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Perturbation by Decomposition: A New Approach to Singular Initial Value Problems with Mamadu-Njoseh Polynomials as Basis Functions

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Abstract: This paper focuses on the application of Mamadu-Njoseh polynomials (MNPs) as basis functions for the solution of singular initial value problems in the second-order ordinary differential equations in a perturbation by decomposition approach. Here, the proposed method is an hybrid of the perturbation theory and decomposition method. In this approach, the approximate solution is slihtly perturbed with the MNPs to ensure absolute convergence. Nonlinear cases are first treated by decomposition. The method is,easy to execute with well-posed mathematical formulae. The existence and convergence of the method is also presented explicitly. Resulting numerical evidences show that the proposed method, in comparison with the Adomian Decomposition Method (ADM), Homotpy Pertubation Method and the exact solution is reliable, efficient and accuarate.

Keywords: Perturbation method, Orthogonal polynomials, Mamadu-Njoseh polynomials, Chebychev polynomials, singular initial value problems, ordinary differential equation (ODE).

1. Introduction

The perturbation theory is a mathematical model employed to estimate an approximate solution which previously was unable to solve accurately by starting from a complete solution of a related problem. It was first proposed in 1927 by Paul Dirac when he investigated the emission of particles radioactive elements in the laboratory. In recent times, practical applications of the perturbation theory have been well noted in many areas of science and technology including the Newton's law of universal gravitation. Also, many Nonlinear equations in the literature have been solved by the application of perturbation theory, such as Refs. [1-3].

Applications of polynomials as basis functions for seeking the approximate solution of many mathematical problems have been on the increase. For instance, Mamadu and Njoseh [4] used certain orthogonal polynomials as basisfunctions for solving integral equations. In like manner, Njoseh and Mamadu [5] applied the Mamadu-Njoseh polynomials for the solution of fifth order boundary value problems. Also, Njoseh and Mamadu [6] used the Chebychev polynomials as basisfunctions for the solution to twelfth order boundary value problems. For more references [7-9].

The central theme of this research is to explore the numerical solution of the singular initial value problems in the second-order differential equations of the form [10]:

$$y'' + \frac{2n}{x}y' + \frac{n(n-1)}{x^2}y + f(x,y) = g(x), \; n \geq 0, \quad (1)$$

with initial conditions

$$y(0) = \alpha, \qquad y'(0) = \beta,$$

where y is the unknown function, $f(x,y)$ is nonlinear and$g(x)$ is the source term.

To solve (1), we have developed a mathematical algorithm called "perturbation by decomposition". The method employs the MNPs (which are A-stable in the interval [-1,1]) as trial functions in the approximation of the analytic solution of Eq. (1).

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Basically, well structured mathematical formula for evaluating both linear and nonlinear cases is derived. Also, the nonlinear term $f(x, y)$ is first decomposed with a well-posed formula with given exact solution as the initial approximation in the course of implementation.

2. Perturbation by Decomposition

In this section, we derive some eloquent formulations for solving Eq. (1).

Now consider the linear case of Eq. (1) given as:

$$y'' + \frac{2n}{x}y' + \frac{n(n-1)}{x^2}y = g(x), \; n \geq 0, \quad (2)$$

with initial conditions

$$y(0) = \alpha, \qquad y'(0) = \beta.$$

Define a power series solution for Eq. (2) as

$$y_m(x) = \sum_{r=0}^{m} a_r x^r, \quad (3)$$

where a_r, r =0(1)m are the unknown constants.

Differentiating Eq. (3) twice and substituting into (2) to obtain:

$$\sum_{r=2}^{m} a_r r(r-1)x^r + 2n\sum_{r=1}^{m} r a_r x^r + n(n-1)\sum_{r=0}^{m} a_r x^r = g(x) \quad (4)$$

This implies that

$$\sum_{r=0}^{m} a_{r+2}(r+1)(r+2)x^{r+2} + 2n\sum_{r=0}^{m} a_{r+1}(r+$$
$1xr+1+$nn$-1r=0marxr=g(x)$ (5)

Now, slightly perturbing Eq. (5) with the parameter $H_m(x)$, that is,

$$\sum_{r=0}^{m} a_{r+2}(r+1)(r+2)x^{r+2} + 2n\sum_{r=0}^{m} a_{r+1}(r+$$
$1xr+1+$nn$-1r=0marxr=gx+Hm(x)$ (6)

where

$$H_m(x) = \sum_{r=1}^{m} \alpha_r \varphi_r(x), -1 \leq x \leq 1 \quad (7)$$

where $\alpha_r, r = 1,2, \ldots m$, are constants to be determined, $\varphi_r(x), r = 1,2, \ldots, m$, are the Mamadu-Njoseh polynomials [5].

Note that Eq. (6) is an identity in x. Thus, equating to zero the coefficients of various power of x to get system of linear equations for $0 \leq r \leq 2$ as follows:

$$a_{r+2} = \frac{(g(x) + {}^{5}/_{3}\alpha_2)}{(r+1)(r+2)} \;,$$

$$a_{r+1} = \frac{\alpha_1}{2n(r+1)} \;,$$

$$a_r = -\frac{2\alpha_2}{3n(n-1)}.$$

On using the initial conditions, we arrive at,

$$a_0 = \alpha, \; a_1 = \beta.$$

$$\Rightarrow a_2 = {}^{g(x)}/_{12}.$$

Substituting these estimates into (3), we get our approximate solution as:

$$y_2(x) = \alpha + \beta x + {}^{g(x)}/_{12}\, x^2 \quad (8)$$

Case 2: When the problem is nonlinear, then it has the form

$$y'' + \frac{2n}{x}y' + \frac{n(n-1)}{x^2}y + f(x, y) = g(x), n \geq 0$$

with initial conditions

$$y(0) = \alpha, \qquad y'(0) = \beta$$

Here, $f(x, y)$ is the nonlinear term, and must be decomposed using the formulation,

$$f(x, y) \equiv A_m = \sum_{k=0}^{m} y_{m-k}(x) y_k(x), \quad (9)$$

where the initial approximation is the exact solution given. Thus, the system of linear equations defined in $0 \leq r \leq 3$ are as follows:

$$a_{r+2} = \frac{(g(x) - A_m + {}^{5}/_{3}\alpha_2)}{(r+1)(r+2)} \;,$$

$$a_{r+1} = \frac{\alpha_1}{2n(r+1)} \;,$$

$$a_r = -\frac{2\alpha_2}{3n(n-1)}.$$

Thus, the required solution becomes:

$$y_2(x) = \alpha + \beta x + \frac{g(x) - A_m}{20} x^2 \quad (10)$$

3. Existence and Convergence of Solutions

Theorem 3.1 Let $2 \leq r \leq 3$, then the singular initial value problem (1) has a unique solution.

Proof. Let y and y^* defines two separate solutions of Eq. (1), then

$$|y - y^*| = \left| \sum_{r=0}^{m} a_r x^r - \sum_{k=0}^{m} a_k x^k \right|$$

$$\leq \sum_{r=0}^{m} x^r\,|a_r - a_r^*| + \sum_{r=0}^{m} x^r\,|a_k - a_k^*|$$

$$\leq r(l_1 + l_2)|y - y^*|$$

$$= r|y - y^*|$$

Thus, we get $(3-r)|y - y^*| \leq 2$. Since $2 \leq r \leq 3$ then $|y - y^*| = 0$. Therefore, $y = y^*$.

Theorem 3.2 The solution (8) converges to the analytic solution of Eq. (2) when $0 \leq r \leq 2$.

Proof. Set $y(x) = \lim_{m\to\infty} y_m(x)$. According to Eqn. (1.8), we have

$$y(x) = \alpha + \beta x + {}^{g(x)}/_{12}\,x^2 \qquad (11)$$

Subtracting (11) from (8),

$$y_m(x) - y(x) =$$

$$\sum_{r=0}^{m} a_r x^r - \left(\alpha + \beta x + {}^{g(x)}/_{12}\,x^2\right) \qquad (12)$$

If we set $e_{m+1}(x) = y_{m+1}(x) - y_m(x)$, $e_m(x) = y_m(x) - y(x), |e_m^*| = \max_x |e_m(x)|$, then since e_m is monotone decreasing from (12), applying the mean value theorem we have:

$$e_{m+1} = e_m - \sum_{r=0}^{m} a_r x^r$$

$$\Rightarrow |e_{m+1} - e_m + \textstyle\sum_{r=0}^{m} a_r x^r| \leq (\textstyle\sum_{r=0}^{m} x^r\,|a_r -$$

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$$\leq b(l_1 + l_2)|e_m(x)|$$

$$= br_1\|e_m(x)\|$$

Since $e_m(x,\mu) \geq e_m(x,r)$ for $0 \leq \mu \leq r$, $a \leq r \leq b$, then

$$|e_{m+1}(x,r) - e_m(x,r) + re_m(x,r)| \leq br_1\|e_m(x)\|$$

$$\Rightarrow |e_{m+1}(x,r) - |2-r|\|e_m(x)\|| \leq br_1\|e_m(x)\|$$

$$\Rightarrow |e_{m+1}(x,r) \leq |2-r| + br_1\|e_m(x)\||$$

$$\leq \begin{cases} (2 - a + br_1)\|e_m(x)\|, & 0 < r \leq 2 \\ (b(1+r_1) - 2)\|e_m(x)\|, & r > 2 \end{cases}$$

Thus,

$$\|e_{m+1}(x)\| = \max_{x\in J}|e_{m+1}(x)|$$

$$\leq r_1 \max_{x\in J}|e_m(x)| = r_1\|e_m(x)\|$$

Since $0 \leq r \leq 2$, then $\|e_m(x)\| \to 0$.

This complete the proof.

Theorem 3.3 The solution (1.10) converges to the exact solution of problem (1.1) when$0 \leq r \leq 3$.

Proof. The proof is similar to the the previous theorem.

4. Numerical Applications

We next consider some numerical applications in this section. The error formulation for the problems is defined explicitly as

$$e_r = |y(x) - y_i(x)|, \qquad r = 1,2,3\ \ldots,$$

where $y(x)$ is the analytic or exact solution as available in literature, and $y_i(x)$ is the computed solution.

Example 4.1 Consider the linear singular initial value problem

$$y'' + \tfrac{4}{x}y' + \tfrac{2}{x^2}y = 12, \qquad (13)$$

$$y(0) = 0,\ y'(0) = 0,$$

by the perturbation method for case $n = 2$ using Mamadu-Njoseh polynomials of degree 2 as perturbation type. The exact solution is given as $y(x) = x^2$.

Here, Eq. (13) is linear, and such $\alpha = \beta = 0$, $g(x) = 12$. Using Eq. (8), we obtained the required solution as

$$y(x) = x^2,$$

which is the exact solution.

Example 4.2 Consider the nonlinear singular initial value problem

$$y'' + \tfrac{6}{x}y' + \tfrac{6}{x^2}y + y^2 = 20 + x^4 \qquad (14)$$

$$y(0) = 0,\ y'(0) = 0,$$

by the perturbation method for case $n = 2$ using Mamadu-Njoseh polynomials of degree 2 as perturbation type. The exact solution is given as $y(x) = x^2$.

Here, Eq. (14) is nonlinear, and such $\alpha = \beta = 0$, $g(x) = 20 + x^4$.

Also,

$$a_{r+2} = \frac{\left(g(x) - A_m + {}^{5}/_{3}\alpha_2\right)}{(r+1)(r+2)},$$

$$\mathrm{A_m} = \sum_{k=0}^{m} \mathrm{y_{m-k}(x) y_k(x)}.$$

$$\text{For } \mathrm{m} = 0: \quad \mathrm{A_0} = \mathrm{y_0^2(x)} = \mathrm{x}^4$$

This implies that

$$a_2 = \frac{(20 + x^4 - x^4)}{20}, \ r = 3, \ \alpha_2 = 0$$

Using Eq. (10), we obtained the required solution as

$$y(x) = x^2$$

which is the exact solution.

5. Discussion of Results

We have applied the proposed method for the solution of both linear and nonlinear singular initial value problems in the second-order ordinary differential equations using Mamadu-Njoseh polynomials (MNPs) as trial functions. We observed from the numerical applications that the Mamadu-Njoseh polynomials as basis functions yield the exact solution itself. This is so because the proposed method being a semi-analytic method coupled with A-stable MNPs will ensure absolute convergence.

6. Conclusion

In this paper, we have derived and implemented a new mathematical algorithm method for solving both linear and nonlinear singular initial value problems in the second-order ordinary differential differential equations employing the Mamadu-Njoseh polynomials as basis/trial functions. Here, the proposed method converges absolutely to the exact solution without any hidden transformation or linearization.

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