D gyroscope is used for the measurement of the subsequent incremental change of the chassis orientation. The goal of the proposed stabilization is to maintain the aiming geo coordinates generated by the fire control system while moving in an uneven terrain.

Key words: Quaternions, aiming stabilization, gyro compassing, inclinometer, Earth rotation compensation.

1. Introduction

Driving a vehicle on uneven terrain causes unwanted sudden changes of the observed object position within the camera's field of view. The first aim of stabilization is to minimize such changes. Current fire control systems have the separate azimuth angle stabilization and the separate elevation angle stabilization. Such 2D stabilization helps to maintain the observed object in the field of view of the camera but fails to compensate for the lateral tilt of the vehicle chassis.

The chassis platform in an uneven terrain is nonhorizontal, but the ballistic trajectory is curved in respect of the horizontal plane. The motivation is to predict the hit point in geo coordinates, while servo coordinates are continuously recalculated according to the nonhorizontal chassis and the generated geo coordinates.

The 3D gyroscope offers the angular velocity of the chassis along with the rotation of the Earth. To generate the geo coordinates with respect to the terrain, a compensation for the Earth's rotation is needed [1].

The visual subsystem of day and night vision detects and recognizes objects of interest [2]. The problem with finding objects while driving a vehicle is to keep the object of interest in the field of view of the camera. To this end, it is necessary to stabilize the direction of the camera optical axis in respect of the horizontal plane by compensating for changes in the vehicle chassis orientation. Stabilization facilitates the process of finding a target and is even an essential part of aiming while driving. The fire control system is tracking the recognized object, measures its distance, motion parameters, and predicts the hit point [3]. It automatically performs ballistic aiming at the predicted hit point before firing [4]. Stabilization allows the operator to search for targets and automatically aim even when the vehicle is leaning and changing direction to avoid obstacles. The aim of the proposed 3D automatic stabilization is to maintain the generated orientation of the optical axis of the weapon in respect of the horizontal plane even in case of the lateral tilt of the vehicle chassis.

Two degrees of freedom are sufficient to achieve the desired direction of the optical axis when tracking the target or to achieve the desired ballistic aiming. The desired direction can thus be controlled by two rotational axes of the azimuth and elevation servo systems. The third degree of freedom is the twist around the optical axis. The geo twist angle does not

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affect the direction of the optical axis, but the value is calculated for the gun-barrel parallax compensation. These are the reasons why it is necessary to differentiate between the azimuth and the elevation angle relative to the vehicle chassis platform, and the azimuth and the elevation angle relative to the horizontal plane. The vehicle chassis orientation has three degrees of freedom. An inclinometer is used to measure the initial 2D tilt of the chassis in respect of the horizontal plane. Subsequently we use a gyro compassing to measure the initial orientation in respect of the geographical north. Finally, the same 3D fiber optical gyroscope is used to measure incremental changes of the vehicle chassis orientation.

2. Representations of Orientation

The orientation of the orthonormal dextrorotatory vector base a: a_1, a_2, a_3 with respect to vector base b: b_1, b_2, b_3 is expressed by the tensor $q(a, b) = \sum a_i b_i$, where $a_i b_i$ are dyads of base vectors. It is customary to write the coordinates of the tensor in the form of a matrix. In general, the coordinate matrix of the tensor $q(a, b)$ with respect to any base c: c_1, c_2, c_3 is a real matrix 3 × 3:

$$Q_c(a, b) = [c_i.\ q(a, b).\ c_j] \tag{1}$$

We can verify that the coordinates of the tensor $q(a, b)$ in respect of the bases a and b are equal matrices and thus we can write them without index indicating the base:

$$Q_a(a, b) = Q_b(a, b) = Q(a, b) = [b_i.\ a_j] \tag{2}$$

In practical applications, the matrix $Q(a, b)$ is known as Euler-Rodrigues rotation matrix $R(f, v)$ [5]:

$$\mathrm{Q(a,b) = R(f,v) =} \begin{bmatrix} (1-C)v_1^2 + C, (1-C)\,v_1v_2 - Sv_3, (1-C)v_1v_3 + Sv_2 \\ (1-C)v_1v_2 + Sv_3, (1-C)v_2^2 + C, (1-C)v_2v_3 - Sv_1 \\ (1-C)\,v_1v_3 - Sv_2, (1-C)\,v_2v_3 + Sv_1, (1-C)\,v_3^2 + C \end{bmatrix} \tag{3}$$

where $f \in <0, \pi>$ is the angular distance of orientations a and b, $S = \sin(f)$, $C = \cos(f)$. The coordinates $v = (v_1, v_2, v_3)$, $|v| = 1$, of the rotation axis direction are the same in vector bases a and b. Particularly, for directions v = (1, 0, 0), (0, 1, 0), and (0, 0, 1), the matrix $R(f, v)$ has the form of the elementary rotation matrices:

$$\mathrm{X(f)} = \begin{bmatrix} 1, 0, 0 \\ 0, C, -S \\ 0, S, C \end{bmatrix}$$

$$\mathrm{Y(f)} = \begin{bmatrix} C, 0, S \\ 0, 1, 0 \\ -S, 0, C \end{bmatrix} \tag{4}$$

$$\mathrm{Z(f)} = \begin{bmatrix} C, -S, 0 \\ S, C, 0 \\ 0, 0, 1 \end{bmatrix}$$

A composition of orientation tensors is a scalar product of tensors:

$$q(a, c) = q(a, b)\cdot q(b, c) \tag{5}$$

A composition of real coordinates in respect of the relevant bases is a product of matrices in a reverse order:

$$Q(a, c) = Q(b, c)\ Q(a, b) \tag{6}$$

The above matrix representation of orientation has been used in Ref. [1]. The quaternion representation of orientation is based on the fact that any nonzero quaternion q defines an automorphism $b_i = q^{-1} a_i q$, where the quaternion generators 1, a_1, a_2, a_3 and 1, b_1, b_2, b_3 represent the two orthonormal right handed vector bases a: a_1, a_2, a_3 and b: b_1, b_2, b_3. We have to distinguish between the quaternion and the coordinates of this quaternion in different bases [6]. Any quaternion may be expressed as a linear combination of quaternion generators. We call the coefficients of the relevant linear combination the coordinates of the quaternion in respect of the relevant quaternion generators. If the two sets of generators a and b are arbitrary, we define the relative orientation of a in respect of b by the quaternion $q(a, b)$ in Eq. (7), where $a_i b_i$ are quaternion products:

$$2q^2(a, b) = -1 - \sum a_i b_i \tag{7}$$

In other words, quaternion $q(a, b)$ defines the automorphism $b_i = q^{-1}(a, b) a_i q(a, b)$ between the generators 1, a_1, a_2, a_3 and 1, b_1, b_2, b_3 [6]. We have to point out that quaternions $q(a, b)$ and $-q(a, b)$ represent the same relative orientation of the base a in respect of the base b.

The coordinates of orientation quaternion are the same in two vector bases, of which ordered relation is being considered. Thus we can write, analogously with Eqs. (2) and Eq. (3), for coordinates of the orientation quaternion $q(a, b)$ in base a or b:

$$Q(a, b) = R(v, f) = [c, sv_1, sv_2, sv_3] \tag{8}$$

The tuple of coordinates in Eq. (8) is the quaternionic analogy of the Euler-Rodrigues rotation matrix in Eq. (3). The coordinates $v = (v_1, v_2, v_3)$, $|v|= 1$, of rotation axis in Eqs. (3) and (8) are the same, but $s = \sin(f/2)$, $c = \cos(f/2)$, where $f \in <0, \pi>$ is the angular distance of orientations a and b. Analogously with Eq. (4), we denote elementary rotation tuples:

$$\begin{aligned} X(f) &= [c, s, 0, 0] \\ Y(f) &= [c, 0, s, 0] \\ Z(f) &= [c, 0, 0, s] \end{aligned} \tag{9}$$

A composition of orientations has the same form as Eq. (5), where the scalar product of tensors should be replaced by the product of quaternions:

$$q(a, c) = q(a, b)q(b, c) \tag{10}$$

A composition of the relevant 4-tuples of real coordinates is a quaternionic product of 4-tuples in a reverse order having the same form as in Eq. (6).

Quaternionic product of real 4-tuples is defined as:

$a = (a_0, a_1, a_2, a_3)$
$b = (b_0, b_1, b_2, b_3)$
$c = (c_0, c_1, c_2, c_3)$
$c = ab$

$$\begin{aligned} c_0 &= a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \\ c_1 &= a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 \\ c_2 &= a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1 \\ c_3 &= a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0 \end{aligned}$$

3. Established Vector Bases

EARTH: e_1, e_2, e_3	angular velocity $w = |w|e_3$
GROUND: g_1, g_2, g_3	horizontal plane g_1, g_2, $(g_1 = e_1)$
HEADING: h_1, h_2, h_3	horizontal plane h_1, h_2, $(h_3 = g_3)$
CHASSIS: c_1, c_2, c_3	chassis platform c_1, c_2
OPTICS: o_1, o_2, o_3	camera optical axis o_2

The vector base EARTH: e_1, e_2, e_3 fixed to the Earth is rotating around e_3. The vector base GROUND: g_1, g_2, g_3 is rotating around e_3 together with the Earth, but rotated around $g_1 = e_1$ by a constant angle $f_0 = \pi/2$—latitude of the battlefield. The vectors g_1, g_2 define the horizontal plane.

The vector base HEADING: h_1, h_2, h_3 may rotate an angle f_1 around $h_3 = g_3$ while h_1, h_2 lie in the horizontal plane with h_2 pointing collinearly with the longitudinal side of the chassis in horizontal position (Fig. 1). The vector base CHASSIS: c_1, c_2, c_3 is fixed to the chassis with c_2 pointing collinearly with the longitudinal side of the chassis. The unit vectors c_1, c_2 are respectively (transversely and longitudinally) collinear with the chassis platform.

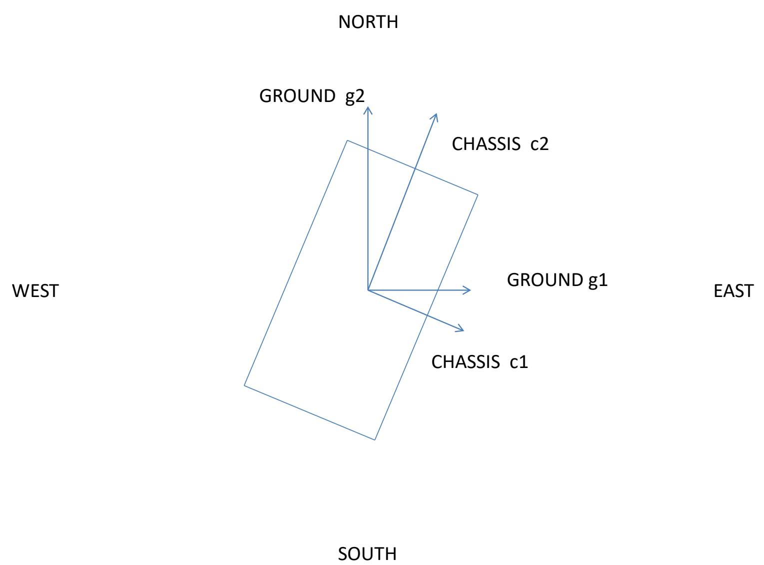

Fig. 1 The base HEADING coincides with the base CHASSIS in horizontal position.

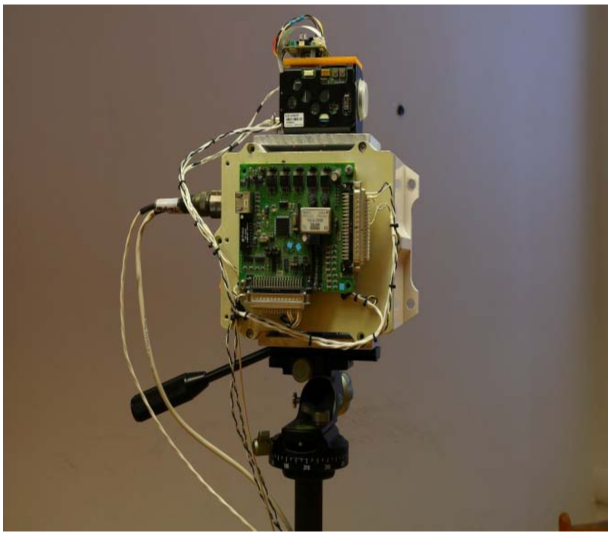

Fig. 2 Laboratory chassis positioning.

The coordinates of the chassis orientation quaternion are the subject of the measurement. Translatory motion does not enter the issue. The graphical representation is just a choice of bases location.

The vector base OPTICS: o_1, o_2, o_3 at the end of the kinematic chain of the mobile weapon is rotated around c_3 and then around o_1. Mutual orientation (OPTICS, CHASSIS) is controlled by two servo systems: azimuth around c_3 and elevation around o_1. The unit vector o_2 is representing the orientation of the camera optical axis.

4. Chassis Initial Inclination

The initial inclination of the vector base CHASSIS: c_1, c_2, c_3 with respect to the vector base HEADING: h_1, h_2, h_3 is measured by an accelerometer used in static measurement as an inclinometer. The device is fixed to the chassis platform collinearly with the base vectors c_1, c_2.

Let us denote U, V the values read from the inclinometer expressed in angular units. Based on the values U, V, we define an inclination angle f_2 and coordinates u of the horizontal inclination axis expressing the right handed rotation of the base CHASSIS in respect of the base HEADING:

$$\begin{aligned} f_2 &= \mathrm{sqrt}(U^2+V^2) \\ u &= (V, -U, 0)/f_2 \end{aligned} \tag{11}$$

Let us define the coordinates of the initial inclination quaternion by Eq. (8) with values from Eq. (11):

$$Q(\mathrm{CHASSIS}_0, \mathrm{HEADING}_0) = R(u, f_2) \tag{12}$$

The initial inclination is measured when the chassis has been stationary for at least half a minute so that the tilt measurement relative to the horizontal plane is not disturbed by acceleration (Fig. 2).

5. Gyro Compassing

Digital compass data used in Ref. [1] may not be reliable because of ferromagnetic noise. Gyro compassing is the ability of high-performance gyroscopes to determine the heading of the vehicle without external aiding. The gyroscope module contains three optical gyroscopes firmly connected to form a dextrorotatory vector base c: c_1, c_2, c_3 attached to the chassis. In an incremental measurement mode, the registers of 3 gyroscopes provide three rotations about the respective axes over a period of 40 ms since the previous measurement.

When chassis is static in respect of the Earth, the coordinates $w_c = (X, Y, Z)$ of the Earth's angular velocity vector w are measured by 3 fiber optic gyroscopes. Since its magnitude is relatively small, steady medians of at least 500 data are needed to overcome the noise (Fig. 3).

After achieving satisfactory stability of steady medians, we use initial inclination from Eq. (12) for recalculation of pure quaternion coordinates $w_c = (0, X, Y, Z)$ in vector base CHASSIS to coordinates $w_h = (0, A, B, C)$ in vector base HEADING:

$$w_h = R(u, f_2)\, w_c\, R(-u, f_2) \tag{13}$$

Finally, we calculate the initial values of angles f_0, f_1 introduced in chapter 3:

$$f_0 = \mathrm{atan2}(C, \mathrm{sqrt}(A^2+B^2); \quad f_1 = \mathrm{atan2}(B, A) \tag{14}$$

The resulting tuple of initial chassis coordinates in respect of the battlefield is:

$$Q(\mathrm{CHASSIS}_0, \mathrm{GROUND}_0) = Z(f_1)\, R(u, f_2) \tag{15}$$

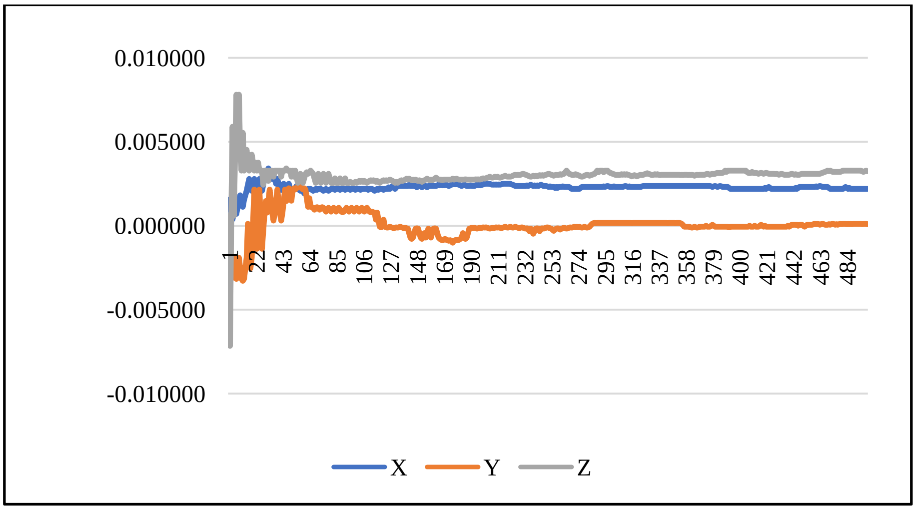

Fig. 3 The steady medians of the Earth's angular velocity coordinates after 20 s.

6. Chassis Orientation Development

The chassis orientation development is measured by the same 3 fiber optic gyroscopes being used for gyro compassing.

Let us denote X, Y, Z values read from registers of individual gyroscopes. We define the non-negative angle f and the coordinates $v = (v_1, v_2, v_3)$ of the unit vector defining the instant direction of chassis rotation axis:

$$f = \mathrm{sqrt}(X^2+Y^2+Z^2) \quad v = (X, Y, Z)/f \tag{16}$$

The non-negative angle f is the magnitude and the coordinates v define the axis of the dextrorotatory rotation of the vector base CHASSIS: c_1, c_2, c_3 with respect to its previous orientation:

$$Q(\mathrm{CHASSIS}_{i+1}, \mathrm{CHASSIS}_i) = R(v, f) \tag{17}$$

The coordinates $R(v, f)$ define the change of chassis orientation since the last measurement in the gyroscopes incremental mode. Recall that the values of coordinates are the same in bases $\mathrm{CHASSIS}_i$ and $\mathrm{CHASSIS}_{i+1}$.

7. Chassis to Ground Relative Orientation

For the purposes of stabilization, we need to continuously measure the orientation of the chassis with respect to the battlefield. Nevertheless, a 3D gyroscope mounted on the chassis does not measure the angular velocity with respect to the battlefield, but the angular velocity with respect to distant stars. Therefore, we must compensate for the Earth's rotation. Applying Eq. (10), we can write for the chain of orientation quaternions

$$q(C_{i+1}, G_{i+1}) = q(C_{i+1}, C_i)\, q(C_i, G_i)\, q(G_i, G_{i+1}) \tag{18}$$

where C = CHASSIS, G = GROUND, and $i = 0, 1, \ldots, n$-1. We have to use the reverse order of the relevant 4-tuples of coordinates like for matrices in Eq. (6):

$$Q(C_{i+1}, G_{i+1}) = Q(G_i, G_{i+1})\, Q(C_i, G_i)\, Q(C_{i+1}, C_i). \tag{19}$$

The coordinates of the initial orientation of the vehicle chassis with respect to the geographical north and the horizontal plane are given by Eq. (15). The orientation of the vehicle's chassis is subject to change due to off-road movement and Earth's rotation. Orientation changes are measured discretely in steps of 40 ms using three gyroscopes connected perpendicular to each other and attached to the chassis. The relevant coordinates in Eq. (16) and Eq. (17) measured in incremental measurement mode represent the change in chassis orientation since the last measurement:

$$Q(\mathrm{CHASSIS}_{i+1}, \mathrm{CHASSIS}_i) = R(v, f) \tag{20}$$

Orientation changes of the vector base GROUND will not be measured but calculated as a result of the constant rotation. The Earth's angular velocity vector $w = |w|e_3$ is a constant pure quaternion with $|w|$ = 0.000167123 [deg/40 ms]. And 40 ms is a period of reading the data from gyroscopes in the native incremental mode. Its 4-tuple of quaternion coordinates in vector base EARTH is $w_e = [0, 0, 0, |w|]$. The vector base GROUND: g_1, g_2, g_3 of the battlefield is dextrorotatory rotated in respect of the base EARTH: e_1, e_2, e_3 by an angle of magnitude f_0 around $e_1 = g_1$. Both bases are rotating together with the Earth. With regards to Eqs. (8) and (9), we have for coordinates w_g of pure quaternion w in vector base GROUND:

$$w_g = X(-f_0)w_eX(f_0) \tag{21}$$

The incremental change of orientation coordinates of vector base GROUND: g_1, g_2, g_3 with respect to its previous orientation is:

$$Q(\text{GROUND}_{i+1}, \text{GROUND}_i) = R(w_g/|w|, |w|) \tag{22}$$

Finally, the discrete steps of the chassis orientation with respect to the battlefield are:

$$Q(\text{CHASSIS}_{i+1}, \text{GROUND}_{i+1}) = R(-w_g/|w|, |w|)\, Q(\text{CHASSIS}_i, \text{GROUND}_i)\, R(v, f) \tag{23}$$

8. Geo Coordinates Definition

The established vector bases:

$$\text{GROUND: } g_i,\ \text{CHASSIS: } c_i,\ \text{OPTICS: } o_i \tag{24}$$

form a kinematic chain of the mobile weapon. In accordance with Eq. (6), a composition of the relevant coordinates of orientation quaternions is:

$$Q(\text{OPTICS, GROUND}) = Q(\text{CHASSIS, GROUND})\, Q(\text{OPTICS, CHASSIS}) \tag{25}$$

Let us define the geo coordinates as three Euler angles pan, tilt, twist by the equation:

$$Q(\text{OPTICS, GROUND}) = Z(\text{pan})X(\text{tilt})Y(\text{twist}) \tag{26}$$

Coordinates Q(OPTICS, CHASSIS) are given by position angles of the two servo systems: azimuth f_3 and elevation f_4. The corresponding tuple of quaternion coordinates is:

$$Q(\text{OPTICS, CHASSIS}) = Z(f_3)\, X(f_4) \tag{27}$$

The initial coordinates in Eq. (15) and the subsequent discrete steps of coordinates in Eq. (23) allow us to join Eqs. (25)-(27) into the equation:

$$Z(\text{pan})\, X(\text{tilt})\, Y(\text{twist}) = Q(\text{CHASSIS}_i, \text{GROUND}_i)\, Z(f_3)\, X(f_4) \tag{28}$$

The stabilization of geo coordinates is a process that can be turned on or off at any time. When the operator activates the stabilization, the superior control system is waiting until the conditions of the initial static measurement of coordinates $Q(\text{CHASSIS}_0, \text{GROUND}_0)$ are met. In the first step of the stabilization process, the solution of Eq. (28) with respect to unknown initial angles pan, tilt, twist is calculated. In all subsequent steps of the stabilization process, the solution of Eq. (28) with respect to unknown angles f_3, f_4, and twist is repeatedly calculated. In both cases the decomposition equation of the same type is solved, namely:

$$Z(\text{pan})\, X(\text{tilt})\, Y(\text{twist}) = Q \tag{29}$$

where $Q = Q(\text{CHASSIS}_0, \text{GROUND}_0)\, Z(f_3)\, X(f_4)$, and for $i > 0$:

$$Z(f_3)\, X(f_4)\, Y(-\text{twist}) = Q \tag{30}$$

where $Q = Q^{-1}(\text{CHASSIS}_i, \text{GROUND}_i)\, Z(\text{pan})\, X(\text{tilt})$.

Geo coordinates pan and tilt remain either constant or may track the object recognized by the FCS (Fire Control System). The servo coordinates f_3, f_4 are continuously calculated from Eq. (30) in accordance with the measured (CHASSIS_i, GROUND_i) orientations. Twisting of the optical axis o_2 does not change its direction. The value of the geo coordinate twist is calculated for the gun barrel parallax compensation.

The control of servo coordinates f_3, f_4 for constant aiming geo coordinates pan = tilt = 0 while turning the chassis from the north to east and south, and moving simultaneously in an uneven terrain, is illustrated in Fig. 4.

When the operator deactivates the stabilization, the superior control system begins to ignore data from the inclinometer and the 3D gyroscope. Subsequently, the new values of the geo coordinates are calculated from Eq. (29) with $Q(\text{CHASSIS}_0, \text{GROUND}_0) = I$ (identity matrix). Geo coordinates pan, tilt become to be equal

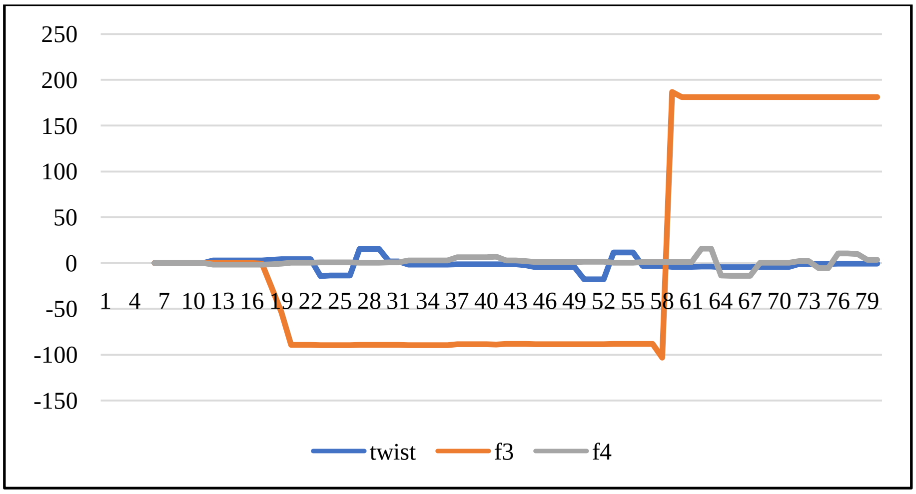

Fig. 4 Angular coordinates generated during stabilization.

to servo coordinates f_3, f_4, and twist becomes to be zero. For $i > 0$, the values f_3, f_4 remain equal to generated pan, tilt, and twist = 0 until the stabilization process remains deactivated.

It should be emphasized that 3D stabilization is a new subsystem of the superior control system. 3D stabilization of the defined geo coordinates pan, tilt does not mean keeping them at constant values, but at values generated by the superior control system, regardless of the orientation of the chassis during driving in an uneven terrain.

9. Quaternion to Euler Angles Decomposition

Any 4-tuple of unit quaternion coordinates $Q = [q_0, q_1, q_2, q_3]$ may be decomposed into the sequence of 3 elementary rotations in Eq. (9). We use the decomposition of quaternion coordinates Q to Euler angles defined by the equation:

$$Z(\alpha)\ X(\beta)\ Y(\gamma) = Q \tag{31}$$

The solution of Eq. (31) is:

$$\begin{aligned} &\alpha = r + s; && -\pi < \alpha \le \pi \\ &\beta = 2\ \mathrm{atan2}(u + v, u - v); && -\pi/2 < \beta < \pi/2 \\ &\gamma = r - s; && -\pi < \gamma \le \pi \end{aligned} \tag{32}$$

where

$$\begin{aligned} r &= \mathrm{atan2}(q_0 + q_1, q_3 + q_2) \\ s &= \mathrm{atan2}(q_0 - q_1, q_3 - q_2) \\ u &= \mathrm{sqrt}((q_0 + q_1)^2 + (q_3 + q_2)^2) \\ v &= \mathrm{sqrt}((q_0 - q_1)^2 + (q_3 - q_2)^2) \end{aligned} \tag{33}$$

10. Conclusion

An advanced original method of the mobile weapon aiming stabilization has been presented. The 2D inclinometer in combination with 3D optical gyroscope is used to measure the initial orientation of the vehicle chassis. Subsequently the same 3D gyroscope is used to measure the development of the initial chassis orientation. The original quaternion-based Earth rotation compensation algorithm was introduced to express the relative orientation of the chassis with respect to the horizontal plane and the geographical north. The geo coordinates as a subject of control are finally defined by the decomposition of orientation quaternion coordinates to Euler angles. The cooperation of the superior FCS with the stabilization module is being verified.

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