

Analysis of Logistic Model with Constant Harvesting in a View of Non-Integer Derivative

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Abstract: The conformable fractional derivative method has been utilized in order to examine the logistic model with constant harvesting. Such method introduces a generalization to the classical analysis of Logistic model, and hence the features of the Logistic model, such as subcritical and supercritical harvesting, have been investigated in a view of fractional calculus. The positive auxiliary parameter, σ , with dimension of time is implemented to maintain the dimensionality of the system. The significant information of such parameter to the population has been discussed. The population expressions, obtained by conformable description, are compared with the expressions of the classical derivative. This comparison shows that the non-integer expressions are in a parallel line with that of the classical one.

Keywords: Logistic model, conformable description, non-integer derivative, conformable differential equations, fractional calculus.

1. Introduction

A method of conformable fractional derivative has been considered to solve a logistic model with constant harvesting. Such method is well-known in the applications of applied sciences and engineering. In general, a fractional derivative is appeared after an inquiry, from L'Hôpital to Leibniz in 1695, about a definition of non-integer order derivative (order $\frac{1}{2}$) [1]. Then, in 1826, Abel [2, 3] investigated and solved the tautochrone integral equation using the fractional calculus (FC). In last decades, an extensive applications of FC have been published in variety of disciplines, such as biological, physical and engineering models [2, 4-6] and we will cite in this work some pioneer investigations. For instance, Battaglia et al. [7] adopted FC in order to examine the heat transfer equations, while the water movement through fractured materials is considered by Benson et al. [8]. In rheology, Bagley et al. [9, 10] investigated viscoelastic models utilizing FC method. Their

theoretical analysis shows a very good agreement with the experimental studies.

Sophisticated Biological models, characterized by dynamical systems, require a complicated mathematical technique and in most cases a numerical technique must be adopted. However, in some special cases, the applications of FC may successfully be applied. For example, a modeling of bio-impedance for human skin as well as a model that describes the behaviour of arterial viscoelasticity and hysteresis are successfully investigated in a view of FC [11, 12].

Thus, such method is of interest topic in the research community and has several developments, such as Riemann–Liouville, Hadamard, Erdélyi–Kpber, Caputo and conformable formulations, in order to be appropriate in solving particular problems [2, 13-19]. Considering the conformable fractional derivative, obtained by Khalil et al. [17] and then generalized by Katugampola [16], we have the following formulation as

$$\frac{d^\alpha}{dt^\alpha} f(t) = t^{1-\alpha} \frac{d}{dt} f(t) \quad (1)$$

where $\alpha \in (0,1]$.

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Thus, applying a simultaneous line of reasoning of those in Refs. [20, 21], we reach to a definition of the fractional derivative as follows

$$\frac{d}{dt} \rightarrow \frac{1}{\sigma^{1-\alpha}} \frac{d^\alpha}{dt^\alpha}, \quad \alpha \in (0,1] \quad (2)$$

where σ is an auxiliary positive parameter and has a dimension of time.

Applying (1) in (2) yields,

$$\frac{d}{dt} \rightarrow \frac{t^{1-\alpha}}{\sigma^{1-\alpha}} \frac{d^\alpha}{dt^\alpha} \quad (3)$$

The logistic model with harvesting was analyzed in several studies [22-27]. According to these investigations, two cases for the harvested rate have been recognised, which are a subcritical and a supercritical harvesting. For the subcritical harvesting, the populations survives to the equilibrium state, while for the supercritical harvesting and for any selection value of the initial population, the population dies at a finite time. In a view of classical derivatives, a comprehensive investigation for the sandhill crane population as well as for the fish harvesting have been carried out by Banks [22].

In the following sections, we investigate the logistic model with constant harvesting in a view of the conformable fractional derivative. Then, the subcritical and the supercritical harvesting will be examined. These results will be compared with the solutions of the classical derivative. Finally, the influence of the auxiliary parameter, σ , on the population rate will be discussed.

2. Analytic Solutions

The logistic model with harvesting, which is represented by the initial value problem, is given as

$$\frac{dp(t)}{dt} = rp(t)\left(1 - \frac{p(t)}{k}\right) - h, \quad p(0) = p_0 \quad (4)$$

where, $p(t)$ is the population at times $t \geq 0$, while r, k and h are positive constants and represent the growth rate, carrying capacity and removal rate. p_0 represents the initial population.

Applying (3) in (4), on rearranging, yields

$$\frac{dp(t)}{dt} = \sigma^{1-\alpha} t^{\alpha-1} \left(rp(t)\left(1 - \frac{p(t)}{k}\right) - h \right) \quad (5)$$

the initial value problem in the form of conformable fractional derivative. Thus, it is easily seen that this equation is separable and the integral form is given by,

$$\frac{k}{r} \int \frac{dp}{\frac{k}{r}(\gamma-h) - (p-\frac{k}{2})^2} = \int \sigma^{1-\alpha} t^{\alpha-1} dt \quad (6)$$

where, $\gamma = 4kr$.

As mentioned earlier, two distinguished situations can be detected from (6). The subcritical harvesting, which occurs when $\gamma > h$, while the supercritical harvesting is existed when $\gamma < h$.

2.1 Subcritical Harvesting

When $\gamma > h$, there is a subcritical harvesting and the solution of the initial value problem (6) along with using the initial condition, given in (4), is written as

$$p(t) = \frac{1}{2} \left[k + \sqrt{\frac{k(\gamma-h)}{4r}} \tanh \left(\frac{\sigma^{1-\alpha} t^\alpha}{\alpha} \sqrt{\frac{r(\gamma-h)}{k}} + c \right) \right] \quad (7)$$

where c is given by

$$c = \operatorname{arctanh} \left((2p_0 - k) \sqrt{\frac{4r}{k(\gamma-h)}} \right) \quad (8)$$

Note that the relationship between the two parameters c and p_0 leads us to consider the following two points:

Firstly, a real value for c is obtained when the following inequality is met

$$\left| (2p_0 - k) \sqrt{\frac{4r}{k(\gamma-h)}} \right| < 1 \quad (9)$$

and hence,

$$\frac{1}{2} \left(k - \sqrt{\frac{k(\gamma-h)}{4r}} \right) < p_0 < \frac{1}{2} \left(k + \sqrt{\frac{k(\gamma-h)}{4r}} \right) \quad (10)$$

Note that as $t \rightarrow \infty$, the population, (7), reaches to equilibrium state and is given by

$$p(t) = \frac{1}{2} \left(k + \sqrt{\frac{k(\gamma-h)}{4r}} \right) \quad (11)$$

In addition, when $0 < \alpha < 1$ and $\sigma \rightarrow \infty$, the population (7) survives to finite state (11) in a short of time. Therefore, σ may play an important role in controlling the population rate when the conformable fractional derivative is utilized in the analysis.

Secondly, when p_0 does not meet the inequality (9), c becomes a complex and can be expressed as $c = c_R \pm i\pi$, where c_R is a real part of the complex. Hence, the $\tanh(\dots)$ function in (7) becomes

$$\tanh\left(\frac{\sigma^{1-\alpha}t^\alpha}{\alpha}\sqrt{\frac{r(\gamma-h)}{k}} + c_R \pm i\pi\right) = \coth\left(\frac{\sigma^{1-\alpha}t^\alpha}{\alpha}\sqrt{\frac{r(\gamma-h)}{k}} + c_R\right) \quad (12)$$

Thus, (7) becomes

$$p(t) = \frac{1}{2}\left[k + \sqrt{\frac{k(\gamma-h)}{4r}}\coth\left(\frac{\sigma^{1-\alpha}t^\alpha}{\alpha}\sqrt{\frac{r(\gamma-h)}{k}} + c_R\right)\right] \quad (13)$$

where c_R is given by

$$c_R = \operatorname{arccoth}\left((2p_0 - k)\sqrt{\frac{4r}{k(\gamma-h)}}\right) \quad (14)$$

From (14), it can clearly seen that the population survival is governed by c_R through p_0 . Therefore, there are again two situations may be considered

1) when p_0 satisfies

$$p_0 > \frac{1}{2}\left(k + \sqrt{\frac{k(\gamma-h)}{4r}}\right) \quad (15)$$

then, c_R lies in the interval $(0, \infty)$ and hence $p(t)$ in (13) reaches to finite state, given in (11), as $t^\alpha \rightarrow \infty$.

2) when p_0 satisfies

$$p_0 < \frac{1}{2}\left(k - \sqrt{\frac{k(\gamma-h)}{4r}}\right) \quad (16)$$

then, c_R lies in the interval $(-\infty, 0)$, hence $p(t) \rightarrow -\infty$ as $t^\alpha \rightarrow -\alpha\sigma^{\alpha-1}c_R\sqrt{\frac{k}{r(\gamma-h)}}$. Thus, the population dies at finite time.

2.2 Supercritical Harvesting

When $\gamma < h$, there is a supercritical harvesting and the solution of the initial value problem (6) can be written as

$$p(t) = \frac{1}{2}\left[k - \sqrt{\frac{k(h-\gamma)}{4r}}\tan\left(\frac{\sigma^{1-\alpha}t^\alpha}{\alpha}\sqrt{\frac{r(h-\gamma)}{k}} + c_2\right)\right] \quad (17)$$

where c_2 is given by

$$c_2 = \arctan\left((k - 2p_0)\sqrt{\frac{4r}{k(h-\gamma)}}\right) \quad (18)$$

3. Discussion

Analytic solutions of the logistic model with

constant harvesting are obtained in a view of conformable fractional derivative, where $0 < \alpha \leq 1$. The expressions of the population for the subcritical and supercritical harvesting are explicitly obtained in (7, 13) and (17), respectively. These expressions are equivalent to the solutions of classical derivative when $\alpha \rightarrow 1$ [22].

For $\gamma > h$, the subcritical harvesting is occurred and the expression (7) and (13) describe the population survival to the equilibrium state (11) as $t^\alpha \rightarrow \infty$. It is important to mention that, when the initial population, p_0 , meets the inequality (16), the population will be extinct although the harvesting is subcritical. In contrast, for $h > \gamma$, the supercritical harvesting is existed and (17) represent the population extinct at finite time.

Figs. 1 and 2 were created using the expressions (7) and (13) respectively, for appropriated selected values of σ , p_0 , k , r and h . These Figures show the conformable derivative profiles compared with the classical derivative one, at $\alpha = 1$. When $t^\alpha \rightarrow \infty$, the population of our solutions together with the classical one reach to the finite state (11).

Figs. 3 and 4 were generated using the expansions (13) along with (16) and (17) respectively, for given values of σ , p_0 , k , r and h . Fig. 3 describes that when the initial population p_0 is so low, the population distinct at finite time and for any given values of α .

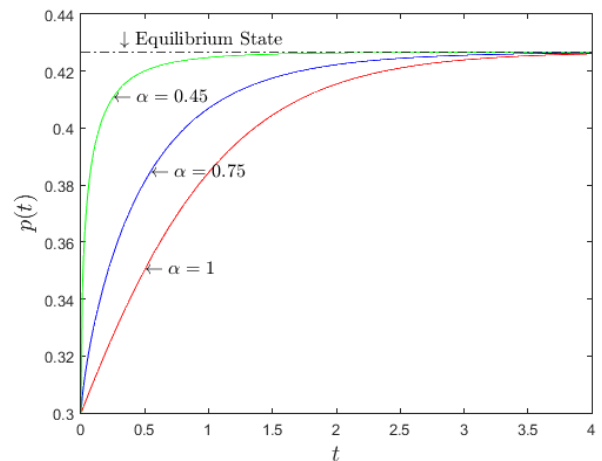


Fig. 1 The population profile for subcritical harvesting is given by (7), where $\alpha = 0.45, 0.75, 1$, $\sigma = 2$, $p_0 = 0.3$, $k = \frac{1}{2}$, $r = \frac{1}{2}$, and $h = 0.5$ ($\gamma = 1 > h$).

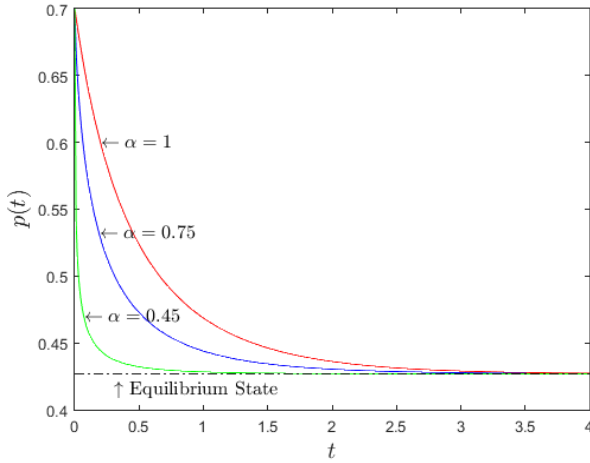


Fig. 2 The population profile for subcritical harvesting is given by (13), where $\alpha = 0.45, 0.75, 1$, $\sigma = 2$, $p_0 = 0.7$, $k = \frac{1}{2}, r = \frac{1}{2}$ and $h = 0.5$ ($\gamma = 1 > h$).

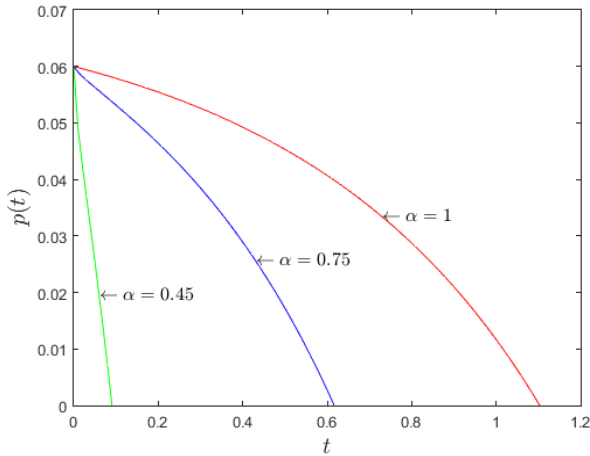


Fig. 3 The population distinct for subcritical harvesting is given by (13), where $\alpha = 0.45, 0.75, 1$, $\sigma = 2$, $p_0 = 0.06$, $k = \frac{1}{2}, r = \frac{1}{2}$ and $h = 0.5$ ($\gamma = 1 > h$).

This is equivalent to what can be predicted from biological grounds. Fig. 4 regards the conformable profile for the supercritical harvesting, which displays the distinction of the population at finite time and for selected values of α .

Figs. 5 and 6 were obtained from the expressions (7) and (13) for different values of σ . These Figures show the variation of the population profiles as σ varies. In addition, the population tends to the equilibrium state markedly as σ increases. This result is corresponding to what can be expected from (7) and (13). Therefore, in a view of conformable fractional derivative, the axillary parameter σ provides a significant information to the population.

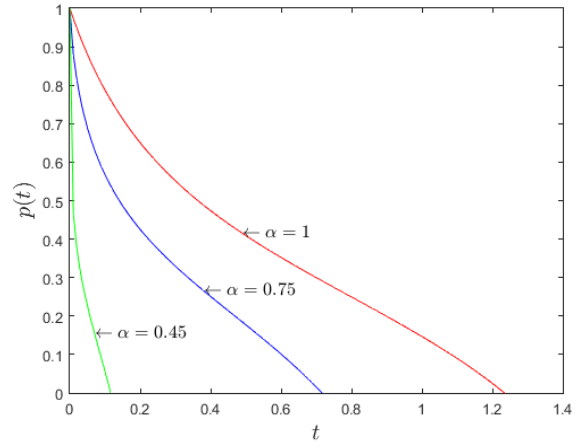


Fig. 4 The population profile for supercritical harvesting is given by (17), where $\alpha = 0.45, 0.75, 1$, $\sigma = 2$, $p_0 = 1$, $k = \frac{1}{2}, r = \frac{1}{2}$ and $h = 3$ ($\gamma = 1 < h$).

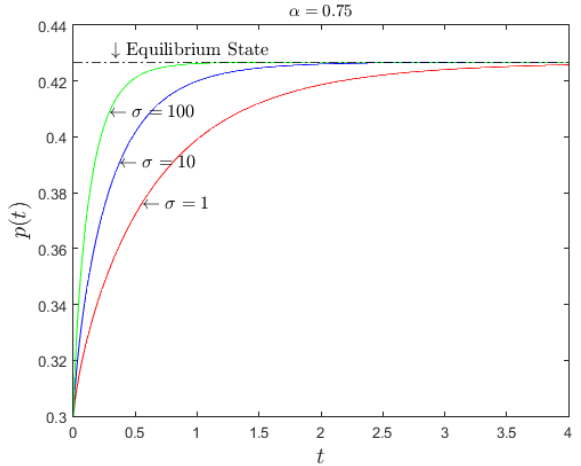


Fig. 5 The population variation as σ varies, which is given by (7) for $\alpha = 0.75$, $p_0 = 0.3$, $k = \frac{1}{2}, r = \frac{1}{2}$ and $h = 0.5$ ($\gamma = 1 > h$).

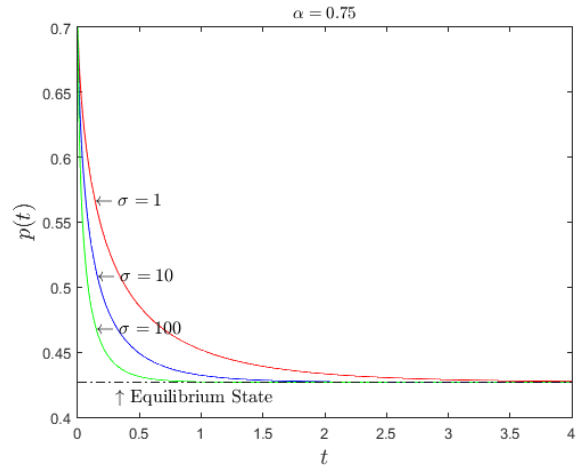


Fig. 6 The population variation as σ varies, which is given by (13) for $\alpha = 0.75$, $p_0 = 0.7$, $k = \frac{1}{2}, r = \frac{1}{2}$ and $h = 0.5$ ($\gamma = 1 > h$).

4. Conclusions

This paper aimed to investigate the logistic model with constant harvesting utilizing the conformable fractional description. This analysis, obtained above, generalizes the classical analysis of the Logistic model with constant harvesting using the conformable derivative and hence the features of the logistic model such as subcritical and supercritical harvesting have been discussed. It is seen that the auxiliary parameter, σ , gives an important information to the population. This analysis may extend by considering the parameters r, k and h vary with time, which leads to shifting the initial value problem to nonlinear system and the numerical technique must be used. However, using an appropriated approximate method combined with the conformable description may obtain valuable approximate solutions that describe the behaviour of the population as the environment changes.

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