

Development of a Notion of Limit as Proximity From Discourse Analysis in Secondary-School Function Classes

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Mathematical teaching and learning of an introductory calculus conception of limit in secondary school before students learn the formal definition of limit in higher education are discussed. In this study, applications of the theory of instruction based on "Formalizing Introductory Notions (FIN)" (Nagle, 2013), in which a theory incorporating new pedagogical approaches was introduced to describe static notions of limit, without ignoring dynamic notions of limit, are presented to help foster an informal limit conception better aligned with the formal definition. A qualitative discourse analysis based on students' utterances, including students' drawing pictures on graphs, was done. In the results of the investigation, it was found that the students' utterances drawings on a graph of the secants (segments) used static notions of limit supported by dynamic notions of limit according to the operating activities. There were learning activities in which students developed a notion of limit as the proximity of the predicted tangent line of a function. Consequently, students' discussions changed focus to the validation of limit candidates with static notions of limit. To overcome the contradiction of their explanations of the operating activities with dynamic notions of limit, the students changed to an explanation with static notions of limit. In light of the findings, this study suggests adapting the pedagogical approach used by the Nagle (2013).

Keywords: limit conception, Nagle's (2013) theory, static notions of limit, dynamic notions of limit, learning activities

Background

This paper is a study of learning for the limit (proximity) concept, a mathematical concept. The limit concept at the secondary education stage, before formal learning in higher education mathematics, was investigated.

As for the limit concept, the importance of learners' understanding of the limit in social practice and in calculus learning has been pointed out. On the other hand, learners' difficulty and misconception in understanding the limit concept have also been pointed out, and there are studies for overcoming the difficulty (e.g., Cornu, 1991). In those, learners' understanding of the limit concept is viewed from a perspective of constructing the concept of a limit, and the focus is placed on learners' notions of a limit, as a research approach. It has been shown that there is an inconsistency between learners' notions of a limit (Cottrill et al., 1996). Especially for the last 10 years, research on learners' smooth transition to static notions for understanding the limit concept has drawn attention (e.g., Boester, 2010). The reason for the focus on the research on the learners' translation to static notions of a limit is that the inconsistency between dynamic notions, which represent the movement of points of a graph that appeals mainly to the learners' vision in the

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development of understanding of the introductory limit, and static notions, which mainly represent the terms and symbols of functions based on the formal ε - δ definition in higher education stage has been pointed out (Nagle, 2013).

Position of This Study

In order to resolve this inconsistency between notions, Nagle (2013) organized previous research on notions of limit and introduced "Formalizing Introductory Notions (FIN)," a new model of constructing notions of limit.

The FIN developed by Nagle is not a transition from learners' dynamic notions to static notions of the construction of a limit concept, but a new guidance principle in the learning of introductory differentiation. It incorporates dynamic notions into static notions of the limit and is a guidance principle and learning activity for constructing an introductory limit concept that is better adapted to the formal definition by students' static notions, supported by their dynamic notions. Nagle, Tracy, Adams, and Scutella (2017), in practice based on the FIN, suggest a transition to static notions through a discussion activity that incorporates dynamic notions into students' static notions, the verbal use of dynamic predicates by students, based on the function definition, has been shown.

On the other hand, factors and circumstances that dynamic notions supported static notions, as to why and how dynamic notions supported static notions, have not been explored. In other words, Nagle et al.'s (2017) future research challenge is to clarify the reality of the activity—what aspects of dynamic notions support the transition to static notions in what way in the activity that incorporates dynamic notions into static notions.

If the FIN is followed, it will be possible for students to make a transition from the notions of the proximity of limit to static notions supported by dynamic notions. Nagle (2003) also suggests that guidance by the FIN allows learners to start forming what lies in the formal definition of a limit at the secondary education stage before higher education. However, no experimental study has been conducted for secondary education classes using Nagle's (2013) guidance principle based on the FIN. Therefore, this study attempts to investigate the possibility of the application of the FIN to practice.

Purpose

The purpose of this study is to explore the possibility of the application of the guidance principle of "Formalizing Introductory Notions (FIN)" developed by Nagle (2013) to class practice. Therefore, the function classes in the third year of secondary school, which is a stage before learning the formal limit concept in higher education, are investigated as a case study.

In addition, what is essential when investigating the application of the FIN to class practice is the transition of learners' notions of a limit. The reason is that the FIN is a model that incorporates learners' dynamic notions of a limit into static notions and that views the transition of static notions supported by dynamic notions as the construction of the limit concept. Also in the research in the discussion activity encouraged based on the FIN developed by Nagle et al. (2017), the purpose of the activity, which is also a characteristic of the activity, is to make a smooth translation to static notions by incorporating learners' dynamic notions of a limit into static notions (Nagle et al., 2017, p. 574). Therefore, in order to explore the application of the FIN in class practice, which is the purpose of this study, it is necessary to capture the existence of learners' dynamic and static notions during the activity and investigate the transition process to

static notions. Thus, the following research questions are set and investigated in order to accomplish the purpose of this study:

(1) What notions of limit do students have in discussions during class activities? Do dynamic notions and static notions exist or not?

(2) If dynamic notions and static notions exist, do students' dynamic notions of a limit support the transition to static notions or not?

In this study, when investigating learners' transition process to static notions, the support status of learners' dynamic notions of limit for the transition to static notions, which was suggested by Nagle et al. (2017), is adopted as a research perspective. The term "support status" in this study refers to the status of notions viewed as a preparatory stage for static notions or as a review of dynamic notions. This way of supporting notions of limit is based on the fact that Cottrill et al. (1996) suggested the existence of dynamic notions as a stage in the spontaneous generation of the limit concept and as a stage before the transition to static notions.

Methods

Survey

A classroom observation was conducted in the mathematical function area of a third-grade (15-year-old age level) class (20 students) at a certain secondary school in Tokyo, Japan. This case is based on the records of six hours of classes out of the observation period, which is a total of 20 hours for about four months.

For the survey data, what was written on the whiteboard and the utterance and appearance of the students were recorded at the same time as an image at the time of class observation, with two video cameras placed in front of and in the rear of the window, respectively. After the class, handouts and notes used during class were photocopied to collect what the students had written. The author took field notes every hour to make a class record.

Method of Analysis

The method of analysis is a qualitative discourse analysis that identifies students' notions of the proximity of limit from their utterance and that interprets their utterance.

First, the utterance in the surveyed class scene is generated in the protocol. During the utterance protocol, the protocol associated with activities related to the use of a limit and a graph is compared with students' drawings, field notes collected by the author and the proximity metaphor (Oehrtman, 2009), and identified as a protocol representing the concept of the proximity of limit. The reason why the metaphor is adopted here is that it was considered that the metaphor would be suitable for classification and extraction by the metaphor to embody the concept of the proximity of limit during activities, in this case, as a method of embodying an abstraction, which is a concept or an object. In addition, the reason why the dynamic metaphor introduced by Oehrtman (2009) was adopted is that it was considered to be suitable for identification of notions of limit by dynamic operational activities in this case. It was considered that the metaphor by Oehrtman (2009) could concretely express the concept maintained for students' problem-solving, in that it deals with students' intuition inherent in learning of the introductory limit. This made it possible to interpret abstract notions as concrete notions, and it was adopted because it was considered to be suitable for the application to the analysis in this case.

Then, the protocols of dynamic notions and static notions are extracted for the protocol of the identified the proximity of limit. They are divided into protocols that represent dynamic notions based on movements, such as movement of points or operations of a graph and static notions based on ideas, terms, and formulas of functions, and other protocols. For these divided protocols, in order to explore the relation to the support of students' dynamic notions of the proximity of limit for the transition to static notions, a protocol analysis is conducted on dynamic notions and static notions. In the protocol analysis, the state supporting the transition from dynamic notions to static notions is described with students' drawings and field notes collected by the author, from the interpretation of students' notions of limit by their activities.

Subjects of Analysis

For the utterance of the students, who are the subjects of the analysis, the focus was placed on the utterance protocol of Mk (alias), the first student who submitted an idea for problem-solving in the class, and on the utterance protocols of students, who participated in the discussion for problem-solving afterwards. These protocols are also used as subjects of the analysis.

In addition, the target learning task is a task that models the cross-section surface of the radio wave reflection surface of the parabolic antenna as a shape in the graph of the quadratic function, $y = 0.125x^2$. To capture the concept of the proximity of limit by exploring and unraveling the mechanism in which the reflected radio waves (hereafter, reflection lines) gather at one point is the students' learning objective. In order to solve the problem, the teacher started having the students describe the reflection (Figure 1) in a graph paper (Figure 2). A part of the image used in learning activities and the graph chart paper drawn by a student are shown below.





Results

First, the results of the activities are shown in the transcript in Table 1 for 11 scenes (The three-digit numbers in the table are the utterance protocol numbers, and the students' behaviors were added by the author in ()). Then, because the existence of students' dynamic notions and static notions of limit has been confirmed, each notion is identified. Finally, the support status for the transition from dynamic notions to static notions is shown. In this paper, due to space limitations, only Scenes 5, 6, 7, and 8, where the notions of limit of Mk (the student focus), the first student who submitted an idea for problem-solving in the class, make a transition, are reported.

DEVELOPMENT OF A NOTION OF LIMIT

		Table 1: Transcript table for reflection baselines	
Date & time	Scene	Focused discourse of each scene concerning the reflection baselines	Exhibited graphs & formulae
2013 3 PM	1	In order to explore the mechanism of reflection, which is a learning task, we are using a graph of of a fu square of the reflection state as an imaging activity.	nction proportional to the
Dec. 17, 2013 4 PM	2	Students participated in a discussion, and although the angle of reflection was not limited to 90 degrees base line was necessary to reflect [this].	s, the students all agreed that a
Dec. 17, 2013 4 PM	3	1033:Teacher: "What is the line for reflection?" 1034:Mt: "We need to draw a tangent to the parabola." 1035:Pm: "We need to determine the tangent and draw a perpendicular line." 1035:Mk: "The graph is curved, so we can't draw a perpendicular line that perfectly follows the icontours. I did, however, somehow manage to draw a vertical line that follows the contours." 1038:Mk: "Because it's curved, it's impossible to draw a line that is perpendicular or intersects this line unless we shift it a little like this (draws a straight black line on the graph). So, that's how I came up with this line."	Figure 3: MK's drawing of the tangent as a base line for reflection
Dec. 17, 2013 4 PM	4	041:Mk: "So I guess this is a circle. If you make it smaller, you can draw it with almost the same line." 042:Mk: "If I draw a circle (demonstrates by drawing a circle while speaking)" - Omitted $-045:Mk:"If I this is the center (points to the tangent point A), when I make a circle, I connect thesetwo. When I do this, I can pretty much draw a line that runs perpendicular to it. (Draws a circle, andthen draws a line running through each intersection and tangent point on the graph.) So, if we makethis circle smaller and draw a line through the thick line on the graph, it pretty much runsperpendicular. Oh, no, that's not right."$	Figure 4: Mk drew a line and tangent connecting the intersection of the graph and the circle
Dec. 17, 2013 4 PM	5	051:Mk: "If you connect it, you can draw a straight line that pretty much follows the graph, right?" 1052:Teacher: "You connect them like this (Connects B and C to create a line segment.)" 053:Mk: "So, as the diameter of the circle is made smaller and smaller, the difference in width becomes smaller, so it pretty much follows the graph." 1054:Teacher: "No comment." 1055:Ty: (Pointing to Mk's graph) "Anyway, this won't work unless we can draw a straight line on the curved graph. It is easy to write and understand it in this way if equidistant points are connected centering on point A. Also, if the circle is made smaller, error is reduced."	Figure 5: This diagram shows that as the line segment BC becomes smaller, it approaches the curved line.
Dec. 20, 2013 5 PM	6	061:Mk: "I couldn't do it when the circle was big." 062:Teacher: "Eh? So it's no good when the circle is big, huh?" 063:Mk: "Somehow, if it's too big, the conjunction points – here BC – of the antenna of the parabola for this will be off when connected by BC."	
Dec. 20, 2013 5 PM	7	071:Teacher: "In the ball experiment, this worked well. If you all do a good job of drawing your gaps, you should be able to figure it out. So, this graph should make sense to you. By the way, I see something that needs fixing." 072:Mix. "Humn. So we decrease the probability of mistakes, making it the same as the probability of having a difference." 073:Teacher: "Ah, I see. You want to make the probability the same." 074:Mix." Yeah, because it is impossible to get it just right." 075:Teacher: "Oh, I like it! I like that a lot! So, that's another way of dealing with this. Good job!"	Figure 6: Mk draws a small circle to
Jan. 9, 2014 6 PM Jan. 9, 2014 6 PM	8	 083:Teacher: "If this is the case, the fact that we drew a perpendicular line for BC tells us that we can use this figure. It's perpendicular to BC. Do you think we can improve on this by drawing the line more accurately?" 084:Mix. "We could make BC equal to zero. That's impossible, though." 085:Thm: "Yeah, that's impossible." 086:Mix. "However, if we don't do that, we won't be able to get it just right." 086:Teacher: "Well, let's hear what Mk has to say. Go ahead, Mk. (prompts Mk to speak so the whole class can hear.)" 088:Mix. "Uh, well" 088:Teacher: "If the size of the circle is fixed (indicates the diagram written at the beginning of class), then, first of all, it would seem that the part that is off is fixed to some degree. Well, that's my opinion, anyway." 090:Nm: "Can't we just get rid of BC?" 092:Mix.""."UH, hoeque the longeth of the circle smaller?" 	Figure 7: Mk assumes the circle to be zero and makes the reflection line coincide with this.
	9	 Jose John. Weil, because therein the farther size of in it were possible to draw [11] at zero, that would be the origin of the perpendicular line." Jose Jose Jose Jose Jose Jose Jose Jose	
Jan, 13, 2014 7 PM	10	 101:Teacher: "With regards to the goal of today's lesson, I hope you understand the inclination at the point BC and at the red line, but if you can also wrap your head around the formula, that is of course great. Even if you don't understand the formula, if you understand the slope, that's fine. So, Zu, you've got it, Go ahead." 102:Zu: "Uh, so even if you don't know the formula, if you understand the slope, that's enough?" 103:Teacher: "If you understand the slope, this means that, for example, you can figure out the point for BC. What should you understand about BC?" 104:Unn: "The line for BC." 105:Teacher: "Yep, Either a graph or a line are okay. [As long as you understand that] the smaller it gets, the more this red line and graph" 107:Teacher: "Yep, Either a graph or a line are okay. [As long as you understand that] the smaller it gets, the more this red line and graph" 107:Teacher: "Yep, Sthey as they get closer - I guess you can call it the slope of the graph - but anyway because it is curved, as the image shows, the difference becomes smaller. You can then make your calculations where they are the easiest. Because y = 1/8 x 2, this is an easy calculation to make. For example, it's good if we can get rid of any fractions, so 2 or 4 are good to work with." 	Figure 8: A diagram in which the slope of the line connecting the intersection of the graph and the oricle approaches the slope of the tangent line, and the formula for this diagram.
Jan. 18, 2014 8 PM	11	By calculating the slope of line segment BC by appropriately reducing the width of the value of x, and of that the slope approaches 1, the existence of base lines for virtual reflections and their virtual validity v relassroom.	confirming through calculation vere confirmed in the

Identification of Dynamic and Static Notions

(a) Scene 5 and Figure 5 ... dynamic notions

The students draw a small circle as a method of problem-solving and explore the secant BC connecting points at the intersections with the curve of the graph as a reference line of reflection.

(b) Scenes 6, 7 and Figure 6 ... dynamic notions

The students draw a smaller circle to make the line segment shorter, but the reflection lines do not intersect at one point though they approach the point.

(c) Scenes 8, 9 and Figure 7 ... static notions

Since the reflection lines do not seem to intersect at one point, the students imagined a tangent line of the curve of the graph that the secant BC would supposedly approach and used it as the reference line of reflection.

(d) Scene 10 and Figure 8 ... static notions

The students calculated the slope of the secant, and because the value of the slope approaches the value of the slope of the tangent line, the validity of the assumption was confirmed.



Figure 5. How to draw a reflection line.



Figure 6. Diagram in which the reflection lines approach one point.



Figure 7. Diagram in which the reflection lines gather at one point.

Figure 8. Figure and formula to calculate the value of the slope.

Support of Dynamic Notions for the Transition to Static Notions

In this case, the students could not predict the reference line with dynamic notions based on graph operations for problem-solving (Scenes 6, 7), but because static notions avoiding the misconception of the proximity of limit occurred (Scenes 8, 9), they were able to predict the reference line or the tangent line to solve the problem. In other words, it is considered that graph operations gave the students dynamic notions, and that the students made a transition to static notions with the support of those notions. The support status of dynamic notions for the transition to a static notion is shown from three points in the following discussion, with the utterance protocol in the result scene.

Discussion

Support in Preparation of Static Notions

Dynamic notions can serve to confirm the expected directionality of student problem-solving. With the verification of this directionality and teacher support, as in 087 "Well, let's hear what Mk has to say. Go ahead, Mk (following omitted)", preparations for static notions can be thought of as being arranged by dynamic notions.

Specifically, the learning activity in this case can be thought of as having shifted to a static notion because the students have experienced the dynamic notion. This is due to the fact that because students experienced the dynamic notion—i.e., it was confirmed to students as per the operations on the graph that the reflection line gradually approaches a certain point—it can be said that the validity of the proximity method for problem-solving has been confirmed. Because of the presence of this dynamic notion, it became possible to confirm relevance by assuming a notion of problem-solving that forms a static line of reference. Also, because the teacher provided support, this can be thought of as transitioning over to a static notion.

In summary, because a series of operational activities can confirm the validity of the directionality of problem-solving as a result of a dynamic notion of the proximity of limit, preparations are laid for the transition to a static notion, helping to make this transition smooth. It can thus be said that this serves to support the dynamic notion that facilitates the transition to the static notion.

Supporting a Static Notion Transition That Avoids Misconceptions

Mk's utterance protocol 074—"Because it's impossible to get it just right"—can be thought of as positioning the dynamic notion so that students are aware of the limits of this notion when it comes to problem-solving.

In this particular case, problem-solving is not possible with dynamic notions, but rather requires use of static notions. This point serves to support static notions where misconceptions are avoided even when the

student has experienced the dynamic notion. The key here is that given the notion the point gets closer, as a result of a dynamic (bringing one point closer to another) operation activity process by which the dividing line (line segment) is decreased, the line segment disappears, and the reflection cannot be explained. On the other hand, because the reflection lines are all concentrated at one point, this is relevant as an operation. In an effort to overcome this contradiction, students can be thought of as shifting over from explanatory activities, such as approaching the tangent of a graph that originally existed as a function target, to a static notion of proximity.

Support for Determining Static Notions of Extreme Proximity

Mk's utterance protocol 084—"We could make BC equal to zero. That's impossible, though"—and 092—"If it were possible to draw it at zero"—can be thought of as a shift to the static notion due to inherit limitations to the static notion.

In the present case, Mk's protocols 084 and 086 in Scene 8 show the student promoting the use of the static notion. This can be thought of as supporting a shift to the static notion. This is because he participated in a discussion utilizing the dynamic notion, but was unable to solve the problem. It can thus be said that, in order to solve the problem, the student shifted from the situation where he had no choice but to use the static notion, as it was the only remaining option after the dynamic notion, the student shifted to the static notion. Because the situation called for the elimination of the dynamic notion, it was critical that the static notion be incorporated into the activity, and thus Mk can be thought of as having no choice but to shift from the dynamic to the static notion. In other words, because the situation was such that the student had no choice but to eliminate the dynamic notion. In other words, because the situation was such that the student had no choice but to eliminate the dynamic notion. In other words, because the situation was such that the student had no choice but to eliminate the dynamic notion is that there was a shift to a static notion. The transition to the static view here was due to the fact that, while the student began with the dynamic notion, there was a need to eliminate this. The implication is that this served to support the transition from the dynamic notion to the static notion and that the shift to the static notion did not occur suddenly.

Conclusion

In the investigation of secondary-school third-grade function classes, a static notion supported by a dynamic notion was discovered in students' utterances during a class looking at secants (line segments) connecting two different points on the graph of a quadratic function. First, the students explained the proximity between a graph of the secants and the curve with dynamic notions of limit. However, there was a contradiction in their explanations of the operating activities with dynamic notions of limit, and the explanations could not provide mathematical validation of the concept of limit. Therefore, to overcome the contradiction in their explanations of the operating activities with dynamic notions of limit, the students changed their explanations to ones involving static notions of limit. There was also an activity in which students developed a static notion predicting the tangent of a function for extreme proximity. Consequently, students began to discuss what the most mathematically relevant limit would be, with the implication being that a transition to a static notion was supported by dynamic notions. Here, it is meaningful to incorporate a dynamic notion into the activities encouraged by Nagle et al. (2017). This significance can be summarized in the three points below:

Experiencing Dynamic Notion

Predicting the directionality of problem-solving by dynamic notion, students could confirm its validity, with the shift to a static notion being found to be significant.

Mutual Complementation of the Notion of Limits

Finding the significance of a static notion that avoids misconceptions (Tirosh, 1991) caused by the dynamic notion of the learner and overcomes the difficulty of transitioning to a static notion (Coitrill et al., 1996).

Setting the Scene for Problem-Solving

The finding that, since it was confirmed that the dynamic notion resulting from having students drawing straight lines and segments on a graph and then holding discussion activities based on these did not lead to students' solving the problem, the shift to a static notion was significant.

These results suggest that, although dynamic interpretations of limit may be intuitive for many students, a dynamic conception that is both useful at the introductory calculus level and in line with the formal notion of limits learned in advanced mathematics is to be fostered. To this end, demonstrations of the teaching principles based on FIN developed by Nagle (2013) have been suggested for function classes at the secondary educational level.

Matters for Future Consideration

In addition to the transition in notions resulting from the student activities suggested in this paper, it is also suggested that a factor in the shift to static notions was interactions resulting from exchanging opinions with others, including teachers.

Future topics to be covered by this research include interactions with others when engaging in learning activities. This should make more detailed analysis of the transition to static notions for limits possible. With regards to interacting with others, clarification of classroom practices of teaching principles based on discussion activities FIN recommended by Nagle et al. (2017) should be further advanced by clarifying the process of what makes up a learning activity, and how this affects the notions of the learner, resulting in a shift to static notions.

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