Journal of Mathematics and System Science 10 (2020)
doi: 10.17265/2159-5291/2020.01.04

Finslerian Mechanical Systems on Manifolds

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Abstract: The goal of this paper is to present Finslerian mechanical systems on manifolds. In conclusion, some differential geometrical and physical results on the related mechanic systems have been given.

Key words: Finsler metric, Finslerian manifold, Finslerian mechanical systems.

1. Introduction

The present paper devoted to the Analytical Mechanics of the Finslerian Mechanical systems. These systems are defined by a triple $\sum_F = (\mathcal{M}, \mathrm{F}^2, F_e)$ where $\mathcal{M}$ is the configuration space, $F(x, y)$ is the fundamental function of a semi definite Finsler space $\mathrm{F^n} = (\mathcal{M}, F(x,y))$ and Fe(x, y) are the external forces. Of course, F^2 is the kinetic energy of the space. The fundamental equations are the Lagrange equations:

$$E_i(\mathrm{F}^2) = \frac{d}{dt}\left(\frac{\partial \mathrm{F}^2}{\partial \dot{\mathrm{x}}^{\mathrm{i}}}\right) - \frac{\partial \mathrm{F}^2}{\partial \mathrm{x}^{\mathrm{i}}} = F_i\left(x, \dot{\mathrm{x}}^{\mathrm{i}}\right)$$

2. Finsler Metrics and Finsler Spaces

Definition 2.1[1] *A Finsler metric on a smooth manifold* $\mathcal{M}$ *is given by a positive function* $F : T\mathcal{M} \longrightarrow \mathbb{R}$ *such that*

F_1 : F is of C^∞ — class on $\widetilde{T\mathcal{M}}$ and only continuous on the null section of the projection $\pi : T\mathcal{M} \longrightarrow \mathcal{M}$.

F_2: F is positive homogeneous of order one with respect to the fibre coordinates $\dot{x}^i = y^i$, i.e.,

$$F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0.$$

F_3 the fundamental tensor $g_{ij}(x, y)$

F_4: For any $(x;\, y) \in \widetilde{T\mathcal{M}}$ the symmetric bilinear form $g_{(x;y)}$ is nondegenerate and has constant signature, where

$$g_{(x;y)}(v, w) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(x, y + sv + tw)]_{s=t=0}, \quad y, v, w \in T_x\mathcal{M} \tag{1}$$

Definition 2.2 [2] If F is a Finsler metric on a manifold $\mathcal{M}$, the pair $F^n = (\mathcal{M}, F)$is called a Finsler space and the bilinear form $g_{(x;y)}$is called the metric (or the fundamental) tensor of the Finsler space. Also F is called the fundamental function of the Finsler space F^n.

Theorem 2.3 [3] If the base manifold $\mathcal{M}$ is Para compact, then there exist functions F: $T\mathcal{M} \rightarrow \mathbb{R}$. which are the fundamental functions for Finsler spaces.

Proposition 2.4 A Finsler space is reducible to a semi Riemannian space if and only if the fundamental function F of the Finsler space satisfies the following parallelogram identity holds:

$$F^2(x;\, y + v) + F^2(x;\, y ¡\, v) = 2F^2(x;\, y) + 2F^2(x;\, v);\ \forall\, y;\, v \in T_x\mathcal{M}$$

Theorem 2.5 **[4]** For a Finsler space F^n the following properties hold:

1) The functions

$$p_i = \frac{1}{2}\frac{\partial F^2}{\partial y^i} = F\frac{\partial F}{\partial y^i} \tag{2}$$

are the components of a d-covector field on the manifold $\widetilde{T\mathcal{M}}$.

2) The functions

$$C_{ijk} = \frac{1}{4}\frac{\partial^2 F^2}{\partial y^i \partial y^j \partial y^k} = \frac{1}{2}\frac{\partial g_{jk}}{\partial y^i} \tag{3}$$

are the components of a $(0;\, 3) -$type completely symmetric d-tensor field on $\widetilde{T\mathcal{M}}$. This tensor field is called the Cartan tensor field of the Finsler space.

3) The 1-form

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$$\theta = p_i dx^i = \frac{1}{2}\frac{\partial F^2}{\partial y^i} dx^i = F\frac{\partial F}{\partial y^i} dx^i = \frac{1}{2} J^*(dF^2) \qquad (4)$$

is globally defined on the manifold $\widetilde{T\mathcal{M}}$ and it is called the Cartan 1-form of the Finsler space F^n. Here J^* is the cotangent structure (1.10).

4) The 2-form

$$\omega = d\theta = dp_i \wedge dx^i = \frac{1}{2} d(J^*(dF^2)) \qquad (5)$$

is globally defined on the manifold $\widetilde{T\mathcal{M}}$, it is a symplectic structure on $\widetilde{T\mathcal{M}}$ and it is called the Cartan 2-form of the Finsler space F^n.

5) The tangent structure J and the symplectic structure ! satisfy:

$\omega(J(X), Y) + \omega(X, J(Y)) = 0$; $\forall X, Y \in \mathcal{X}(T\mathcal{M})$

Proposition 2.6 [5] For a Finsler space $F^n = (\mathcal{M};\ F(x;\ y))$ the following properties are true:

1) $p_i y^i = F^2$;

2) $y_i = g_{ji} y^j =: g_{0i} = p_i$ (the subscript 0 means contraction with y);

3) $C_{0ih} := y^i C_{ijk} = 0$; $C_{i0h} = C_{j0h} = 0$;

4) $F^2(x, y) = g_{ij}(x, y) y^i y^j$:

Proposition 2.7

(1) For a Finsler space F^n, Cartan forms µ and ! are homogeneous of order one with respect to y. Following properties are also true:

$$i_C\theta = 0 ;\ i_C\omega = 0 \qquad (7)$$

(2) For a Finsler space F^n, the angular metric has rank $(n-1)$ so it is degenerate. The angular metric and the metric tensor of a Finsler space are related by the following formula:

$$g_{ij} = h_{ij} + l_i l_j \qquad (8)$$

3. Finslerian Mechanical Systems

For a manifold M, that is the configuration space of a Finslerian dynamical system, we consider the tangent bundle $T\mathcal{M}$ to which we shall refer to as the phase space. Suppose that there is a Finsler function F on $T\mathcal{M}$ and $F_i(x, y) dx^i$ is a globally defined d-covector field on the phase space.

A Finslerian mechanical — which is a natural extension of the Riemannian one presented in the previous section — is defined by a triple $\Sigma = (\mathcal{M}; E_{F^2}, F_i)$. Here E_{F^2} is the energy of the Finsler space $F^n = (\mathcal{M};\ F)$, that is

$$E_{F^2} = y^i \frac{\partial F^2}{\partial y^i} - F^2 \qquad (9)$$

Since the fundamental function F of the Finsler space is homogeneous of order one with respect to y, we have that F^2 is homogeneous of order two, while the metric tensor g_{ij} is zero homogeneous. Consequently, we have that the energy E_{F^2} coincides with F^2, that is

$$E_{F^2}(x, y) = F^2(x, y) = g_{ij}(x.y)\, y^i y^j \qquad (10)$$

Exactly, as for the Riemannian case, the Lagrange equations of the Finslerian mechanical system Σ are given by

$$\frac{d}{dt}\left(\frac{\partial F^2}{\partial y^i}\right) - \frac{\partial F^2}{\partial x^i} = F_i(x, y),\ y^i = \frac{dx^i}{dt} \qquad (11)$$

Using expression (11), one can write an equivalent form of the Lagrange equations (10) as a system of second order differential equations, given by

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x, y)\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{1}{2} F^i(x, y), \quad y^i = \frac{dx^i}{dt} \quad (12)$$

Where

$$F^i(x, y) = g^{ij}(x, y) F_j(x, y) \qquad (13)$$

Here $\gamma^i_{jk}(x, y)$ are the Christoffel symbols of the metric tensor $g_{ij}(x.y)$, given by expression (10). Equation (11) are called the equations of evolution of the Finslerian mechanical system Σ. Solution curves $c(t) = (x^i(t))$ of Lagrange equations (11) or (12) are called evolution curves.

The system of equation (12) locally determine a dynamical system on the phase space $T\mathcal{M}$. If the external force field $F_i(x, y)$ is globally defined on $T\mathcal{M}$, we shall prove that there exists a globally defined vector field S on $\widetilde{T\mathcal{M}}$, whose integral curves are given by the equations of evolution (12) of the dynamical system. In order to do this, we consider the following functions defined on domains of induced local charts of $\widetilde{T\mathcal{M}}$:

$$2G^i(x, y) = \gamma^i_{jk}(x, y)\, y^j y^k - \frac{1}{2} F^i(x, y) \quad (14)$$

Under a change of coordinates (11) on $T\mathcal{M}$, functions G^i transform according to (1), which means that they are local coefficients of a semispray:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i} \tag{15}$$

The equations of evolution (12) are the equations for the integral curves of the semispray S. As the semispray S is determined by the mechanical system $\sum$ only, we shall refer to S as the evolution semispray .

Theorem 3.2 The variation of the kinetic energy $E_{\mathrm{F}^2} = \mathrm{F}^2$ along the evolution curves (12) of the mechanical system $\sum$ is given by

$$\frac{d\ E_{\mathrm{F}^2}}{dt} = \frac{dx^i}{dt}\mathrm{F_i} \tag{16}$$

Proof. we obtain

$$\frac{d\mathrm{F}^2}{dt} = \frac{\partial \mathrm{F}^2}{\partial x^i}\frac{dx^i}{dt}\frac{\partial \mathrm{F}^2}{\partial x^i}\frac{dy^i}{dt}$$

$$= \left[\frac{d}{dt}\left(\frac{\partial \mathrm{F}^2}{\partial y^i}\right) - \mathrm{F_i}\right]\frac{dx^i}{dt} + \frac{\partial \mathrm{F}^2}{\partial x^i}\frac{dy^i}{dt}$$

$$= \frac{d}{dt}\left(\frac{\partial \mathrm{F}^2}{\partial y^i}y^i\right) - \mathrm{F_i}\frac{dx^i}{dt} = 2\frac{d\mathrm{F}^2}{dt} - \mathrm{F_i}\frac{dx^i}{dt}$$

$$\frac{d\mathrm{F}^2}{dt} = 2\frac{d\mathrm{F}^2}{dt} - \mathrm{F_i}\frac{dx^i}{dt}$$

$$\frac{d\mathrm{F}^2}{dt} = \mathrm{F_i}\frac{dx^i}{dt} = \frac{dx^i}{dt}\mathrm{F_i}$$

Examples3.3 *of Finslerian Mechanical Systems*

1) The systems $\sum_F = (\mathcal{M},\ E_{\mathrm{F}^2}, F_e)$ given by $F^n = (\mathcal{M}, \alpha + \beta\ )$ as a Randers space and $F_e = \beta\mathbb{C} = \beta y^i \frac{\partial}{\partial y^i}$. Evidently F_e is 2-homogeneous with respect to y^i.

2) $\sum_F$ determined by $F^n = (\mathcal{M}, \alpha + \beta\ )$ and $F_e = \alpha\mathbb{C}$.

3) $\sum_F$ with $F^n = (\mathcal{M}, \alpha + \beta\ )$ and $F^n = (\alpha + \beta\ )\mathbb{C}$.

4) $\sum_F$ defined by a Finsler space $F^n = (\mathcal{M}, F)$ and $F_e = a^i_{jk}\, y^i y^k \frac{\partial}{\partial y^i}, a^i_{jk}\,(x)$ being a symmetric tensor on the configuration space M.

4. Conclusions

The geometry of the Finslerian mechanical system $\sum$ is determined by the geometry of the Lagrange space $L^n = (M;\ \mathrm{F}^2(x;\ y))$ endowed with the evolution semispray S.

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