

Dimensional Analysis and Similarity Method Driving Self-similar Solutions of the First and Second Kind Inducing Energy

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Abstract: Nearly all scientists, at conjunction with simplifying a differential equation, have probably used dimensional analysis. Dimensional analysis (also called the Factor-Label Method or the Unit Factor Method) is an approach to the problem that uses the fact that one can multiply any number or expression without changing its value. This is a useful technique. However, the reader should take care to understand that chemistry is not simply a mathematics problem. In every physical problem, the result must match the real world. In physics and science, dimensional analysis is a tool to find or check relations among physical quantities by using their dimensions. The dimension of a physical quantity is the combination of the fundamental physical dimensions (usually mass, length, time, electric charge, and temperature) which describe it; for example, speed has the dimension length/time, and may be measured in meters per second, miles per hour, or other units. Dimensional analysis is necessary because a physical law must be independent of the units used to measure the physical variables in order to be general for all cases. One of the most derivation elements from dimensional analysis is scaling and consequently arriving at similarity methods that branch out to two different groups namely self-similarity as the first one, and second kind that through them one can solve the most complex none-linear ODEs (Ordinary Differential Equations) and PDEs (Partial Differential Equations) as well. These equations can be solved either in Eulearian or Lagrangian coordinate systems with their associated BCs (Boundary Conditions) or ICs (Initial Conditions). Exemplary ODEs and PDEs in the form of none-linear can be seen in strong explosives or implosives scenario, where the results can easily be converted to induction of energy in a control forms for a peaceful purpose (i.e., fission or fusion reactions).

Key words: Renewable, nonrenewable source of energy, fusion and fission reactors, small modular reactors and generation four system, nuclear micro reactor, space reactor, dynamic site, return on investment, total cost of ownership.

1. Introduction

In physics and science, dimensional analysis is a method of analysis to find or express the relations among physical quantities in terms of their dimensions. The dimension of a physical quantity is the combination of the basic physical dimensions (usually mass, length, time, electric charge and temperature) which describe it; for example, speed has the dimension length/time, and may be measured in meters per second, miles per hour, or other units [1].

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Dimensional analysis is routinely used to check the plausibility of derived equations and computations as well as forming reasonable hypotheses about complex physical situations. That can be tested by experiment or by more developed theories of the phenomena, which allow categorizing the types of physical quantities. In this case, units are based on their relations or dependence on other units or dimensions, if any.

Isaac Newton (1686), who referred to it as the “Great Principle of Similitude”, understood the basic principle of dimensional analysis. Nineteen-century French mathematician Joseph Fourier made significant contributions based on the idea that physical laws like

$F = MA$ should be independent of the units employed to measure the physical variables. This led to the conclusion that meaningful laws must be homogeneous equations in their various units of measurement, a result that was eventually formalized by Edgar Buckingham with the π (Pi) theorem. This theorem describes how every physically meaningful equation involving n variables can be equivalently rewritten as an equation of $n-m$ dimensionless parameters, where m is the number of fundamental dimensions used. Furthermore, and most importantly, it provides a method for computing these dimensionless parameters from the given variables [1].

A dimensional equation can have the dimensions reduced or eliminated through nondimensionalization, which begins with dimensional analysis, and involves scaling quantities by characteristic units of a system or natural units of nature [1].

The similarity method is one of the standard methods for obtaining exact solutions of PDEs (Partial Differential Equations) in particular non-linear forms. The number of independent variables in a PDE is reduced one-by-one to make use of appropriate combinations of the original independent variables as new independent variables, called “similarity variables”.

In some cases, the dimensional analysis does not provide an adequate approach to establish a solution of a certain eigenvalue problem in nonlinear form, which gives rise to the need to discuss the similarity method as another approach. In particular, for a simple case of dealing with a nonlinear partial differential, which can be reduced to an ordinary differential, so it can be solved for a closed solution, in ordinary way of methods that we have learned in any classical text books of applied mathematics.

In more complex scenarios, dealing with boundary-value problems for a system of ordinary equations with conditions at different ends of an infinite interval requires constructing a self-similar solution that is a more efficient way of solving such

complex boundary value problems the system of ordinary equations directly.

In a specific, instance the passage of the solution into a self-similar intermediate asymptotic allows not to have a need to return to the partial differential equations, indeed, in many cases, the self-similarity of intermediate asymptotic can be, established and the form of self-similar intermediate asymptotic obtained from dimensional considerations [1, 2].

In this short review under the presented title of this article, we are looking at some specific aspect of dimensional analysis for dealing with a certain aspect of explosion and implosion, which is in a very controlled way of inducing energy by solving the most sophisticated ODEs (Ordinary Differential Equations) or PDEs of shock movement generated by either of these processes. For this matter, we are introducing method of similarity and self-similarity of first and second kind respectively.

The human partner in the interaction of a man and a computer often turns out to be the weak spot in the relationship. The problem of formulating rules and extracting ideas from vast masses of computational or experimental results remains a matter for our brains, our minds. This problem is closely connected with the recognition of patterns. The word “obvious” has two meanings, not only something easily and clearly understood, but also something immediately evident to our eyes. The identification of forms and the search for invariant relations constitute the foundation of pattern recognition; thus, we identify the similarity of large and small triangles, etc.

A time-developing phenomenon is called self-similarity if the spatial distributions of its properties at various moments of time can be obtained from one another by a similarity transformation and the fact that we identify one of the independent variables of dimension with time is nothing new from the subject of dimensional analysis point of view. However, this is where the boundary of dimensional analysis goes beyond Pi theorem and steps into a new arena known as

self-similarity, which has always represented progress for researchers.

In recent years, there has been a surge of interest in self-similar solutions of the first and second kind. Such solutions are not newly discovered; they had been identified and, in fact, named by Zel'dovich, famous Russian Mathematician in 1956, in the context of a variety of problems, such as shock waves in gas dynamics and filtration through elasto-plastic materials [3].

Self-similarity has simplified computations and the representation of the properties of phenomena under investigation, and it handles experimental data and reduces what would be a random cloud of empirical points to lie on a single curve or surface, construct procedure that is, known to us as self-similar where variables could be, chosen in some special way.

The self-similarity of the solutions of partial differential equations, either linear or non-linear form, has allowed their reduction to ordinary differential equations, which often simplifies the investigation. Therefore, with the help of self-similar solutions, researchers and scientists have attempted to envisage new phenomena' characteristic properties.

Nonlinearity plays a major role in understanding most physical, chemical, biological, and engineering sciences. Nonlinear problems fascinate scientists and engineers but often elude exact treatment. However elusive may be the solutions do exist—if only one perseveres in seeking them out [2].

2. Eulerian and Lagrangian Coordinate Systems

Before we go forward with the subject of dimensional analysis and utilization of similarity or self-similarity, we have to pay attention to coordinate systems, which are known to engineers and scientists as either Eulerian or Lagrangian coordinate systems. In dealing with the complexity of partial differential equations and quest for their exact solutions analytically, one needs a certain defined boundary

condition that describes the problem at hand. These boundary conditions need to be defined either in the Eulerian or Lagrangian coordinate system when time is varying for the problem of interest. Therefore, we have to have a grasp of Eulerian and Lagrangian coordinate systems and the difference between them as well.

To have a concept for the time, we need a motion, and motion is always determined with respect to some reference system known as the coordinate system in three-dimensions. A correspondence between numbers and points in space is established with the aid of a coordinate system. For three-dimensional space we assume three numbers x_1 , x_2 and x_3 correspond to points as three components of X , Y and Z coordinate system in the Cartesian coordinate system and accordingly for Curvilinear coordinate system for its own designated components according to Figs. 1a and 1b and they are called the coordinates of the point.

In Figs. 1a and 1b, for lines along with any two coordinates remain constant, are called coordinate lines. For example, the line for which x_2 is constant and x_3

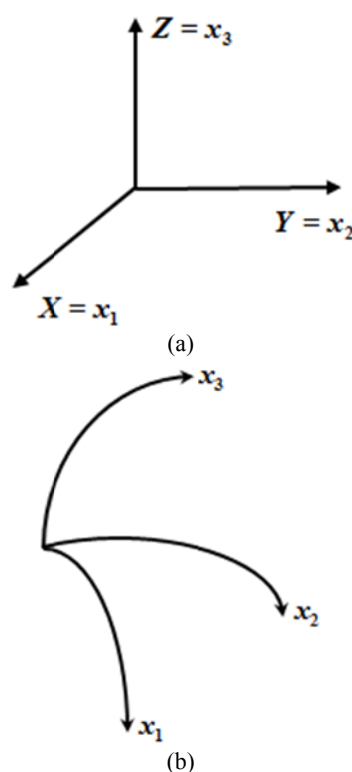


Fig. 1 Cartesian and Curvilinear coordinate systems.

is constant, defines the coordinate line x_1 , along which different points are fixed by the values of x_1 ; the direction of increase of the coordinate x_1 defines the direction along this line. Three coordinate lines may be depicted through each point of space. However, for each point, the tangents to the coordinate lines do not lie in one plane and, in general, they form a non-orthogonal trihedron. Now let us assume, these three points mathematically are presented as x_i for $i = 1, 2, 3$, and the coordinate lines x_i are straight, then the system of coordinates is rectilinear; and if not, then the system is curvilinear. For our purpose of discussion on subject of motion of a continuum anywhere that we encounter it, would be necessary to present the curvilinear coordinate system, which is essential in continuum mechanics.

Now that we start our introduction with concept of time t , then we need to make a notation of time and coordinate system x_i . Therefore, the symbols x_1 , x_2 and x_3 will denote coordinates in any system which may also be Cartesian, the symbols of X , Y and Z in orthogonal form, are presentation Cartesian coordinate system while the fourth dimension time is designated with symbol of t . Thus, if a point moves relative to the coordinate systems x_1 , x_2 and x_3 , while its coordinate changes in time, then we can mathematically present motion of point as follow:

$$x_i = f_i(t)$$

for

$$i = 1, 2, 3 \quad (1)$$

With this notation, the motion of the point will be known if one knows the characteristic and behavior of Eq. (1), providing that the moving point coincides with different points of space at different instants of time. This is, referred to as the Law of Motion of the point, and by knowing this law, we can now define the motion of a continuum. A continuous medium represents a continuous accumulation of points, and by definition, knowledge of the motion of a continuous medium

result in knowledge of the motion of all points. Thus, in general, as one can see, for the study of the motion of a volume of a continuous body as whole, it is insufficient proposition.

For the above situation, one must treat each distinct point individually in order to form a geometrical point of view that is entirely identical points of the continuum. This is, referred to as individualization of the points of a continuum, and below how this law is used in theory and is, determined by the fact that the motion of each point of a continuous medium is subject to certain physical laws that we need to take under consideration [4].

Let the coordinates of points at the initial time t_0 be denoted by ξ_1 , ξ_2 , and ξ_3 or for that matter denoted by ξ_i for $i = 1, 2, 3$ and the coordinated of points at an arbitrary instant of time t by x_1 , x_2 , and x_3 , or in general noted as x_i for $i = 1, 2, 3$ as we have done it before. For any point of a continuum, specified by the coordinates ξ_1 , ξ_2 , and ξ_3 one may write down the law of motion which contains not only functions of a single variable, as in the case of the motion of a point, but of four variables (i.e. all three coordinates plus time), therefore the initial coordinates ξ_1 , ξ_2 , and ξ_3 as well as the time t , we can write [1, 2]:

$$\begin{cases} x_1 = x_1(\xi_1, \xi_2, \xi_3, t) \\ x_2 = x_2(\xi_1, \xi_2, \xi_3, t) \\ x_3 = x_3(\xi_1, \xi_2, \xi_3, t) \end{cases} \Rightarrow x_i = x_i(\xi_1, \xi_2, \xi_3, t) \quad (2)$$

If in Eq. (2), ξ_1 , ξ_2 , and ξ_3 are fixed and t varies (Eulerian), then Eq. (2) describes the law of motion of one selected point of the continuum. If ξ_1 , ξ_2 , and ξ_3 vary and the time t is fixed, then Eq. (2) gives the distribution of the points of the medium in space at a given instant of time (Lagrangian). If ξ_1 , ξ_2 , and ξ_3 including time t vary, then one may interpret Eq. (2) as a formula which determines the motion of the continuous medium and, by definition, the functions in Eq. (2) yield the Law of Motion of the continuum. The coordinates ξ_1 , ξ_2 , and ξ_3 or

sometime definite functions of these variables, which individualize the points of a medium, and the time t are referred to as Lagrangian coordinates. In case of, continuum mechanics, the fundamental problem is to determine the functions presented in Eq. (2).

To expand the above discussions into fluid mechanics in order to analyze fluid flow, the different viewpoints can be taken, very similar to using different coordinate systems. For this matter, two different points of view will be discussed for describing fluid flow. They are called Lagrangian and Eulerian viewpoints.

2.1 Lagrangian Viewpoint

The flow description via the Lagrangian viewpoint is a view in which a fluid particle is followed. This point of view is widely used in Dynamics and Statics and easy to use for a single particle. As the fluid particle travels about the flow field, one needs to locate the particle and observe properties' change [1, 2].

That is:

$$\begin{cases} \vec{r}(t) & \text{Position} \\ T(\xi_1, \xi_2, \xi_3, t) & \text{Temperature} \\ \rho(\xi_1, \xi_2, \xi_3, t) & \text{Density} \\ P(\xi_1, \xi_2, \xi_3, t) & \text{Pressure} \end{cases} \quad (3)$$

where ξ_1 , ξ_2 and ξ_3 represent a particular particle or object. One example of Lagrangian description is the tracking of whales (position only). In order to better understand the behavior and migration routes of the whales, they are commonly tagged with satellite-linked tags to register their locations, diving depths and durations as illustrated in Figs. 3a and 3b [1, 2].

In summary, in Lagrangian schema, frame of reference is moving along with boundary and initial conditions (see Fig. 2).

2.2 Eulerian Viewpoint

The first approach to describe fluid flow is through the Eulerian point of view. The Eulerian viewpoint is

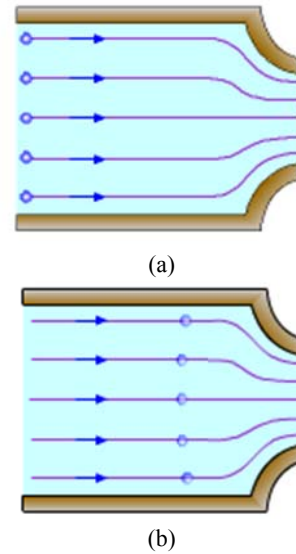


Fig. 2 Lagrangian viewpoint.



(a) Tracking of Wales



(b) Weather Balloon

Fig. 3 Lagrangian examples [1, 2].

implemented by selecting a given location in a flow field (x_1, x_2, x_3) , and observes how the properties (e.g., velocity, pressure and temperature) change as the fluid passes through this particular point. As such, the properties at the fixed points generally are functions of time, such as what is written in Eq. (4) and illustrated in Figs. 4a and 4b [1, 2]:

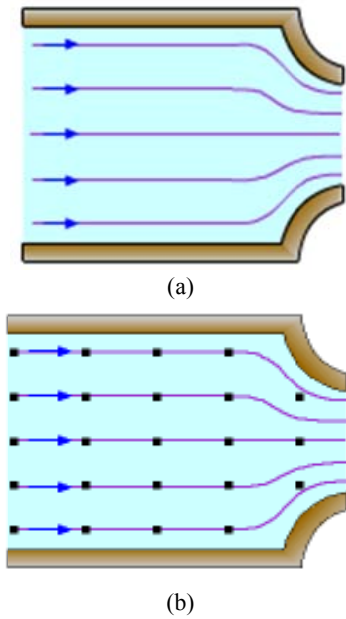


Fig. 4 Eulerian viewpoint.

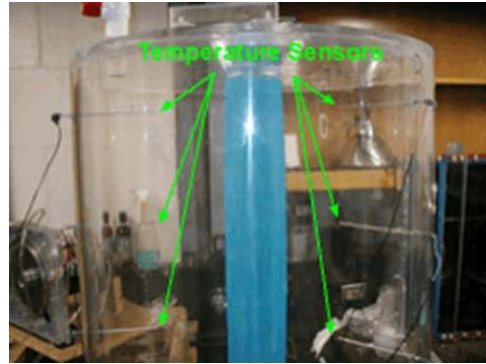
$$\begin{cases} \vec{V}(x_1, x_2, x_3, t) & \text{Velocity} \\ T(x_1, x_2, x_3, t) & \text{Temperature} \\ \rho(x_1, x_2, x_3, t) & \text{Density} \\ P(x_1, x_2, x_3, t) & \text{Pressure} \end{cases} \quad (4)$$

It should be noted that the position function $\vec{r}(t)$ is not used in Eulerian viewpoint. This is a major difference from the Lagrangian viewpoint, which is, used in particle mechanics (i.e., Dynamics and Statics).

However, if the flow is steady, then the properties are no longer the function of time. The Eulerian viewpoint is commonly used, and it is the preferred method in the study of fluid mechanics. Taking the experimental setup, as shown in Fig. 5a, for example, thermocouples (temperature sensors) are usually attached at fixed locations to measure the temperature as the fluid flows over the non-moving sensor location.

Another intuitive explanation can be given in terms of weather stations. The Eulerian system can be thought as land-based weather stations that record temperature, humidity etc. at fixed locations at different time.

In summary, in Eulerian schema, the frame of reference is fixed with respect to the boundary and initial conditions.



(a)



(b)

Fig. 5 Eulerian examples [1, 2].

In general, both Lagrangian and Eulerian viewpoints can be used in the study of fluid mechanics. However, the Lagrangian viewpoint is seldom used since it is not practical to follow large quantities of fluid particles to obtain an accurate portrait of the actual flow fields. The Lagrangian viewpoint is commonly used in dynamics, where the position, velocity, or acceleration over time are important to describe in a single equation. As it turns out, there is a big difference in how we express the change of some quantity depending on whether we think in the Lagrangian or the Eulerian sense.

In summary, there are two different mathematical representations of fluid flow:

(1) The Lagrangian picture in which we keep track of the locations of individual fluid particles. Picture a fluid flow where each fluid particle carries its own properties such as density, momentum, etc. As the particle advances, its properties may change in time. The procedure of describing the entire flow by recording each fluid particle's detailed histories is the Lagrangian description. In other words, pieces of the

fluid are “tagged”. The fluid flow properties are determined by tracking the particles’ motion and properties as they move in time. A neutrally buoyant probe is an example of a Lagrangian measuring device.

The particle properties at position $\vec{r}(t)$ such as temperature, density, pressure, ... can be mathematically represented as follows: $T(\xi_i, t)$, $\rho(\xi_i, t)$, $P(\xi_i, t)$, ... for $i = 1, 2, 3$. Note that ξ_i is representation of a fixed point in three-dimensional space at given time t , which it may include initial time t_0 .

The Lagrangian description is simple to understand conservation of mass and Newton’s laws apply directly to each fluid particle. However, it is computationally expensive to keep track of all the fluid particles’ trajectories in a flow, and therefore, the Lagrangian description is used only in some numerical simulations [1, 2].

(2) The Eulerian picture in which coordinates are fixed in space (the laboratory frame). The fluid properties such as velocity, temperature, density and pressure are written as space and time functions.

The flow is determined by analyzing the behavior of the functions. In other words, rather than following each fluid particle, we can record the evolution of the flow properties at every point in space as time varies. This is the Eulerian description. It is a field description. A probe fixed in space is an example of a Eulerian-measuring device [1, 2].

This means that the flow properties at a specified location depend on the location and on time. For example, the velocity, temperature, density, pressure, ... can be mathematically represented as follows: $\vec{V}(x_i, t)$, $T(x_i, t)$, $\rho(x_i, t)$, $P(x_i, t)$, ... for $i = 1, 2, 3$. Note that x_i is the location of fluid at time t .

The Eulerian description is harder to understand: how do we apply the conservation laws?

However, it turns out that it is mathematically simpler to apply. For this reason, we use mainly the Eulerian description in fluid mechanics.

The aforementioned locations are described in coordinate systems [1, 2].

3. ALE (Arbitrary Lagrangian Eulerian) Systems

The Arbitrary Lagrangian-Eulerian that is noted as ALE is a formulation in which computational system is not a prior fixed in space (e.g., Eulerian Based Formulation) or attached to material or fluid stream (e.g., Lagrangian Based Formulations). ALE Based Formulation can alleviate many of the drawbacks that the traditional Lagrangian-based and Eulerian-based formulation or simulation have.

When using the ALE technique in engineering simulations, the computational mesh inside the domains can move arbitrarily to optimize the shapes of elements, while the mesh on the boundaries and interfaces of the domains can move along with materials to precisely track the boundaries and interfaces of a multi-material system.

ALE-based finite element formulations can reduce either Lagrangian-based finite element formulations by equating mesh motion to material motion or Eulerian-based finite element formulations by fixing mesh in space. Therefore, one finite element code can be used to perform comprehensive engineering simulations, including heat transfer, fluid flow, fluid-structure interactions and metal manufacturing.

Some applications of ALE in finite element techniques that can be applied to many engineering problems are:

- Manufacturing (e.g., metal forming/cutting, casting);
- Fluid-structure interaction (combination of pure Eulerian mesh, pure Lagrangian mesh and ALE mesh in different regions);
- Coupling of multi-physics fields with multi-materials (moving boundaries and interfaces).

Another important application of ALE is the PIC (Particle-In-Cell) analyses, particularly is plasma physics. The PIC method refers to a technique used to

solve a certain class of partial differential equations. In this method, individual particles (or fluid elements) in a Lagrangian frame are tracked in continuous phase space, whereas moments of the distribution such as densities and currents are computed simultaneously on Eulerian (stationary) mesh points.

PIC methods were already in use as early as 1955 [5], even before the first FORTRAN compilers were available. The method gained popularity for plasma simulation in the late 1950s and early 1960s by Buneman, Dawson, Hockney, Birdsall, Morse and others. In plasma physics applications, the method amounts to following the trajectories of charged particles in self-consistent electromagnetic (or electrostatic) fields computed on a fixed mesh [6].

4. Similar and Self-similar Definitions

We need to understand better similar and self-similar methods and their definition in the subject of dimensional analysis. Once we have these methods appropriately defined, we can extend it to Motion of a Medium, particularly from a self-similarity point of view. Also, we are able to deal with the complexity of partial differential equations of conservation laws, such as Mass-Conservation Law, the Momentum-Conservation Law, and finally, the Energy-Conservation Law of non-linear type both in Eulerian and in Lagrangian schemes using all three coordinates systems that we are familiar with. These coordinates are, i.e., Cartesian, Cylindrical and Spherical coordinate systems. Further, this allows us to have a better understanding of the self-similarity of first and second kind, their definitions, the differences between them, and where and how they get applied to our physics and mathematics problems at hand. A few of these examples that we can mention here are, self-similar motion of a gas with central symmetry, both sudden explosion [7, 8] and sudden implosion [9] problems. The first one is, considered self-similarity of first kind while the lateral is considered as self-similarity of second kind. Through these

understandings, we can have a better grasp of gas-dynamics differential equations and their properties in a medium. In addition, the analysis of such differential equations for a gas motion with central symmetry becomes much easier by utilizing a self-similar method.

The self-similar motion of a medium is one in which the parameters that are characterizing the state and motion of the medium vary in a way as the time varies, the spatial distribution of any of these parameters remains similar to itself. However, the scale characterizing this perturbation/distribution can also vary with time in accordance with definite rules. In other words, if the variation of any of the above parameters with time is specified at a given point in space, then the variation of these parameters with time will remain, the same at other points lying on a definite line or surface, providing, that the scale of the given parameter and the value of the time are suitably changed [8].

The analytical conditions for self-similar motion lead to one or more relations between the independent variables, defining functions, which play the role of new independent variables using dimensional analysis and self-similarity approach [7]. This approach follows that, in the case of self-similar motion, the number of independent variables in the fundamental systems of equations is correspondingly reduced. This technique considerably simplifies the complex and non-linear partial differential equations to sets of ordinary differential equations. Thus, sometimes, this makes it possible to obtain several analytical solutions describing, for example, the self-similar motion of the medium; as it was said, in the case of two independent variables, and sometimes even in the case of three independent variables, the fundamental system of equations becomes a system of ordinary rather than partial differential equations [8].

Applications of self-similar approach can be seen to all unsteady self-similar motions with symmetry, all steady plane motions and certain axial symmetrical

motions as well. These types of approaches have solved problems of self-similarity of first kind [7, 8] and second kind [9] in past, where complex partial differential equations of conservations law are described by systems of ordinary differential equations.

Investigation of the most important modern gas dynamics motions or plasma physics such as laser-driven pellet for fusion confinement via self-similar methods enables us to produce very useful conclusions by solving the conservation law equations in them, using self-similarity model. To be concerned about more general types of motion of the medium also allows us to develop and establish laws of motion in various cases of practical interest.

They may include the propagation of strong shock waves in case of explosion and implosion events, propagation of soliton waves, and the reflection of shock waves are few examples that can fall into the category of self-similarity methods. To further have a better understanding of subject similarity and self-similarity requires knowledge of the fundamental equation of gas dynamics, where we can investigate a compressible liquid or gas. Therefore, the next few sections of this chapter are allocated to this matter and related thermodynamics aspect of the state of medium equations.

For this, we also need to understand the difference between compressible and incompressible flows. In addition, the details analysis of similarity can be found in the books by the second author of this article and short review, so we do not have to repeat the same information here [1, 2].

5. Dimensional Analysis and Scaling Concept

Scaling is the branch of measurement that involves constructing an instrument that associates qualitative constructs with quantitative metric units, and the term describes a very simple situation. S.S. Stevens came up with the simplest and most straightforward definition of scaling. He said:

“Scaling is the assignment of objects to numbers according to a rule”.

However, what does that mean?

Most physical magnitudes characterizing nano-scale systems differ enormously from those familiar with macro-scale systems. However, estimating some of these magnitudes can apply to scale laws to the values for macro-scale systems. There are many different scaling laws. At one extreme, there are simple scaling laws that are easy to learn, easy to use, and very useful in everyday life. This has been true since Day One of modern science. Galileo presented several important scaling results in 1638 [10].

The existence of a power-law relationship between certain variables y and x :

$$y = Ax^\alpha \tag{5}$$

where A and α are constants values. This type of relationship often can be seen in the mathematical modeling of various phenomena, not only in mechanical engineering and physics, but also in other science fields such as biology, economics and other engineering discipline.

Distribution of Power-Law is unique and has certain interesting features and graphically can be presented as a log-log scale as a straight line. This can methodically be shown, if we take the base 10 of logarithm of Eq. (6):

$$\left\{ \begin{array}{l} \log(y) = \log(Ax^\alpha) \\ \log(y) = \log A + \log x^\alpha \\ \text{Assume} \\ \log A = B \\ \text{Then} \\ \log y = B + \alpha \log x \end{array} \right. \tag{6}$$

Last relationship in Eq. (6) has a general form of a linear function as presented by $\log y$, and the slope of this linear logarithmic function is the exponential of power law α and it is known as Hausdorff-Besicovitch or fractal dimension [11].

6. Energy in a High Intense Explosion

When the United States exploded the first nuclear bomb under code name “Trinity” as part of Manhattan Project at a location about 100 miles south of Los Alamos in the aptly-named Jornada del Muerto (Journey of Death) in the Alamogordo Desert of southern New Mexico in 1945, the site was selected to test the implosion device. Oppenheimer named the site “Trinity” after a poem that he had been reading.

The event of this explosion and everything associated with it was under shroud of secrecy, and both data and the motion picture of explosion footage from the explosion were highly classified. Two years later, when US government released a series of pictures of the explosion, along with a size scale and time stamps of the explosion (See Figs. 6 and 7, Trinity explosion and its sequence). Taylor [7, 8] managed to determine the energy released during the explosion using only the expansion radius of the blast wave at time t and the principle of dimension analysis. Later on, when data were declassified, his analysis and calculations were remarkably accurate to what these data were showing.

Taylor first treated the problem in 1941, with a numerical solution for an explosion in air. Taylor’s initially classified paper on the subject was published in 1949 as the formation of a blast wave by a very intense explosion [7] “Theoretical Discussion”. The theoretical discussion of 1941 included some fairly, crude experimental comparisons, based on the blasts of conventional weapons, and Taylor also added some better conventional weapons data for the 1949 publication.

As a better experimental test, Taylor produced a companion article “The Formation of a Blast Wave by a Very Intense Explosion II. The Atomic Explosion of 1945” [8]. Using a time series of then recently declassified photographs of the Trinity explosion near Alamogordo, New Mexico, Taylor tested his scaling hypothesis and calculated the energy of the blast.



Fig. 6 Trinity explosion after $t = 0.025$ sec.

The frequently quoted strength of the explosion—18.6 kilotons (the equivalent of 18.6 thousand tons of trinitrotoluene) is quite close to Taylor’s calculated value of 16.8 kilotons.

While Taylor’s scaling law was correct, and his numerical solution worked remarkably well for the case of air, a more general solution was needed to study a blast wave propagating into a cold gas with a more general equation of state. This is important, for example, in the study of supernova explosions. This need had been already answered by the time Taylor published his paper in 1949. Sedov and von Neumann independently found an analytic solution to the problem for the same set of approximations that Taylor used. For this reason, the solution to the blast problem is conventionally known as the “Sedov solution” [4].

Basically, for his calculation, first, he needed two assumptions as follows:

- (1) The energy E was released in a small space;
- (2) The shock wave was spherical.

Then the questions by him were that:

We have the size of the fireball radius (R) as a function of time (t) at several different times. How does the radius (R) depend on:

- Energy (E);
- Time (t);
- Density of the surrounding medium (ρ_0 initial density of air).

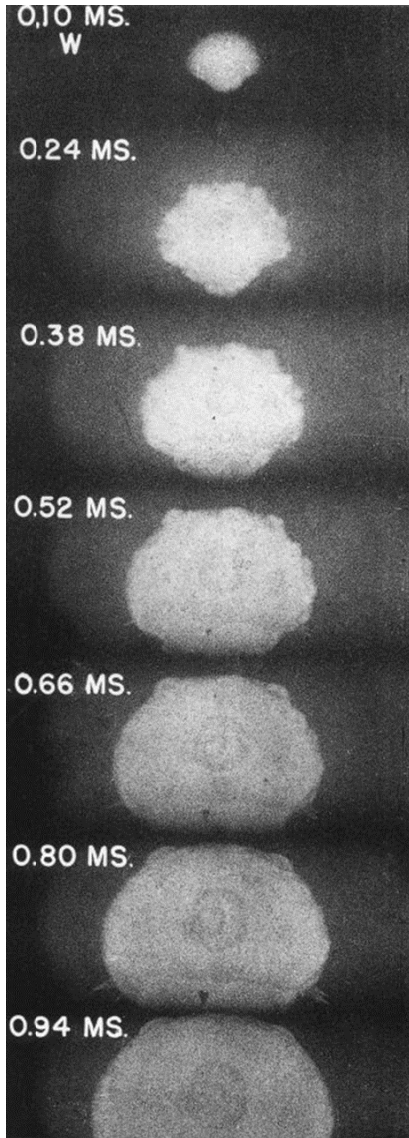


Fig. 7 Trinity explosion sequence.

In a nuclear explosion, there is an essentially instantaneous release of energy E in a small region of space. This produces a spherical shock wave, with the pressure inside the shock wave thousands of times greater than the initial air pressure, which may be neglected. How does the radius R of this shock wave grow with time t ? The relevant governing variables and $[R]=L$ as function of these variable are E, t , and the initial air density ρ_0 , with dimensions $[E] = [E]=ML^2T^{-2}$, $[t]=T$, and $[\rho_0]=ML^{-3}$ [2].

This set of variables has independent dimensions, so $n=3$, $k=3$. We next determine the exponents by

substituting the following [2]:

$$R = f(E, \rho_0, t)$$

$$\begin{aligned} [R] &= L = [E]^a [\rho_0]^b [t]^c \\ &= [ML^2T^{-2}]^a [ML^{-3}]^b [T]^c \end{aligned}$$

or

$$L = M^{a+b} L^{2a-3b} T^{-2a+c}$$

M is to the $a+b$ power because energy and density are both dependent on M .

L is to the $2a-3b$ power because energy is dependent on the square of distance and density is dependent on one over the cube of distance. T is to the $-2a+c$ power because energy is dependent on one over the square of time and time is dependent on time. Therefore, this provides three simultaneous equations as follows:

$$a + b = 0, \quad 2a - 3b = 1, \quad -2a + c = 0$$

with solution $a=1/5$, $b=-1/5$ and $c=2/5$. The radius R is given by Eq. (7) as:

$$R = CE^{1/5} \rho_0^{-1/5} t^{2/5} \quad (7)$$

with C an undetermined constant. If we could plot the radius R of the shock as a function of time t on a log-log plot, the slope of the line should be $2/5$. The intercept of the graph would provide information about the energy E released in the explosion, if the constant C could be determined. By solving a model shock-wave problem, Taylor estimated C to be close to unity ($C=1.033$ for adiabatic index of air $\gamma=1/4$); he was able to take declassified movies of nuclear tests, and using his model, infer the yield of the bombs. The final version of his analysis and calculation is, summarized in Eq. (8).

$$R = E^{1/5} \rho_0^{-1/5} t^{2/5} \quad (8)$$

Again, we have assumed the constant C is approximately 1 ($C \approx 1$) and this implies that, to a good approximation radius of shock wave R can be presented by above equation.

We can now convert to a linear function in order to compute the energy. Taking the logarithm of both sides produces:

$$\begin{cases} \log_{10} R = \frac{2}{5} \log_{10} t + \frac{1}{5} \log_{10} \left(\frac{E}{\rho_0} \right) \\ \frac{5}{2} \log_{10} R = \log_{10} t + \frac{1}{2} \log_{10} \left(\frac{E}{\rho_0} \right) \end{cases}$$

where $\rho_0 = 1.2 \text{ kg/m}^3$. Figs. 6 and 7 illustrate the transformed data from Taylor's original values given in Table 1.

A least-square fit of these data gives an estimate of $(1/2) \log_{10}(E/\rho_0) = 6.90$ so that we have $E = 8.05 \times 10^{13}$ (Fig. 8).

Using the conversion factor of 1 Kiloton = $4.168 \times$

10^{12} Joules gives the strength of Trinity as 19.2 Kilotons. It was later revealed that the actual strength of Trinity explosion was 21 Kilotons. This demonstrates the predictive power of dimensional analysis.

Solving the equation for E , we get:

$$E = (R^5 \rho_0) / t^2 \tag{9}$$

Looking at Fig. 6 and using the number on the slide, we see that at $t = 0.025$ seconds, the radius of the shock wave front was approximately 100 meters. The density of air ρ_0 typically is $\rho_0 = 1.2 \text{ kg/m}^3$. Plugging these values into the energy equation gives:

$$\begin{aligned} E &= (100)^5 \times 1.2 / (0.025)^2 \left[\text{kg} \cdot \text{m}^2 / \text{s}^2 \right] \\ &= 1.92 \times 10^{10} \left[\text{kg} \cdot \text{m}^2 / \text{s}^2 \right] \\ &= 1.92 \times 10^{17} \text{ ergs} \end{aligned}$$

Table 1 Taylor's original data with the time t measured in milliseconds and the radius R in meter [13].

t	$R(t)$	t	$R(t)$	t	$R(t)$	t	$R(t)$	t	$R(t)$
0.10	11.1	0.80	34.2	1.50	44.4	3.53	61.1	15.0	106.5
0.24	19.9	0.94	36.3	1.65	46.0	3.80	62.9	25.0	130.0
0.38	25.4	1.08	38.9	1.79	46.9	4.07	64.3	34.0	145.0
0.52	28.8	1.22	41.0	1.93	48.7	4.34	65.6	53.0	175.0
0.66	31.9	1.36	42.8	3.26	59.0	4.61	67.3	62.0	185.0

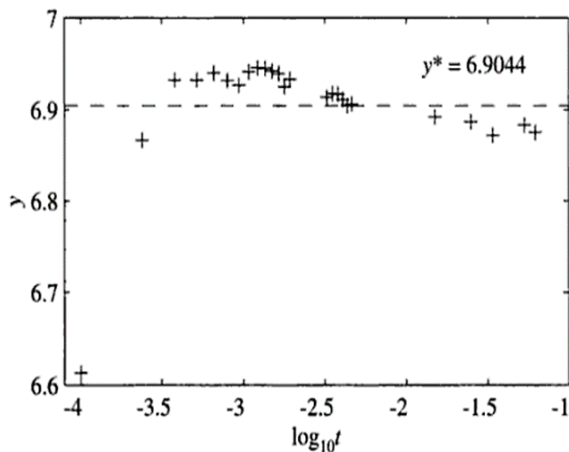


Fig. 8 The graph of $1/2 \log_{10}(E/\rho_0)$, $y = 5/2 \log_{10} R - \log_{10} t$ is identified with the quantity. A least square fit yields $y^* = 6.90$ and because ρ_0 is known, it is possible to solve for E , the energy released during the explosion.

Now, we know that 1 gram of TNT is equal to 4×10^{10} ergs, and hence

$$E \approx 21 \text{ Kilotons of TNT}$$

This datum, of course, was strictly classified; it came as a surprise to the American intelligence communities that this datum was essentially publicly available to those well versed in dimensional analysis. Later, Taylor [7, 8] processing of the photographs taken by H. E. Mack (Figs. 6 and 7) of the first atomic explosion in New Mexico in July 1945 confirmed this scaling law a well-deserved triumph of Taylor intuition.

Note that the entire solution was taking advantages of dimensional analysis along with integration of self-similarity of first kind. For more details refer to references by Zohuri [1, 2].

7. Energy in a High Intense Implosion

The focusing of spherical and cylindrical shock waves has been for various engineering and scientific

applications. It is an effective and economical means of producing high temperature and pressure gas at the implosion center as a focal point. For instance, the converging shock waves process has been successfully adopted in the production of high temperature and density plasma as part of Laser Driven Fusion Program¹⁸⁻²⁰ for imploding pellet of hydrogen isotopes such as deuterium and tritium.

Or for that matter, in man-made explosion of the *Physical Principles of Thermonuclear Explosive Devices* [14] or in nature such as supernovae [15] or other application such as synthetic diamonds from graphite carbide and neutrons [16], they are also used in research related to particles lunched at hypersonic velocity. Moreover, they are associated with research related to substance behavior under server conditions in a high-energy medium [17].

Analytically, Guderley [9] was the first to present a comprehensive investigation involving cylindrical and spherical, shock wave propagation in air, and he obtained a similarity solution. In his solution, shock strength was found to be proportional to R^n where R is the distance of shock from the center of implosion and n , a constant that depends on adiabatic index γ which is specific heat ratio is shown as before ($\gamma = C_p/C_v$). This model clearly implies that theoretically a converging shock wave can increase in strength indefinitely as the radius R approaches zero. In practice, at the center of implosion, temperature and pressure can attain very high but finite values due to experimental limitation. Following Guderley's theory other scientists such as Butler [18] and Stanyukovich [17] focused their effort on the development of similar solutions for the converging process.

Guderley has analyzed the flow behind a converging spherical or cylindrical shock. His treatment of the incoming shock and the flow immediately behind it is complete, but less attention seems to have been paid to the reflected shock and the associated region of disturbance. Butler's paper presented the physical assumptions underlying Guderley's analysis of the

incoming shock are clarified and the reflected shock is treated.

Ashraf [19] is considering imploding spherical and cylindrical shocks near the center (axis) of implosion when the flow assumes a self-similar character. The shock becomes stronger as it converges toward the center (axis) and there is high temperature behind the shock leading to intense exchange of heat by radiation or condition.

His assumption is that the flow behind the shock is not adiabatic but is approximately isothermal, and the time-dependent temperature behind the shock goes on changing as the shock propagates and this temperature is different from that ahead of the shock. The flow behind the shock is likely to have a nearly uniform spatial distribution and that is why the temperature gradient is considered to be zero. This type of flow is known as "homo-thermal flows" and has been dealt by scientists. Except for the idealized intense heat exchange behind the shock, the problem is the same as has been discussed by Guderley (see Fig. 9).

All of them obtained similarity solutions by reducing the problem to nonlinear first order differential equations. The similarity exponent δ of the shock trajectory $R \propto (-t)^\delta$, where t is the time taken by the shock to cover the distance R to reach the origin, cannot be evaluated from dimensional considerations as occurs in the Taylor's explosion problem [7]. The time t taken to be negative before the shock converges to the center (axis) of symmetry and $t = 0$ is instant at which the shock converges to the center (axis).

The shock position is assumed, to be given by Eq. (10):

$$R = A(-t)^\alpha$$

$$\xi = \frac{r}{R} = \frac{r}{A(-t)^\alpha} \quad (10)$$

where R is the radial distance of the shock from center (axis) and A along with α are positive

constants. The interval for variables that are involved with solution that we are seeking includes all of space up to infinity, so that the intervals for the variables are

$$-\infty < t \leq 0, R \leq r < \infty, 1 \leq \xi < \infty$$

Note that the self-similar solution holds only in a region with radius of the order of R , and at large distance, it is connected with the solution of the complete non-self-similar problem in some manner [3].

At the shock front $\xi = 1$ and the front velocity is directed toward the center and is negative, with $\dot{R} = \alpha R/t = -(\alpha R/|t|) < 0$.

The basic equations governing the one-dimensional flow in terms of Lagrangian coordinate η and time t , having temperature as function of time only are Eq. (11):

$$\begin{cases} \frac{\partial r}{\partial \eta} = \frac{1}{\rho r^{j-1}} \\ \frac{\partial u}{\partial t} - r^{j-1} \frac{\partial p}{\partial \eta} \\ \frac{\partial T}{\partial r} = 0 \end{cases} \quad (11)$$

where r is the distance of the particle from the center or axis of symmetry, and u , ρ , p and T are the particle velocity, density, pressure and temperature behind the shock wave respectively. η is the Lagrangian coordinate defined by Eq. (12):

$$d\eta = \rho_0 r_0^{j-1} dr_0 \quad (12)$$

where ρ_0 is the ambient density, r_0 is the value of r at the initial instant of time and $j = 2, 3$ for cylindrical and spherical symmetry respectively.

The continuity equation of (1st set) of Eq. (12) may be expressed in terms of particle velocity u as Eq. (13):

$$\frac{\partial u}{\partial \eta} = -\frac{1}{\rho^2 r^{j-1}} \left[\frac{\partial \rho}{\partial t} + (j-1) \frac{\rho u}{r} \right] \quad (13)$$

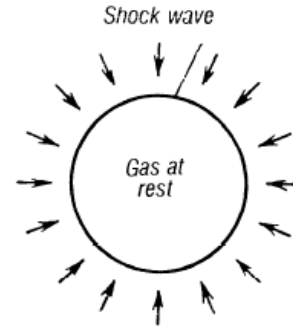


Fig. 9 Diagram of motion of implosion point in front of the core adiabatic compression.

For strong shock, we can now establish the boundary conditions at the shock assuming that the radiative flux across the optically thin shock layer is continuous so that the classical shock condition holds [19], and they are written as:

$$\begin{cases} u_s = (1 - \beta)\dot{R} \\ \rho_s = \rho_0/\beta \\ p_s = (1 - \beta)\rho_0\dot{R}^2 \end{cases}$$

where ρ_0 is the ambient density, \dot{R} the shock velocity and β the density ratio across a strong shock and is equal to $\frac{\gamma-1}{\gamma+1}$.

Ashraf [19] is carrying on a self-similarity solution in this case when he introduces a similarity variable $\mu = \eta/\eta_0$. In his analysis, he seeks a closed form solution via an approximate analytic approach and shows this solution up to second order terms in β .

He also demonstrates for the zeroth order approximation the particle velocity is same as the shock velocity while the density and pressure are linear functions of μ , where implies that the first and second order terms from his established equations contribute more significantly to velocity than to the density and pressure. His solution also describes the Eulerian distance r and Eulerian similarity variable ξ as function of Lagrangian similarity variable μ , which also will indicate that zeroth approximation the Eulerian distance is same as the shock distance.

Analysis of differential equations of gas dynamics associated with this problem allows the similarity

Table 2 Values of α obtained from the analytic expression.

	Cylindrical $j = 2$	Spherical $j = 3$
6/5	0.7767	0.6349
7/5	0.7285	0.5730
5/3	0.6978	0.5359

Table 3 Values of α obtained from the numerical integration.

	Cylindrical	Spherical
6/5	0.7963	0.6553
7/5	0.7530	0.5965
5/3	0.7218	0.5589

exponent α to be, obtained by solving these differential equations either analytically or numerically.

His finding for values of similarity exponent α for the adiabatic flow both analytically and numerically is shown in the Tables 2 and.

He finds that the value of similarity exponent α is smaller for the homo-thermal flows that the adiabatic flows so that the shock velocity $\dot{R} \propto R^{-(1-\alpha)/\alpha}$ is larger in the former case than in the latter as the shock approaches the center of axis. The shock velocity and hence pressure tend to infinity as $t \rightarrow 0$ being the instant of shock implosion.

Zel'dovich and Raizer [3] also show as the shock converges, energy becomes concentrated near the shock front as the temperature and pressure there increase without limit, but the dimensions of the self-similar region decrease with time. They consider a self-similar solution within sphere whose radius decreases in proportion to the radius of the front R .

The effective boundary of this similar region is then considered to be at some constant value

$$(r/R) = \xi = \xi_1.$$

They present an equation for energy contained in spherical implosion situation with the variable radius

$$r_1 = \xi_1 R \text{ as Eq. (14):}$$

$$\begin{aligned}
 E_{S_{\text{implosion}}} &= \int_R^{r_1} 4\pi r^2 dr \rho \left(\frac{1}{\gamma-1} \frac{p}{\rho} + \frac{u^2}{2} \right) \\
 &= 4\pi R^3 \rho_0 \dot{R}^2 \int_1^{\xi_1} g \left(\frac{1}{\gamma-1} \frac{\pi}{g} + \frac{v^2}{2} \right) \xi^2 d\xi
 \end{aligned}
 \tag{14}$$

The integral with respect to ξ from 1 to ξ_1 is a constant, so that the energy $E_{S_{\text{implosion}}} \propto R^3 \dot{R}^2 \propto R^{5-(2/\alpha)}$. The exponent of R is positive for all real values of the specific heat ratio (adiabatic index) γ . For example, for $\gamma = 7/5$, the similar exponent $\alpha = 0.717$, which is close to what is shown by Ashraf calculation [19] in Tables 2 and 3.

$$E_{S_{\text{implosion}}} \propto R^{2.21} \rightarrow 0 \text{ as } R \rightarrow 0 \tag{15a}$$

With integration with respect to ξ extended to infinity ($\xi_1 = \infty$) the integral diverges, thus the total energy in all space is infinite within the framework of the self-similar solution.

In summary, in order to find the value of α numerically a trial and error analysis, where a value of α is assumed and related differential equation is integrated numerically from the initial point A $\xi = 1$, and the behavior of the integral curve is determined. In our value, case for $\delta = 7/5$ the limiting density is (behind the shock front $\rho_1 = 6\rho_0$) about $\rho_{\text{limit}} = 21.6\rho_0$. The density at large distance from the front $r \rightarrow \infty$ before the instant of collapse at the center (or axis) is also $\rho_{\text{limit}} = 21.6\rho_0$, since for $R \neq 0$ and $r \rightarrow \infty$, $\xi = (r/R) \rightarrow \infty$ and $\rho/\rho_0 = G(\xi) \rightarrow G(\infty)$.

Under these conditions, the collapsed energy concentrated at the center of sphere is given as [3]:

$$\int_R^{r_1} 4\pi r^2 dr \rho \left(\frac{1}{\gamma-1} \frac{p}{\rho} + \frac{u^2}{2} \right) \propto r^{5-2/\alpha} \quad (15b)$$

Just as $E_{S_{\text{implosion}}} \propto R^{5-2/\alpha}$ which is seen in above.

Note that again, the entire solution was taking advantages of dimensional analysis along with integration of self-similarity of second kind. For more details refer to references by Zohuri [1, 2].

8. Similarity and Estimating

The notion of similarity is familiar with geometry. Two triangles are similar if all of their angles are equal, even if the two triangles' sides are of different lengths. The two triangles have the same shape; the larger one is simply a scaled-up version of the smaller ones. This notion can be generalized to include physical phenomena. This is important when modeling physical phenomena, such as testing a plane prototype with a scale model in a wind tunnel. The design of the model is dictated by dimensional analysis. The similarity is an extension of geometrical similarity. By definition, two systems are similar if their corresponding variables are proportional at corresponding locations and times. The famous of all and familiar similarity that one can even buy in today's market is Russian nested dolls (Fig. 10). See Refs. [1, 2] for more details.

A Matryoshka doll or a Russian nested doll (often incorrectly referred to as a Babushka doll—babushka means “grandmother” in Russian), is a set of dolls of decreasing sizes placed one inside the other. “Matryoshka” (Матрёшка) is a derivative of the Russian female first name “Matryona”, which was a very popular name among peasants in old Russia. The name “Matryona” in turn is related to the Latin root “mater” and means “mother”, so the name is closely connected with motherhood and in turn, the doll has come to symbolize fertility.



Fig. 10 Russian nested dolls.

A set of matryoshkas consists of a wooden figure, which can be pulled apart to reveal another figure of the same sort inside. It has, in turn, another figure inside, and so on. The number of nested figures is usually five or more. The shape is mostly cylindrical, rounded at the top for the head and tapered towards the bottom, but little else; the dolls have no hands (except those that are painted). Traditionally the outer layer is a woman, dressed in a sarafan. Inside, it contains other figures that may be of both genders, usually ending in a baby that does not open. The artistry is in the painting of each doll, which can be extremely elaborate (see Fig. 10).

Return to the mathematical statement of the Π -Theorem, Eq. (16).

$$a = a_1^p \dots a_k^r \Phi \left(\frac{a_{k+1}}{a_1^{p_{k+1}} \dots a_l^{r_{k+1}}} \right) \quad (16)$$

We can identify the following dimensionless parameters:

$$\begin{aligned} \Pi &= \frac{a_n}{a_1^p \dots a_k^r} \\ \Pi_1 &= \frac{a_{k+1}}{a_1^{p_{k+1}} \dots a_k^{r_{k+1}}} \end{aligned} \quad (17)$$

and so on, such that Eq. (16) can be written as

$$\Pi = \Phi(\Pi_1, \dots, \Pi_{n-k})$$

The parameters $\Pi, \Pi_1, \dots, \Pi_{n-k}$ are known as similarity parameters. Now if two physical phenomena are similar, they will be described by the same function Φ . Denote the similarity parameters of the model and the prototype by the superscripts m and P , respectively. Then if the two are similar, their similarity parameters are equal:

$$\Pi_1^{(P)} = \Pi_1^{(m)}, \dots, \Pi_{n-k}^{(P)} = \Pi_{n-k}^{(m)} \quad (18)$$

So that

$$\Pi^{(P)} = \Pi^{(m)} \quad (19)$$

Therefore, in order to have an accurate physical model of a prototype, we must first identify all of the similarity parameters, and then ensure that they are equal for the model and the prototype.

Finally, we come to estimating. In this course, we will often make order of magnitude estimates, where we try to obtain an estimate to within a factor of ten (sometimes better, sometimes worse). This means that we often drop factors of two, etc., although one should exercise some caution in doing this. Estimating in this fashion is often aided by first doing some dimensional analysis. Once we know how the governed parameter (which we are trying to estimate) scales with other quantities, we can often use our own personal experience as a guide in making the estimate. Similarity is one of the most fundamental concepts, both in physics and mathematics. This first aspect, geometrical similitude is the best known, the best understood, and another one, more abstract, deals with the physical similitude. Since all systems must obey the same physical laws, in addition to the geometrical scaling factors, relations between different physical quantities must be fulfilled in order to make two systems similar. Again, Russian nested dolls (Fig. 10) are very good example of such similarity.

We recognize how central will be these ideas in the theory of modeling. Such reduced models play a central role in shipbuilding, aeronautical engineering, oceanography, etc. In engineering quite often, many different phenomena, belonging to different branches

of science take place simultaneously and conflicts are possible. However, other aspects of similarity can be found in the logic of a machine or in an algorithm. Under its geometrical and logical aspects, similarity and self-similarity appear as rather regular, easy to distinguish patterns. Nature has more fantasy and in some cases, it likes to add some randomness. Self-similarity, in that case, is more difficult to distinguish but is still there.

9. Conclusion

This article was presented as a short review for the readers that are very interested to increase their knowledge, just beyond Pi or Buckingham Theory, and are anxious to understand the mathematics behind energy driven by intense explosion and implosion and how dimensional analysis is driving similarity and self-similarity as well.

More details of this subject would be found in the two books that are written by author Zohuri [1, 2].

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