Journal of Mathematics and System Science 10 (2020)
doi: 10.17265/2159-5291/2020.02.002

Some Conjectures on the Divisor Function

Masatoshi Nakano
Kesennuma College of Technology, 1-174 Ohtougeyama Kesennuma-shi, Miyagi 988-0812, Japan

Abstract: We propose the following conjecture on $\sigma(n)$ the sum-of-divisors function: $\frac{\log(e^{\gamma} n \log\log n - \sigma(n))}{\log(e^{\gamma} n \log\log n)}$ will increase strictly and converge to 1 when n runs from the colossally abundant numbers to infinity. This conjecture is a sufficient condition for the Riemann hypothesis by Robin's theorem, and it is confirmed for n from 10^4 up to 10^{103078}. Further, we present two additional conjectures that are related to Robin's theorem.

Key words: Riemann hypothesis, Robin's theorem, colossally abundant number, divisor function.

1. Introduction

The Riemann hypothesis (RH) has numerous reformulations. In this paper, we investigate the reformulation presented by Robin in which RH is characterized by the sum-of-divisors function as follows [6].

Theorem 1.1 Let $\sigma(n)$ be the sum-of-divisors function, and let γ be the Euler constant.

The Riemann hypothesis is equivalent to the following inequality (we call it Robin's inequality).

$$\sigma(n) < e^{\gamma} n \log\log n \quad \text{for all} \quad n > 5040$$

Previously, Gronwall identified a related asymptotic property of the sum-of-divisors function as follows [3].

Theorem 1.2 $\overline{\lim_{n\to\infty}} \frac{\sigma(n)}{n \log\log n} = e^{\gamma}$

By Theorem 1.2,

$$\underline{\lim_{n\to\infty}} \frac{e^{\gamma} n \log\log n - \sigma(n)}{e^{\gamma} n \log\log n} = 1 - \overline{\lim_{n\to\infty}} \frac{\sigma(n)}{e^{\gamma} n \log\log n} = 1 - 1 = 0$$

On the other hand, from numerical data, it seems that $\lim_{n\to\infty} e^{\gamma} n \log\log n - \sigma(n) = \infty$.

If $\sigma(n) < e^{\gamma} n \log\log n$ does not hold for some n, n must satisfy $\frac{\sigma(n)}{n} \geq e^{\gamma} \log\log n$. Hence, we are interested in the case where $\frac{\sigma(n)}{n}$ is relatively large.

First, we introduce some definitions and theorems related to $\frac{\sigma(n)}{n}$.

Definition 1.1 A natural number n is called superabundant if, for all $m < n$, $\frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}$.

A list of superabundant numbers can be found in The On-Line Encyclopedia of Integer Sequences. The 15 smallest superabundant numbers are listed below [5].

Example 1.1 The 15 smallest superabundant numbers

1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840

The prime decomposition of a superabundant number has the following remarkable characteristics [1].

Theorem 1.3 If n is superabundant and not equal to 1, then there exist natural numbers k and $a_1, a_2, ..., a_k$ such that $n = \prod_{i=1}^{k} (p_i)^{a_i}$, where p_i is the i-th prime number and $a_1 \geq a_2 \geq ... \geq a_k$.

Corresponding author: Masatoshi Nakano, M.D., research field: computational number theory. E-mail: bskr2@arrow.ocn.ne.jp.

Moreover, a_k is equal to 1 unless n is 4 or 36.

By Theorem 1.3, the prime decomposition of a superabundant number n can be uniquely expressed by finite sequences of length a_1, which will be explained in the next section. The time required to calculate $\sigma(n)$ in this manner is short.

The following condition is "stronger" than superabundant.

Definition 1.2 A natural number n is called colossally abundant if and only if there exists $\varepsilon > 0$ such that for all $k > 1$, $\frac{\sigma(n)}{n^{1+\varepsilon}} \geq \frac{\sigma(k)}{k^{1+\varepsilon}}$.

The 15 smallest colossally abundant numbers are listed below [5].

Example 1.2 The 15 smallest colossally abundant numbers

$2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440,$
$4324320, 21621600, 367567200, 6983776800$

An algorithm for generating colossally abundant numbers is described in [2] and [6]. All colossally abundant numbers are also superabundant, but the converse is not true. Robin proved that if a natural number n (>5040, not colossally abundant) does not satisfy Robin's inequality, then neither at least one of the two colossally abundant numbers that put n between [6]. Therefore, to determine whether RH holds, it is sufficient to confirm whether Robin's inequality holds with regard to colossally abundant numbers.

2. Noe Representation of Superabundant Numbers

By Theorem 1.3, the prime decomposition of a superabundant number n greater than 1 contains all primes less than or equal to some prime, and the power of each prime decreases monotonously to1 or 2. Hence, it is uniquely expressed by finite sequences as follows.

Definition 2.1 The prime decomposition of a superabundant number n greater than 1 is uniquely expressed by finite sequences $\{c_1, c_2, ..., c_i, ..., c_k\}$, which we refer to as the Noe representation.

The rules of the Noe representation are as follows.

1) c_i is 0 or prime. If c_i is prime, it is the largest prime that has the power i. Further, if c_i is 0, there is no prime that has the power i $(1 \leq i \leq k-1)$.

2) c_k is prime. c_k is the largest prime that has the power k. The power of a prime that is less than or equal to c_k is k.

The prime decomposition and Noe representation of the 2nd to the 25th superabundant numbers are listed below [5].

Example 2.1 The superabundant numbers are expressed as follows. SA denotes superabundant number, whereas CA denotes colossally abundant number.

2nd SA (1st CA)	{2}	$n = 2$
3rd SA	{0, 2}	$n = 2^2 = 4$
4th SA (2nd CA)	{3}	$n = 3 \cdot 2 = 6$
5th SA (3rd CA)	{3, 2}	$n = 3 \cdot 2^2 = 12$
6th SA	{3, 0, 2}	$n = 3 \cdot 2^3 = 24$
7th SA	{0, 3}	$n = 3^2 \cdot 2^2 = 36$
8th SA	{3, 0, 0, 2}	$n = 3 \cdot 2^4 = 48$
9th SA (4th CA)	{5, 2}	$n = 5 \cdot 3 \cdot 2^2 = 60$
10th SA (5th CA)	{5, 0, 2}	$n = 5 \cdot 3 \cdot 2^3 = 120$

11th SA	$\{5,3\}$	$n=5\cdot 3^2\cdot 2^2=180$
12th SA	$\{5,0,0,2\}$	$n=5\cdot 3\cdot 2^4=240$
13th SA (6th CA)	$\{5,3,2\}$	$n=5\cdot 3^2\cdot 2^3=360$
14th SA	$\{5,3,0,2\}$	$n=5\cdot 3^2\cdot 2^4=720$
15th SA	$\{7,0,2\}$	$n=7\cdot 5\cdot 3\cdot 2^3=840$
16th SA	$\{7,3\}$	$n=7\cdot 5\cdot 3^2\cdot 2^2=1260$
17th SA	$\{7,0,0,2\}$	$n=7\cdot 5\cdot 3\cdot 2^4=1680$
18th SA (7th CA)	$\{7,3,2\}$	$n=7\cdot 5\cdot 3^2\cdot 2^3=2520$
19th SA (8th CA)	$\{7,3,0,2\}$	$n=7\cdot 5\cdot 3^2\cdot 2^4=5040$
20th SA	$\{7,3,0,0,2\}$	$n=7\cdot 5\cdot 3^2\cdot 2^5=10080$
21st SA	$\{7,0,3,2\}$	$n=7\cdot 5\cdot 3^3\cdot 2^4=15120$
22nd SA	$\{7,5,0,2\}$	$n=7\cdot 5^2\cdot 3^2\cdot 2^4=25200$
23rd SA	$\{11,3,2\}$	$n=11\cdot 7\cdot 5\cdot 3^2\cdot 2^3=27720$
24th SA (9th CA)	$\{11,3,0,2\}$	$n=11\cdot 7\cdot 5\cdot 3^2\cdot 2^4=55440$
25th SA	$\{11,3,0,0,2\}$	$n=11\cdot 7\cdot 5\cdot 3^2\cdot 2^5=110880$

3. Conjectures on the Sum-of-Divisors Function

We want to show that $e^{\gamma}n\log\log n-\sigma(n)$ (we call it Robin's difference) strictly increases to infinity, where n is colossally abundant. As the ratio between $e^{\gamma}n\log\log n-\sigma(n)$ and $e^{\gamma}n\log\log n$ converges to 0 by Theorem 1.2, it is not appropriate to compare them directly. Hence, it is natural to take their logarithms, because colossally abundant numbers increase exponentially. Finally, we propose a conjecture on the sum-of-divisors function.

Conjecture 3.1 Let $R_0(n)$ denote $\dfrac{\log(e^{\gamma}n\log\log n-\sigma(n))}{\log(e^{\gamma}n\log\log n)}$, where n is a colossally abundant number.

$R_0(n)$ will increase strictly and converge to 1 when n is greater than or equal to 55440 (9th CA).

Conjecture 3.1 implies that the ratio between the number of digits in Robin's difference and that on the right-hand side of Robin's inequality increases strictly and converges to 1. If this conjecture holds, then RH is true by Robin's theorem (Theorem 1.1).

For validation, we consider the first 21187 colossally abundant numbers in "First 1000000 superabundant numbers" uploaded by Noe in The On-Line Encyclopedia of Integer Sequences. We partially show the colossally abundant numbers with the Noe representation and $R_0(n)$ [5].

Example 3.1 Calculation of $R_0(n)$

9th CA	$\{11,3,0,2\}$	$n=55440$	$R_0(n)=0.6694458330$
10th CA	$\{13,3,0,2\}$	$n=720720$	$R_0(n)=0.7594354261$
11th CA	$\{13,3,0,0,2\}$	$n=1441440$	$R_0(n)=0.7770110271$
12th CA	$\{13,0,3,0,2\}$	$n=4324320$	$R_0(n)=0.7963976299$

13th CA	$\{13,5,3,0,2\}$	$n=21621600$	$R_0(n)=0.8195618298$
14th CA	$\{17,5,3,0,2\}$	$n=367567200$	$R_0(n)=0.8388761842$
15th CA	$\{19,5,3,0,2\}$	$n=6983776800$	$R_0(n)=0.8508321831$
16th CA	$\{23,5,3,0,2\}$	$n=160626866400$	$R_0(n)=0.8654011552$
17th CA	$\{23,5,3,0,0,2\}$	$n=321253732800$	$R_0(n)=0.8690460933$
18th CA	$\{29,5,3,0,0,2\}$	$n=9316358251200$	$R_0(n)=0.8854687820$

. . . . . .

21178th CA $\{237073,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n\approx 2.650577572\times 10^{103030}$ $R_0(n)=0.9999624279$

21179th CA $\{237089,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n\approx 6.284227859\times 10^{103035}$ $R_0(n)=0.9999624297$

21180th CA $\{237091,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n\approx 1.489933867\times 10^{103041}$ $R_0(n)=0.9999624316$

21181st CA $\{237137,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n\approx 3.533184475\times 10^{103046}$ $R_0(n)=0.9999624335$

21182nd CA $\{237143,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n\approx 8.378699660\times 10^{103051}$ $R_0(n)=0.9999624353$

21183rd CA $\{237151,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n\approx 1.987017003\times 10^{103057}$ $R_0(n)=0.9999624372$

21184th CA $\{237157,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n\approx 4.712349914\times 10^{103062}$ $R_0(n)=0.9999624391$

21185th CA $\{237161,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n\approx 1.117585618\times 10^{103068}$ $R_0(n)=0.9999624410$

21186th CA $\{237163,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n\approx 2.650499579\times 10^{103073}$ $R_0(n)=0.9999624428$

21187th CA $\{237173,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n\approx 6.286269366\times 10^{103078}$ $R_0(n)=0.9999624447$

It is confirmed that $R_0(n)$increases strictly on the basis of these data.

Fig. 1 shows the point $(\log n, R_0(n))$, where n is colossally abundant number from the 9th to the 21187th. We do not believe that it is possible for the fundamental shape of the graph to be changed by additional data.

Fig. 2 shows the point $(\log n, \log R_0(n))$, where n is a colossally abundant number from the 21088th to the 21187th. The larger the value of n, the greater is the number of significant digits required to show that $R_0(n)$ increases strictly.

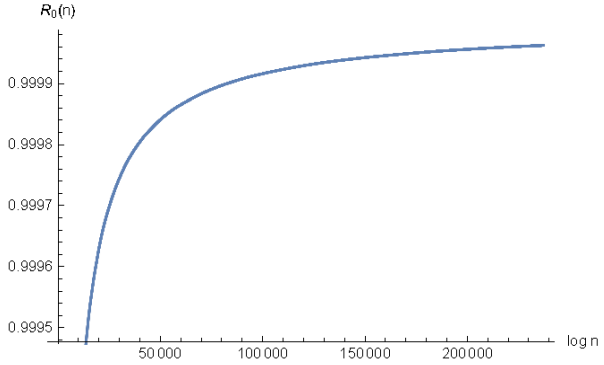

Fig. 1 $(\log n, R_0(n))$ **n is CA(9th-21187th).**

According to Conjecture 3.1, Robin's difference is rather large. Next, we investigate Robin's difference more concretely. Thus, two conjectures follow.

Conjecture 3.2 Let $D(n)$ denote $(e^{\gamma} n \log\log n)(1-\frac{\log\log n}{\log n}) - \sigma(n)$, where n is a colossally abundant number. $D(n)$ will be positive and increase strictly when n is greater than or equal to the 1201st CA.

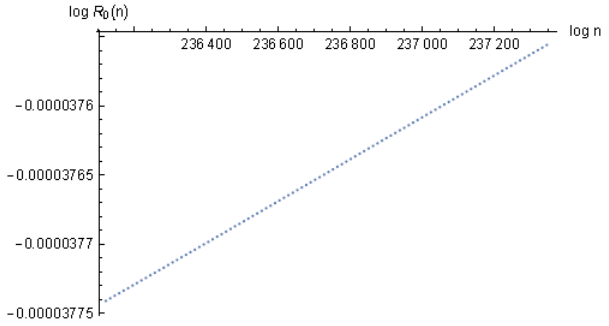

Fig. 2 $(\log n, \log R_0(n))$ **n is CA(21088th-21187th).**

Conjecture 3.2 implies that Robin's difference is greater than $\frac{e^{\gamma} n(\log\log n)^2}{\log n}$.

We partially show the colossally abundant numbers with the Noe representation and $D(n)$.

Example 3.2 Calculation of $D(n)$

1200th CA	{9157,127,23,13,7,5,0,0,3,0,0,0,0,0,2}	$n \approx 9.217564724\times10^{3992}$	$D(n) = -5.369289833\times10^{3986}$
1201st CA	{9161,127,23,13,7,5,0,0,3,0,0,0,0,0,2}	$n \approx 8.444211044\times10^{3996}$	$D(n) = 4.811038585\times10^{3991}$
1202nd CA	{9173,127,23,13,7,5,0,0,3,0,0,0,0,0,2}	$n \approx 7.745874791\times10^{4000}$	$D(n) = 9.749105743\times10^{3995}$
1203rd CA	{9181,127,23,13,7,5,0,0,3,0,0,0,0,0,2}	$n \approx 7.111487645\times10^{4004}$	$D(n) = 1.367233329\times10^{4000}$
1204th CA	{9187,127,23,13,7,5,0,0,3,0,0,0,0,0,2}	$n \approx 6.533323700\times10^{4008}$	$D(n) = 1.645711157\times10^{4004}$
1205th CA	{9199,127,23,13,7,5,0,0,3,0,0,0,0,0,2}	$n \approx 6.010004471\times10^{4012}$	$D(n) = 1.908701875\times10^{4008}$
1206th CA	{9203,127,23,13,7,5,0,0,3,0,0,0,0,0,2}	$n \approx 5.531007115\times10^{4016}$	$D(n) = 2.059124421\times10^{4012}$
1207th CA	{9209,127,23,13,7,5,0,0,3,0,0,0,0,0,2}	$n \approx 5.093504452\times10^{4020}$	$D(n) = 2.140678977\times10^{4016}$
1208th CA	{9221,127,23,13,7,5,0,0,3,0,0,0,0,0,2}	$n \approx 4.696720455\times10^{4024}$	$D(n) = 2.227689666\times10^{4020}$
1209th CA	{9227,127,23,13,7,5,0,0,3,0,0,0,0,0,2}	$n \approx 4.333663964\times10^{4028}$	$D(n) = 2.260652430\times10^{4024}$

1210th CA $\{9227,131,23,13,7,5,0,0,3,0,0,0,0,0,2\}$ $n \approx 5.677099793\times10^{4030}$ $D(n) = 3.096838478\times10^{4026}$

. . . . . .

21178th CA $\{237073,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n \approx 2.650577572\times10^{103030}$ $D(n) = 4.813219343\times10^{103027}$

21179th CA $\{237089,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n \approx 6.284227859\times10^{103035}$ $D(n) = 1.141160095\times10^{103033}$

21180th CA $\{237091,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n \approx 1.489933867\times10^{103041}$ $D(n) = 2.705578329\times10^{103038}$

21181st CA $\{237137,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n \approx 3.533184475\times10^{103046}$ $D(n) = 6.415954971\times10^{103043}$

21182nd CA $\{237143,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n \approx 8.378699660\times10^{103051}$ $D(n) = 1.521503113\times10^{103049}$

21183rd CA $\{237151,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n \approx 1.987017003\times10^{103057}$ $D(n) = 3.608266369\times10^{103054}$

21184th CA $\{237157,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n \approx 4.712349914\times10^{103062}$ $D(n) = 8.557259017\times10^{103059}$

21185th CA $\{237161,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n \approx 1.117585618\times10^{103068}$ $D(n) = 2.029444792\times10^{103065}$

21186th CA $\{237163,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n \approx 2.650499579\times10^{103073}$ $D(n) = 4.813072650\times10^{103070}$

21187th CA $\{237173,661,83,29,13,0,7,5,0,0,0,0,3,0,0,0,0,0,0,0,2\}$

$n \approx 6.286269366\times10^{103078}$ $D(n) = 1.141525622\times10^{103076}$

It is confirmed that $D(n)$ increases strictly on the basis of these data.

Fig. 3 shows the point $(\log n, \log \frac{e^{\gamma} n \log\log n}{\sigma(n)})$, where n is a colossally abundant number from the 9th to the 21187th.

Fig. 4 shows the point $(\log n, \log \frac{(e^{\gamma} n \log\log n)(1-\frac{\log\log n}{\log n})}{\sigma(n)})$, where n is a colossally abundant number from the 9th to the 21187th.

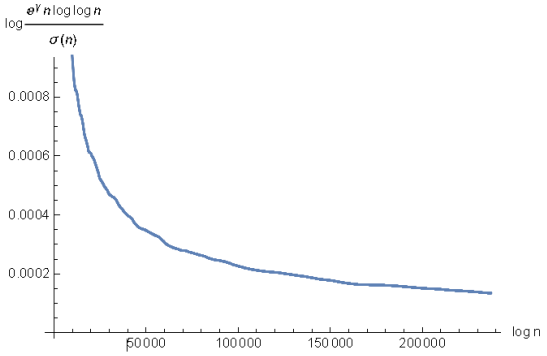

Fig. 3 $(\log n, \log \frac{e^{\gamma} n \log\log n}{\sigma(n)})$ ***n* is CA(9th-21187th).**

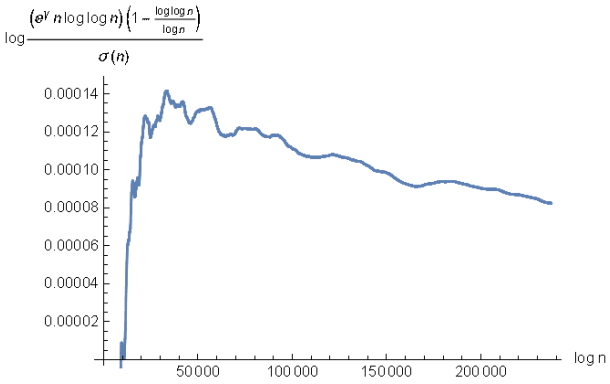

Fig. 4 $(\log n, \log \frac{(e^{\gamma} n \log\log n)(1-\frac{\log\log n}{\log n})}{\sigma(n)})$ **n is CA(9th-21187th).**

In both graphs, the x-axis is an asymptote. The shape of the latter curve is rather irregular.

We investigate the phase of increase in $D(n)$ more precisely.

Conjecture 3.3 Let $R_1(n)$ denote $\frac{\log((e^{\gamma} n \log\log n)(1-\frac{\log\log n}{\log n}) - \sigma(n))}{\log((e^{\gamma} n \log\log n)(1-\frac{\log\log n}{\log n}))}$, where n is a colossally abundant number. $R_1(n)$ will increase strictly and converge to 1 when n is greater than or equal to the 1382nd CA.

If Conjecture 3.3 is true, then $D(n)$ increases strictly at a rather high rate.

We partially show the colossally abundant numbers with the Noe representation and $R_1(n)$.

Example 3.3 Calculation of $R_1(n)$

1382nd CA	{10753, 139, 29, 13, 7, 5, 0, 0, 0, 3, 0, 0, 0, 0, 0, 2}	$n \approx 1.233675857 \times 10^{4705}$	$R_1(n) = 0.9986533059$
1383rd CA	{10771, 139, 29, 13, 7, 5, 0, 0, 0, 3, 0, 0, 0, 0, 0, 2}	$n \approx 1.328792266 \times 10^{4709}$	$R_1(n) = 0.9986583451$
1384th CA	{10781, 139, 29, 13, 7, 5, 0, 0, 0, 3, 0, 0, 0, 0, 0, 2}	$n \approx 1.432570942 \times 10^{4713}$	$R_1(n) = 0.9986644873$
1385th CA	{10789, 139, 29, 13, 7, 5, 0, 0, 0, 3, 0, 0, 0, 0, 0, 2}	$n \approx 1.545600789 \times 10^{4717}$	$R_1(n) = 0.9986682386$
1386th CA	{10799, 139, 29, 13, 7, 5, 0, 0, 0, 3, 0, 0, 0, 0, 0, 2}	$n \approx 1.669094292 \times 10^{4721}$	$R_1(n) = 0.9986730883$
1387th CA	{10831, 139, 29, 13, 7, 5, 0, 0, 0, 3, 0, 0, 0, 0, 0, 2}	$n \approx 1.807796027 \times 10^{4725}$	$R_1(n) = 0.9987075018$
1388th CA	{10837, 139, 29, 13, 7, 5, 0, 0, 0, 3, 0, 0, 0, 0, 0, 2}	$n \approx 1.959108555 \times 10^{4729}$	$R_1(n) = 0.9987300860$
1389th CA	{10847, 139, 29, 13, 7, 5, 0, 0, 0, 3, 0, 0, 0, 0, 0, 2}	$n \approx 2.125045050 \times 10^{4733}$	$R_1(n) = 0.9987490467$
1390th CA	{10853, 139, 29, 13, 7, 5, 0, 0, 0, 3, 0, 0, 0, 0, 0, 2}	$n \approx 2.306311392 \times 10^{4737}$	$R_1(n) = 0.9987630004$
1391st CA	{10859, 139, 29, 13, 7, 5, 0, 0, 0, 3, 0, 0, 0, 0, 0, 2}	$n \approx 2.504423541 \times 10^{4741}$	$R_1(n) = 0.9987734918$

. . . . . .

21178th CA $\{237073, 661, 83, 29, 13, 0, 7, 5, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 2\}$

$n \approx 2.650577572 \times 10^{103030}$ $R_1(n) = 0.9999603601$

21179th CA $\{237089, 661, 83, 29, 13, 0, 7, 5, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 2\}$

$n \approx 6.284227859 \times 10^{103035}$ $R_1(n) = 0.9999603621$

21180th CA $\{237091, 661, 83, 29, 13, 0, 7, 5, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 2\}$

$n \approx 1.489933867 \times 10^{103041}$ $R_1(n) = 0.9999603641$

21181st CA $\{237137, 661, 83, 29, 13, 0, 7, 5, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 2\}$

$n \approx 3.533184475 \times 10^{103046}$ $R_1(n) = 0.9999603662$

21182nd CA $\{237143, 661, 83, 29, 13, 0, 7, 5, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 2\}$

$n \approx 8.378699660 \times 10^{103051}$ $R_1(n) = 0.9999603683$

21183rd CA $\{237151, 661, 83, 29, 13, 0, 7, 5, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 2\}$

$n \approx 1.987017003 \times 10^{103057}$ $R_1(n) = 0.9999603703$

21184th CA $\{237157, 661, 83, 29, 13, 0, 7, 5, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 2\}$

$n \approx 4.712349914 \times 10^{103062}$ $R_1(n) = 0.9999603724$

21185th CA $\{237161, 661, 83, 29, 13, 0, 7, 5, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 2\}$

$n \approx 1.117585618 \times 10^{103068}$ $R_1(n) = 0.9999603744$

21186th CA $\{237163, 661, 83, 29, 13, 0, 7, 5, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 2\}$

$n \approx 2.650499579 \times 10^{103073}$ $R_1(n) = 0.9999603765$

21187th CA $\{237173, 661, 83, 29, 13, 0, 7, 5, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 2\}$

$n \approx 6.286269366 \times 10^{103078}$ $R_1(n) = 0.9999603785$

It is confirmed that $R_1(n)$ increases strictly on the basis of these data.

Fig. 5 shows the point $(\log n, R_1(n))$, where n is a colossally abundant number from the 1382nd to the 21187th. Fig. 6 shows the point $(\log n, \log R_1(n))$, where n is a colossally abundant number from the 21088th to the 21187th. Figs. 5 and 6 are similar to Figs. 1 and 2, respectively.

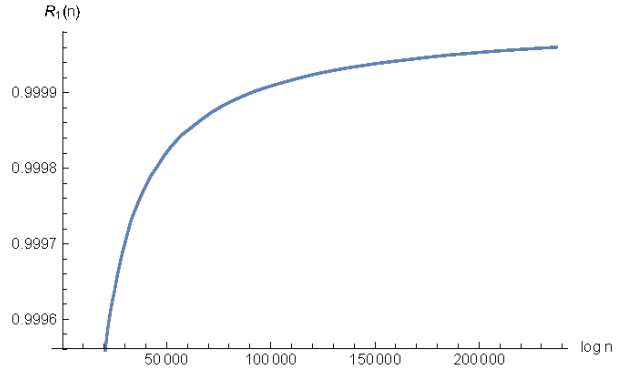

Fig. 5 $(\log n, R_1(n))$ ***n*** **is CA(1382nd-21187th).**

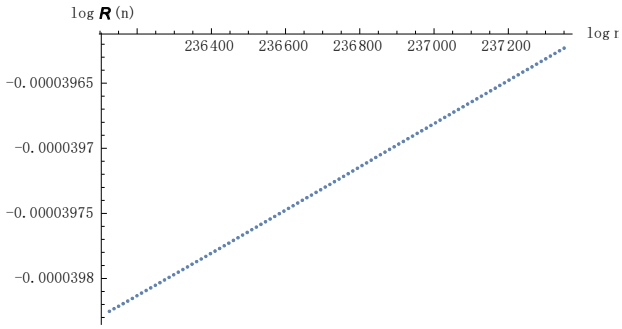

Fig. 6 $(\log n, \log R_1(n))$ ***n*** **is CA(21088th- 21187th).**

4. Conclusions

All the data and conjectures presented herein strongly suggest that RH is true.

To prove that Robin's inequality is a sufficient condition for RH, Robin proved the following theorem [6].

Theorem 4.1 If RH does not hold, then for colossally abundant number *n*, the following equation holds:

$$\frac{\sigma(n)}{n\log\log n} = e^{\gamma}(1+\Omega_{\pm}(\log n)^{-b})$$

where b is some number in the open interval $(1-\theta, \frac{1}{2})$, and θ is the largest number in the real part of the zeros of the ζ function.

Robin used this theorem in proof by contradiction. The conclusion of this theorem is an excessively strong condition for considering Robin's difference. Therefore, it is desirable to have a proposition that has a weakened assumption and proves the weakened conclusion contradictory to Robin's difference. Then, the negation of the weakened assumption holds by proof by contradiction and it implies that a phenomenon stronger than RH holds.

Von Koch characterized RH by the error term of the prime number theorem as follows [4].

Theorem 4.2 Let $\pi(x)$ be the prime-counting function, and let C be some constant.

RH is equivalent to the following inequality.

$$\frac{|\pi(x)-\int_0^x \frac{dt}{\log t}|}{\sqrt{x}\log x} < C$$

Schoenfeld improved upon this theorem as follows [7].

Theorem 4.3 RH is equivalent to the following inequality.

$$\frac{|\pi(x)-\int_0^x \frac{dt}{\log t}|}{\sqrt{x}\log x} < \frac{1}{8\pi} \text{ for all } \quad x \geq 2657$$

Let us consider how small the right-hand side of the inequality can be. The next conjecture is that of the deep Riemann hypothesis.

Conjecture 4.1 $\lim_{x\to\infty} \frac{\pi(x)-\int_0^x \frac{dt}{\log t}}{\sqrt{x}\log x} = 0$

We believe it may be provable that the behavior of Robin's difference is not compatible with

$\pi(x)-\int_0^x \frac{dt}{\log t} = \Omega(\sqrt{x}\log x)$ similarly to Theorem 4.1.

Acknowledgment

We extend our gratitude to Katsuhiko Kakehi for his insightful advice on the subject and Shun-ichi Kurino (College of Science and Technology, Nihon University) for his comprehensive aid in the calculations.

References

[1] Alaoglu, L. and Erdös, P. (1944). "On Highly Composite and Similar Numbers." *Trans. Amer. Math. Soc.* 56 (3): 448-469.

[2] Briggs, K. (2006). "Abundant Numbers and Riemann Hypothesis." *Exp. Math.* 15 (2): 251-256.

[3] Gronwall, T. H. (1913). “Some Asymptotic Expressions in the Theory of Numbers.” *Trans. Amer. Math. Soc.* 14113-14122.

[4] Von Koch, H. (1901). “Sur la distribution des nombres premiers.” *Acta Math.* 24: 159-182.

[5] Noe, T. D. (2009). “First 1000000 Superabundant Numbers.” http://oeis.org/A004394.

[6] Robin, G. (1984). “Grandes valeurs de la function somme des diviseurs et hypothèse de Riemann.” *J. Math. Pures Appl. Neuvième Série* 63 (2): 187-213.

[7] Schoenfeld, L. (1976). “Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$ II.” *Math. Comput.* 30 (134): 337-360.