Water Wave Focusing Using Coordinate Transformation

Takahito Iida[1, 2] and Masashi Kashiwagi[1]

1. Department of Naval Architecture and Ocean Engineering, Osaka University, Osaka 5650871, Japan

2. Research Fellow (DC1) of Japan Society for the Promotion of Science, Osaka 5650871, Japan

Abstract: In this paper, wave focusing based on a coordinate transformation is proposed. It is known that the 2-dimensional wave equation which governs a shallow water problem in a potential theory can keep invariance under coordinate transformation. Once equivalent medium parameters are obtained so as to keep the invariance, wave rays can be arbitrarily designed. We show the design of equivalent medium for wave squeezing to focus waves on a specific domain. Numerical computations are carried out by a finite element based software COMSOL Multiphysics. Results show good agreement between predictions from the theory and computations. It can be applied for a wide range of frequency because the proposed method is able to be applied regardless of the frequency.

Key words: Coordinate transformation, wave focusing, shallow water.

1. Introduction

In order to control water wave propagations, many methods have been proposed. For instance, Murashige et al. [1] designed a submerged wave lens and they attempted to focus water waves at one point. Clément et al. [2] arranged multiple circular cylinders into a triangle section so that the cylinders behave as a prism to bend wave rays. It is well known that the phase velocity or refractive index depends on a depth of sea bed and thus waves are controllable by designing the geometry of bed [3]. Wave cancellation by utilizing scattering waves from floating bodies [4] is also popular way for wave control. However, these proposed methods are not perfect to control water waves; in most cases these methods are dependent on the frequency or difficult to obtain an analytical design.

Iida et al. [5] proposed a different concept of a controlling method based on the coordinate transformation in shallow water, which can manipulate wave rays more freely. The coordinate transformation based method was originally proposed in electromagnetic waves [6] and this method is based on a fact that governing equations (Maxwell's equations) are invariant under coordinate transformation. Iida et al. [5] extended this method to water waves.

Since motivations of wave controlling are in most cases to focus waves at an arbitrary point or to deflect waves to protect structures, the coordinate transformation method is applied to a wave focusing in this paper (a transformation for cloaking a structure from waves was studied [5]). First, invariance of the 2-dimensional wave equation under coordinate transformation is derived. Second, a transformation to squeeze wave rays is applied and required fluid parameters are designed. Numerical computations are carried out by using obtained fluid parameters and computed wave patterns are shown. For computations, a finite element method based software COMSOL Multiphysics is employed and a linear potential flow in shallow water problem is considered.

2. Theory

2.1 Coordinate Transformation and Invariance of Governing Equation

In order to apply the coordinate transformation

Corresponding author: Takahito Iida, Ph.D. student, research fields: ocean engineering, water waves.

method, a governing equation should keep invariance under coordinate transformation. Here we consider the free surface wave problem in shallow water. Small amplitude of incident waves and inviscid flow with irrotational motion are assumed, and then the problem is treated as a linearized potential flow. We focus and take attention on the condition that the wave length λ is sufficiently larger than the water depth h (regularly $\lambda > 20\,h$ is a criterion of long wave approximation with 4% error). Then the governing equation of shallow water can be written as the 2-dimensional wave equation (Helmholtz equation) as follows:

$$\nabla^2 \phi + k^2 \phi = 0 \qquad (1)$$

where ∇ is a vector differential operator, $\nabla^2 = \nabla \cdot \nabla$ is the Laplacian, ϕ is the amplitude of the velocity potential and k is the wave number. Since we consider the shallow water problem, the velocity potential ϕ can be replaced by wave amplitude η and the wave number $k = \omega/\sqrt{gh}$, where ω is the circular frequency, g is the gravitational acceleration and h is the water depth. Here we consider transforming Eq. (1) to an arbitrary coordinate system [7]. First, Eq. (1) is integrated with an arbitrary volume V as

$$\int_V \{\nabla^2 \phi + k^2 \phi\} dV = 0 \qquad (2)$$

Eq. (2) is deformed by applying the Gauss's divergence theorem as follow

$$\int_{\partial V} \nabla \phi \cdot n \, dS + \int_V k^2 \phi \, dV = 0 \qquad (3)$$

where ∂V is a smooth closed surface and n is the outward normal vector to ∂V. Eq. (3) is a form on the Cartesian coordinate system x and let us transform it to an arbitrary coordinate system x', then we obtain

$$\int_{\partial V'} \frac{AA^T}{|A|} \nabla' \phi \cdot n' \, dS' + \int_{V'} \frac{1}{|A|} k^2 \phi \, dV' = 0 \qquad (4)$$

where superscript T represents the transpose of a matrix and A is the Jacobian transformation matrix between the original and transformed systems with following coefficients

$$A_{ij} \equiv \frac{\partial x_i'}{\partial x_j} \qquad (5)$$

The governing equation on an arbitrary system can be obtained by Eq. (4) as

$$\nabla' \cdot \frac{AA^T}{|A|} \nabla' \phi + \frac{1}{|A|} k^2 \phi = 0 \qquad (6)$$

Differences compared between Eqs. (1) and (6) are coefficient terms represented by A, and if a medium realizing these coefficients exists, the wave equation is invariant under coordinate transformation. Note that such media frequently do not exist in nature, and materials artificially developed to satisfy these coefficients are called metamaterials [8]. In electromagnetic waves, these coefficients are controlled by permeability and permittivity [6], or fluid density and bulk modulus [9] are used in acoustic waves. In the shallow water problem, some medium parameters are considered for control, such as fluid density [5], water depth [7] or gravitational acceleration [7]. In this paper, we define these parameters as the water depth and the gravitational acceleration which are proposed by Zareei et al. [7]. Then Eq. (6) is deformed as

$$\nabla' \cdot \frac{AhA^T}{|A|} \nabla' \phi + \frac{1}{|A|} \frac{\omega^2}{g} \phi = 0 \qquad (7)$$

Here the water depth and the gravitational acceleration are redefined as

$$h' = \frac{AhA^T}{|A|}, g' = |A|g \qquad (8)$$

Then, Eq. (7) is rewritten as

$$\nabla' \cdot (h' \nabla' \phi) + \frac{\omega^2}{g} \phi = 0 \qquad (9)$$

Therefore, if the water depth and the gravitational acceleration satisfy Eq. (8), shallow water keeps invariance under coordinate transformations. We can figure out that the water depth h' becomes a 2×2 tensor, that is, the water depth should be anisotropic. Such water depth can be realized by a homogenization method [10] for instance. As in Eq. (8), the gravitational acceleration g' is also required variable, however, it is not easy to change the gravitational acceleration directly. There are some feasible techniques to avoid this problem; limiting the coordinate transformations to satisfy $|A| = 1$ [10] or constant [7], or replacing the variable from the

gravitational acceleration to another parameter. Iida et al. [11] proposed a small water channel network that realized anisotropy by designing the depth and widths along the x and y axes of the small cross joined water channel as an alternative of the gravitational acceleration.

2.2 Coordinate Transformation for Wave Squeezing

In this paper, the coordinate transformation to focus waves at a particular region is considered. Fig. 1A shows an original Cartesian coordinate system and this system is deformed to squeeze at the center domain as shown in Fig. 1B. Transformation equations for colored sections as in Fig. 1B are represented by

$$\begin{cases} x_1' = x_1, \ y_1' = \frac{y_1}{a}\left(\frac{b-a}{\ell}x_1 + a\right), \ z_1' = z_1 & \text{on blue} \\ x_2' = x_2, \ y_2' = \frac{b}{a}y_2, \ z_2' = z_2 & \text{on yellow} \\ x_3' = x_3, \ y_3' = \frac{y_3}{a}\left(\frac{a-b}{\ell}x_3 + a\right), \ z_3' = z_3 & \text{on green} \end{cases}$$
$$(10)$$

where a, b and ℓ are the sizes of the squeezed domains and x_i-y_i, x_i'-y_i' are the local coordinates for colored sections. Required parameters to keep the invariance under the transformation are obtained by substituting Eq. (10) into Eq. (8), as

$$h_i' = \begin{pmatrix} e_{x'x'}^i & e_{x'y'}^i \\ e_{y'x'}^i & e_{y'y'}^i \end{pmatrix} h, \ g_i' = g/e_{x'x'}^i \quad (11)$$

where

$$\begin{cases} e_{x'x'}^1 = \frac{a\ell}{(b-a)x_1'+a\ell} \\ e_{x'y'}^1 = e_{y'x'}^1 = \left(\frac{a\ell}{(b-a)x_1'+a\ell}\right)^2 \frac{b-a}{a\ell}y_1' \\ e_{y'y'}^1 = \left(\frac{a\ell}{(b-a)x_1'+a\ell}\right)^3 \left(\frac{b-a}{a\ell}\right)^2 y_1'^2 + \frac{(b-a)x_1'+a\ell}{a\ell} \end{cases}$$
$$(12)$$

$$\begin{cases} e_{x'x'}^2 = \frac{a}{b} \\ e_{x'y'}^2 = e_{y'x'}^2 = 0 \\ e_{y'y'}^2 = \frac{b}{a} \end{cases}$$
$$(13)$$

$$\begin{cases} e_{x'x'}^3 = \frac{a\ell}{(a-b)x_3'+a\ell} \\ e_{x'y'}^3 = e_{y'x'}^3 = \left(\frac{a\ell}{(a-b)x_3'+a\ell}\right)^2 \frac{a-b}{a\ell}y_3' \\ e_{y'y'}^3 = \left(\frac{a\ell}{(a-b)x_3'+a\ell}\right)^3 \left(\frac{a-b}{a\ell}\right)^2 y_3'^2 + \frac{(a-b)x_3'+a\ell}{a\ell} \end{cases}$$
$$(14)$$

Here Fig. 2 shows the spatial distributions of constative tensors on blue section written in Eq. (12). We find out that the water depth and the gravitational acceleration must be heterogeneous in the space, and the water depth needs to be anisotropic. If such situation is realized, waves in Fig. 1A and waves in Fig. 1B become equivalent. In Fig. 1A, waves propagate straightforwardly along the arrow without scattering. On the other hand, waves in Fig. 1B are focused on the center domain but still propagate along the arrow with no scattering. That is, propagating waves with width a are transformed to the waves with width b ($a > b$).
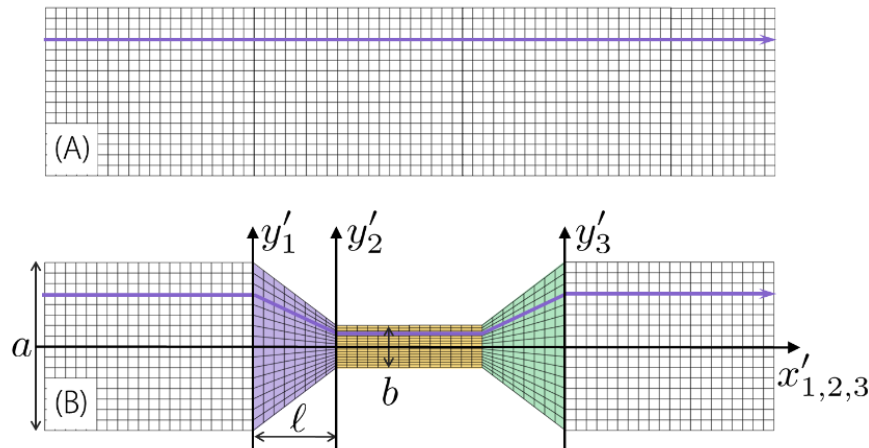

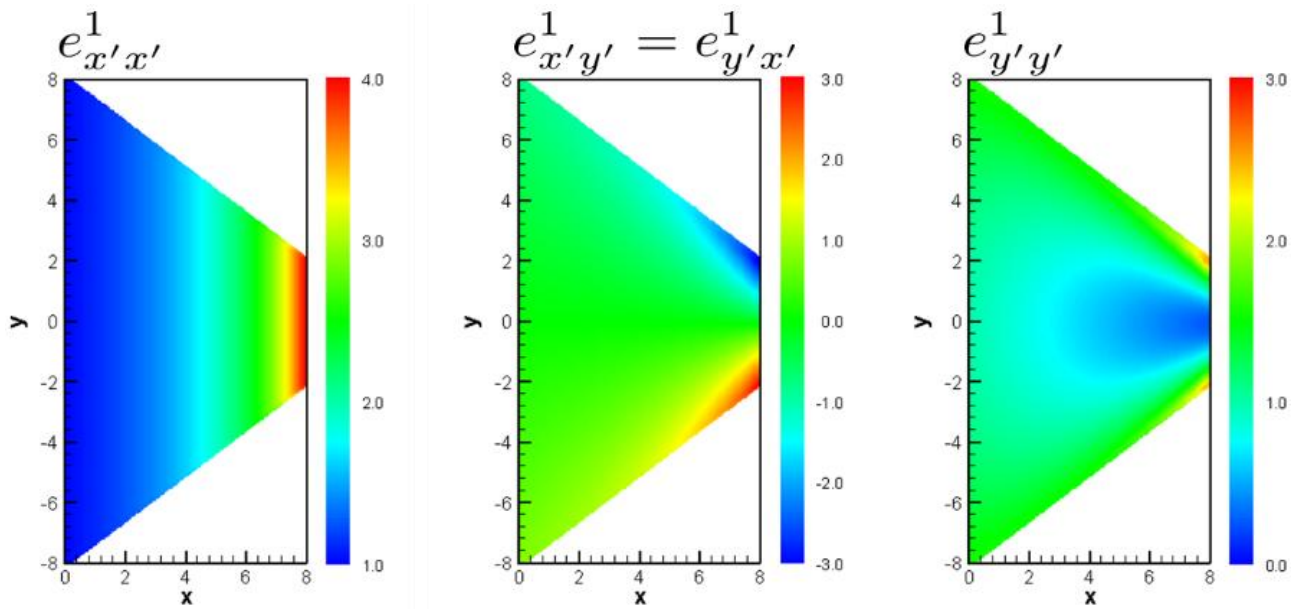**Fig. 1** (A) Cartesian coordinate system. (B) Squeezed coordinate system for wave focusing.

Fig. 2   Spatial distributions of constitutive tensors on blue section.

3. Computation Results and Discussion

In this paper, wave patterns are computed under the assumptions that the water depth and the gravitational acceleration can satisfy Eq. (11). Note that such situation could be manufactured by homogenization methods for instance. To take account of anisotropy and heterogeneity for the fluid analysis, a finite element method based software COMSOL Multiphysics is used, and the shallow water problem with the linear potential theory is solved. Here the fluid domain is designed as in Fig. 1B to be focused on the center as a wave guide. The 2-dimensional wave equation is considered as the governing equation and it is represented by Eq. (9) on the transformed sections. Upper and lower boundaries are walls and waves propagate from the left region to the right region. The water depth of original wave guide is $h = 1.0$ and all values are normalized by the incident wave amplitude η_I, the fluid density ρ and the original water depth h. The width of the wave guide $a = 160$ is squeezed to be $b = 40$, and lengths of these connecters and path are $\ell = 80$.

Wave patterns at wave length $\lambda = 60$ obtained by numerical computations are shown in Fig. 3. Fig. 3A is

a result of a wave pattern without assumption of Eq. (11); the water depth and the gravitational acceleration are constant for all domains. Waves incident from the left boundary are scattered on the connecter and some waves are reflected. Hence only few waves can go through the squeezed section. Next, values of Eqs. (11)-(14) are assumed on the color sections, and the resultant wave pattern is shown in Fig. 3B. Waves propagating the wave guide are not scattered on the connecter and all waves are led to the narrow guide. Moreover, waves after transmitting the narrow guide keep the original wave shapes. It is because the space on Fig. 1B where Eq. (11) is assumed is essentially equivalent to the space on Fig. 1A and thus waves behave as if the space is rectangular in spite of connecting the narrow guide.

Fig. 4 shows a wave pattern at other three different wave lengths. Here the water depth is $h = 1.0$ and thus we show results of $\lambda \geq 20\,h$ which is the condition that the wave length is sufficiently larger than the water depth. Figs. 4(A) and (a) are the result at $\lambda = 20$, (B) and (b) are at $\lambda = 40$ and (C) and (c) are at $\lambda = 80$. Capital letters represent the cases that the water depth and the gravitational acceleration are constant, and small letters show the cases assuming Eq. (11). In the
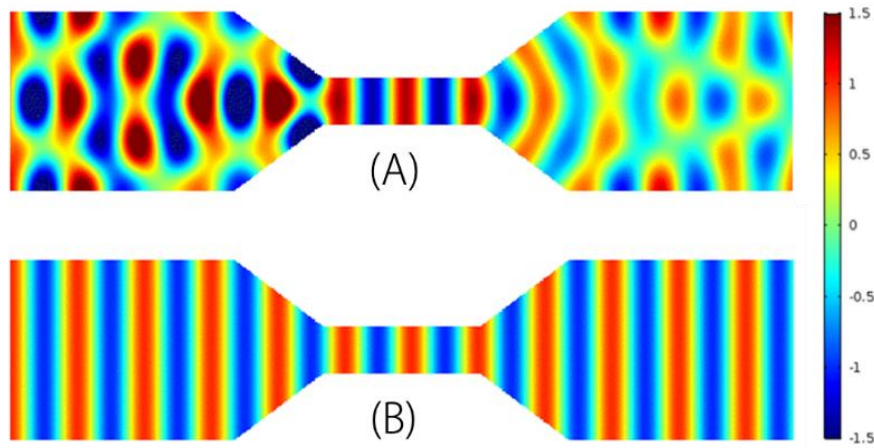
Fig. 3 Spatial wave patterns of $\lambda = 60$: (A) Result withflat water depth and constant gravitational acceleration. (B) Result with special distributed materials represented by Eq. (11).
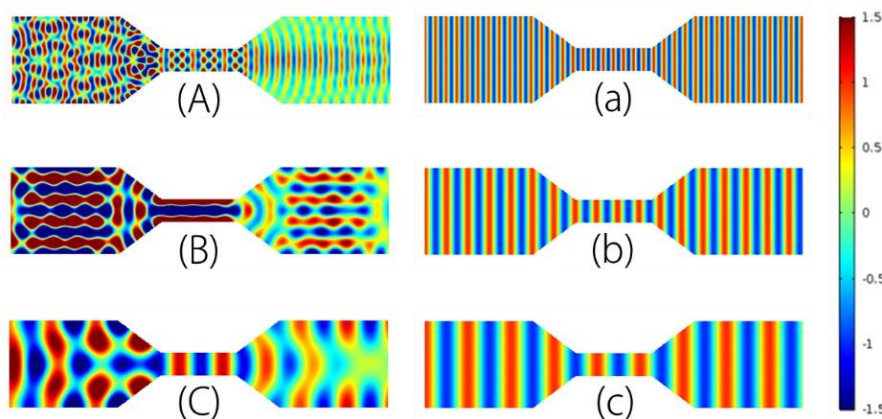

Fig. 4 Spatial wave patterns corresponding to wave length. Capital letters show results with flat water depth and constant gravity and small letters are with special distributed materials represented by Eq. (11): (A) & (a) Wave length $\lambda = 20$; (B) & (b) Wave length $\lambda = 40$; (C) & (c) Wave length $\lambda = 80$.

cases of (a), (b) and (c), waves are not scattered on the connecter and led to the narrow guide, and transmitted to the original guide with keeping wave shapes, because components of the water depth and the gravitational acceleration represented by Eqs. (12)-(14) are independent of frequency. The coordinate transformation based wave controlling can be utilized regardless of the value of frequency, and therefore we expect that the method can be applied for various design of coastal/ocean engineering.

4. Conclusions

In this paper, the coordinate transformation based method for wave controlling is applied for wave focusing to correct waves in a specific domain. The governing equation of shallow water problem can keep the invariance under the coordinate transformation, and should satisfy the following conditions; the water depth and the gravitational acceleration are heterogeneity and the water depth is anisotropy. The space is transformed to focus waves on a central domain, and the required water depth and gravitational acceleration are realized. Numerical computations are carried out assuming these water depth and gravitational acceleration, and it is shown that there is a good agreement between the theory and computation. Waves are not scattered on the

connecter and led to the narrow guide, and transmitted to the original guide with keeping wave shapes. Computation results show that this method is independent of the frequency.

References

[1] Murashige, S., and Kinoshita, T. 1990. "Fundamental Studies on Ocean Wave Focusing." *Journal of the Society of Naval Architects of Japan* 1990 (168): 183-92.

[2] Clément, A. H., and Pianet, G. 2001. "Numerical Measurements of the Index of Wave Refraction through a Group of Vertical Cylinders." In *Proceedings of the 16th International Workshop on Water Waves and Floating Bodies*, Hiroshima.

[3] Bobinski, T., Eddi, A., Petitjeans, P., Maurel, A., and Pagneux, V. 2015. "Experimental Demonstration of Epsilon-Near-Zero Water Waves Focusing." *Applied Physics Letters* 107 (1): 014101.

[4] Iida, T., Kashiwagi, M., and Miki, M. 2015. "Wave Pattern in Cloaking Phenomenon around a Body Surrounded by Multiple Vertical Circular Cylinders." *International Journal of Offshore and Polar Engineering* 26 (1): 635-41.

[5] Iida, T., and Kashiwagi, M. 2016. "Shallow Water Cloaking with Anisotropic Fluid." In *Proceedings of the 31st International Workshop on Water Waves and Floating Bodies*, 65-8.

[6] Pendry, J. B., Schurig, D., and Smith, D. R. 2006. "Controlling Electromagnetic Fields." *Science* 312 (5781): 1780-2.

[7] Zareei, A., and Alam, M. R. 2015. "Cloaking in Shallow-Water Waves via Nonlinear Medium Transformation." *Journal of Fluid Mechanics* 778 (September): 273-87.

[8] Schurig, D., Mock, J. J., Justice, B. J., Cummer, S. A., Pendry, J. B., Starr, A. F., and Smith, D. R. 2006. "Metamaterial Electromagnetic Cloak at Microwave Frequencies." *Science* 314 (5801): 977-80.

[9] Cummer, S. A., and Schurig, D. 2007. "One Path to Acoustic Cloaking." *New Journal of Physics* 9 (3): 45.

[10] Berraquero, C. P., Maurel, A., Petitjeans, P., and Pagneux, V. 2013. "Experimental Realization of a Water-Wave Metamaterial Shifter." *Physical Review E* 88 (5): 051002.

[11] Iida, T., and Kashiwagi, M. 2017. "Design of Small Water Channel Network for Shallow Water Cloaking." In *Proceedings of the 32nd International Workshop on Water Waves and Floating Bodies*, Dalian, China.

[12] Tichit, P. H., Burokur, S. N., and de Lustrac, A. 2010. "Waveguide Taper Engineering Using Coordinate Transformation Technology." *Optics Express* 18 (2): 767-72.

[13] Zhang, K., Wu, Q., Meng, F. Y., and Li, L. W. 2010. "Arbitrary Waveguide Connector Based on Embedded Optical Transformation." *Optics Express* 18 (16): 17273-9.