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The Generalized Roper-Suffridge Extension Operator on the Reinhardt Domain

S. Rahrovi and H. Piri

Department of Mathematics, Basic Science Faculty, University of Bonab, Bonab, Iran.

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Abstract: Let $p_j \in \mathbb{N}$ and $p_j \geq 1$, $j = 2,\ldots,n$ be a fixed positive integer. In this paper a generalized Roper-Suffridge extension operator

$$F(z) = \left(f(z_1) + f'(z_1)\sum_{j=2}^{n} P_j(z_j), (f'(z_1))^{\frac{1}{p_2}} z_2, \ldots, (f'(z_1))^{\frac{1}{p_n}} z_n \right),$$

on Reinhardt domain $\Omega_{n,p_2,\ldots,p_n} = \{z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^{n} |z_j|^{p_j} < 1\}$ is defined. Some different conditions for P_j are established under which the operator preserves an almost spirallike mapping of type β and order α and spirallike mapping of type β and order α, respectively. In particular, our results reduce to many well-known results.

Keywords: Roper-Suffridge extension operator, Reinhardt Domain, Almost spirallike mapping of type β and order α, Spirallike mapping of type β and order α, Minkowski functional.

1. Introduction and Preliminaries

Let $\mathbb{C}^n$ be the vector space of n complex variables $z = (z_1,\ldots,z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and Euclidean norm $\| z \| = \langle z, z\rangle^{1/2}$, where $z, w \in \mathbb{C}^n$. The open ball $\{z \in \mathbb{C}^n : \| z \| < r\}$ is denoted by B_r^n and the unit ball B_1^n by B^n. The closed ball $\{z \in \mathbb{C}^n : \| z \| \leq r\}$ is denoted by $\overline{B}_r^n$, and the unit sphere is denoted by $\partial B^n = \{z \in \mathbb{C}^n : \| z \| = 1\}$. In the case of one complex variable, B^1 is denoted by U. For $n \geq 2$, let $\hat{z} = (z_2,\ldots,z_n) \in \mathbb{C}^{n-1}$ so that $z = (z_1, \hat{z}) \in \mathbb{C}^n$.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of complex linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm,

$$\| A \| = \sup\{\| A(z) \| : \| z \| = 1\},$$

and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. Let Ω be a domain in $\mathbb{C}^n$ and $H(\Omega)$ be the set of holomorphic mappings from Ω into C^n. Let $0 \in \Omega$, a mapping $f \in H(\Omega)$ is called normalized if $f(0) = 0$ and $J_f(0) = I_n$, where $J_f(0)$ is the complex Jacobian matrix of f at the origin and I_n is the identity operator on $\mathbb{C}^n$.

Let $S(\Omega)$ be the set of normalized biholomorphic mappings on Ω. In the case of one complex variable, the set $S(U)$ is denoted by S. A normalized mapping $f \in H(\Omega)$ is said to be convex if its image is a convex domain. Let $0 \in \Omega$, a normalized mapping $f \in H(\Omega)$ is said to be starlike with

Corresponding author: S. Rahrovi, Department of Mathematics, Basic Science Faculty, University of Bonab, Bonab, Iran.

respect to the origin if its image is a starlike domain with respect to the origin. The classes of starlike and convex mappings on Ω will be denoted by $S^*(\Omega)$ and $K(\Omega)$, respectively. In the case of one complex variable $S^*(U)$ and $K(U)$ is denote by S^* and K, respectively. A normalized mapping $f \in H(\Omega)$ is said to be ε starlike if there exists a positive number ε, $0 \le \varepsilon \le 1$, such that $f(B^n)$ is starlike with respect to every point in $\varepsilon f(B^n)$. Assume that $P: \mathbb{C}^n \to \mathbb{C}$ is a homogeneous polynomial of degree n. Then P satisfies $P(\lambda z) = \lambda^n P(z)$ for $\forall z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. It is easy to see that $\nabla P(z) z = nP(z)$, where $\nabla P(z) = \left(\frac{\partial P}{\partial z_1}, \ldots, \frac{\partial P}{\partial z_n} \right)$ is the gradient of $P(z)$.

A domain $\Omega \subset \mathbb{C}^n$ is said to be circular domain if $e^{i\theta} z \in \Omega$ holds for every $z \in \Omega$ and $\theta \in \mathbb{R}$. A domain $\Omega \subset \mathbb{C}^n$ is said to be Reinhardt domain if $(e^{i\theta_1} z_1, e^{i\theta_2} z_2, \ldots, e^{i\theta_n} z_n) \in \Omega$ holds for every $z = (z_1, z_2, \ldots, z_n) \in \Omega$ for all $\theta_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$. The Minkowski functional $\rho(z)$ of a bounded circular convex domain Ω in $\mathbb{C}^n$ is defined as

$$\rho(z) = \inf \left\{ t > 0, \frac{z}{t} \in \Omega \right\}, z \in \mathbb{C}^n.$$

If Ω is a bounded circular convex domain, then Ω is a Banach space in $\mathbb{C}^n$ with respect to this norm, and $\Omega = \{ z \in \mathbb{C}^n : \rho(z) < 1 \}$. Also, The Minkowski functional $\rho(z)$ is C^1 on $\overline{\Omega}$ except for a lower dimensional manifold. Moreover, the Minkowski functional $\rho(z)$ has the following properties of (see [11]):

$$\frac{\partial \rho}{\partial z}(\lambda z) = \frac{\partial \rho}{\partial z}(z), \lambda \in [0, +\infty), z \in \Omega \setminus \{0\}, \quad (1)$$

$$\frac{\partial \rho}{\partial z}(e^{i\theta} z) = e^{-i\theta} \frac{\partial \rho}{\partial z}(z), \theta \in R, z \in C^n \setminus \{0\}.$$

Definition 1. [22] Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded complete convex domain. Its Minkowski functional $\rho(z)$ is C^1 except for a lower dimensional manifold. Assume that $0 \le \alpha < 1$ and $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. A mapping $f \in H(\Omega)$ is said to be almost spirallike mapping of type β and order α if the following condition holds:

$$Re\left[2e^{-i\beta} \frac{\partial \rho(z)}{\partial z} J_f^{-1}(z) f(z) \right] \ge \rho(z) \alpha \cos \beta,$$

$$z \in \Omega \setminus \{0\},$$

where $\frac{\partial \rho(z)}{\partial z} = \left(\frac{\partial \rho(z)}{\partial z_1}, \ldots, \frac{\partial \rho(z)}{\partial z_n} \right)$.

Definition 2. [22] Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded complete convex domain. Its Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. A mapping $f \in H(\Omega)$ is said to be spirallike mapping of type β and order α if

$$\left| \begin{array}{l} 4\alpha e^{-i\beta} \frac{\partial \rho(z)}{\partial z} J_f^{-1}(z) f(z) - \\ \rho(z)\left(\cos\beta - i2\alpha \sin\beta \right) \end{array} \right| \le \rho(z) \cos \beta,$$

$$z \in \Omega \setminus \{0\},$$

for $0 < \alpha < 1$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ and

$$Re\left[2e^{-i\beta} \frac{\partial \rho(z)}{\partial z} J_f^{-1}(z) f(z) \right] \ge 0, z \in \Omega \setminus \{0\},$$

for $\alpha = 0$.

The class $\widehat{S}_\alpha(\Omega, \beta)$ consists of all normalized spirallike mappings of type β and order α on Ω and the class $A\widehat{S}_\alpha(\Omega, \beta)$ consists of all normalized almost spirallike mappings of type β and order α on Ω for $0 \le \alpha < 1$. Then, we have

$$f \in A\hat{S}_{\alpha}(U,\beta) \iff f \in S(U) \ and \ Re\left[e^{-i\beta}\frac{f(\xi)}{\xi f'(\xi)}\right] \geq \alpha\cos\beta, \qquad \xi \in U,$$

and

$$f \in \hat{S}_{0}(U,\beta) \iff f \in S(U) \ and \ Re\left[e^{-i\beta}\frac{f(\xi)}{\xi f'(\xi)}\right] \geq 0, \qquad \xi \in U,$$

and

$$f \in \hat{S}_{\alpha}(U,\beta) \iff f \in S(U) \ and \ \left|2\alpha(1-i\tan\beta)\frac{f(\xi)}{\xi f'(\xi)} - 1 + i2\alpha\tan\beta\right| \leq 1, \xi \in U$$

for $0<\alpha<1$.

The class $S_{\alpha}^{*}(\Omega)$ consists of all biholomorphic starlike mappings of order α on Ω for $0 \leq \alpha < 1$. Let $S_{\alpha}^{*}(\Omega) = \hat{S}_{\alpha}(\Omega,0)$ for $0<\alpha<1$ (we say α-spirallike) and $S_{0}^{*}(\Omega) = S^{*}(\Omega)$, and let $\hat{S}_{0}(\Omega,\beta) = A\hat{S}_{0}(\Omega,\beta) = \hat{S}(\Omega,\beta)$. It is evident that $A\hat{S}_{0}(\Omega,0) = \hat{S}_{0}(\Omega,0) = S^{*}(\Omega)$. From Theorem 1.2.1 in [2], we have $S_{\alpha}^{*}(\Omega) \subset S^{*}(\Omega)$ for $0 \leq \alpha < 1$. Spirallike mappings are important for study because they are natural generalization of starlike mappings which leads to a useful criterion for univalence.

In 1995, Roper and Suffridge [19] introduced an extension operator which gives a way of extending a locally biholomorphic function on the unit disc U to a locally biholomorphic mapping on the unit ball B^n in $\mathbb{C}^n$.

For fixed $n \geq 2$, the Roper-Suffridge extension operator (see [6] and [19]) is defined as follows:

$$[\Phi_n(f)](z) = \left(f(z_1), \sqrt{f'(z_1)}\hat{z}\right), \qquad z \in B^n,$$

where f is a normalized biholomorphic mapping on the unit disc U in C and $z=(z_1,\hat{z})$ belonging to the unit ball B^n in C^n and the branch of the power function is chosen so that $\sqrt{f'(z_1)}\,|_{z_1=0}=1$.

The following results illustrate the important and usefulness of the Roper-Suffridge extension operator

$$\Phi_n(K) \subseteq K(B^n), \qquad \Phi_n(S^*) \subseteq S^*(B^n).$$

The first was proved by Roper and Suffridge when they introduced their operator [19], while the second result was given by Graham and Kohr [5]. Until now, it is difficult to construct the concrete convex mappings, starlike mappings on B^n. By making use of the Roper-Suffridge extension operator, we may easily give many concrete examples about these mappings. This is one important reason why people are interested in this extension operator. A good treatment of further applications of the Roper-Suffridge extension operator can be found in the recent book by Graham and Kohr [6].

In 2002, Gong and Liu [3, 9] introduced the definition of $\varepsilon-$starlike mappings and obtained that the operator

$$[\Phi_{n,\frac{1}{p}}(f)](z) = \left(f(z_1), (f'(z_1))^{\frac{1}{p}}\hat{z}\right),$$

maps the $\varepsilon-$ starlike functions on U to $\varepsilon-$ starlike mappings on the Reinhardt domain $\Omega_{n,p} = \left\{z \in C^n : |z_1|^2 + \sum_{j=2}^{n} |z_j|^p < 1\right\}$, where $p \geq 1$. When $\varepsilon=0$ and $\varepsilon=1$, $\Phi_{n,\frac{1}{p}}(f)$ maps the starlike function and the convex function on U

to starlike mapping and convex mapping on $\Omega_{n,p}$, respectively.

Furthermore, Gong and Liu [4] proved that the operator

$$[\Phi_{n,\frac{1}{p_2},\ldots,\frac{1}{p_n}}(f)](z)=\left(f(z_1),(f'(z_1))^{\frac{1}{p_2}}z_2,\ldots,(f'(z_1))^{\frac{1}{p_n}}z_n\right),$$

maps the $\varepsilon-$ starlike functions on U to $\varepsilon-$ starlike mappings on the Reinhardt domain $\Omega_{n,p_2,\ldots,p_n}=\left\{z\in C^n:|z_1|^2+\sum_{j=2}^{n}|z_j|^{p_j}<1\right\}$, where $p_j\geq 1$, $j=2,\ldots,n$. Also, Liu and Liu [10] proved that this operator preserves starlikeness of order α on the domain $\Omega_{n,p_2,\ldots,p_n}$. On the other hand, Feng and Liu [1] proved that this operator preserves almost starlikeness of order α on the domain $\Omega_{n,p_2,\ldots,p_n}$.

In 2005, Muir [13] modified the Roper- Suffridge extension operator as follows:

$$[\Phi_{n,Q}(f)](z)=\left(f(z_1)+f'(z_1)Q(\hat{z}),\sqrt{f'(z_1)}\hat{z}\right),$$
$$z=(z_1,\hat{z})\in B^n,$$

where $Q(\hat{z})$ is a homogeneous polynomial of degree 2 with respect to $\hat{z}$, and f, z_1 and $\hat{z}$ are defined as above. He proved that this operator preserves starlikeness and convexity if and only if $\|Q\|\leq 1/4$ and $\|Q\|\leq 1/2$, respectively. This modified operator plays a key role to study the extreme points of convex mappings on B^n (see [14, 15]). Later, Kohr [7], Muir [12] and Rahrovi et all [18] used the Loewner chain to study the modified Roper-Suffridge extension operator. Recently, the modified Roper-Suffridge extension operator on the unit ball B^n is also studied by Wang and Liu [21] and Feng and Yu [1] and S. Rahrovi et all [17].

In 2011, Wang and Gao [20] introduced the following extension operator on the Reinhardt domain $\Omega_{n,p_2,\ldots,p_n}$:

$$[\Phi_{n,p_2,\ldots,p_n}(f)](z)=\begin{pmatrix}f(z_1)+f'(z_1)\sum_{j=2}^{n}a_jz_j^{p_j},\\ (f'(z_1))^{\frac{1}{p_2}}z_2,\ldots,(f'(z_1))^{\frac{1}{p_n}}z_n\end{pmatrix},\qquad(2)$$

where where f is a normalized locally biholomorphic function on the unit disc U, p_j are positive integer, a_j are complex constants, $j=2,\ldots,n$ and the branch are chosen such that $(f'(z_1))^{\frac{1}{p_j}}|_{z_1=0}=1$. Some conditions for a_j are found under which the operator preserves the properties of almost starlikeness of order α and starlikeness of order α, on the Renihardt domain $\Omega_{n,p_2,\cdots,p_n}$, respectively.

In contrast to the modified Roper-Suffridge extension operator on the unit ball B^n, it is natural to ask if we can modify the Roper-Suffridge extension operator on the Reinhardt domain $\Omega_{n,p_2,\ldots,p_n}$. In 2014, Li and Feng [8] introduced the following extension operator

$$F(z)=\begin{pmatrix}f(z_1)+f'(z_1)\sum_{j=2}^{n}P_j(z_j),\\ (f'(z_1))^{\frac{1}{p_2}}z_2,\ldots,(f'(z_1))^{\frac{1}{p_n}}z_n\end{pmatrix},$$

on the Reinhardt domain $\Omega_{n,p_2,\ldots,p_n}$. where $P_j(z_j)$ is a homogeneous polynomial of degree n with respect to z_j, and f, z_1 and $\hat{z}$ are defined as above. They proved that this operator can preserve the properties of almost starlikeness of order α, starlikeness of order α and strongly starlikeness of order α on the domain $\Omega_{n,p_2,\ldots,p_n}$ given by different conditions for P_j, $j=2,\ldots,n,$ respectively, where $\Omega_{n,p_2,\ldots,p_n}$ is defined as

$$\Omega_{n,p_2,\ldots,p_n}=\left\{z\in C^n:|z_1|^2+\sum_{j=2}^{n}|z_j|^{p_j}<1\right\}.\qquad(3)$$

In this paper we will establish some different conditions for P_j such that almost spirallikeness of type β and order α and spirallikeness of type β

and order α are preserved under the above generalized Roper-Suffridge operator. Our results enable us to obtain some known results from a unified perspective and also leads many new results.

2. Some Lemmas

In order to prove the main results, we need the following lemmas.

Lemma 1. [16]. Let p be a holomorphic function on U. If $Re\ p(z) > 0$ and $p(0) > 0$, then

$$|p'(z)| \le \frac{2Re\ p(z)}{1-|z|^2}.$$

Lemma 2. [6] (Schwarz-Pick lemma) Suppose that $g \in H(U)$ satisfies $g(0) = 0$ and $g(U) \subset U$, then

$$|g'(\xi)| \le \frac{1-|g(\xi)|^2}{1-|\xi|^2},$$

for each $\xi \in U$.

Lemma 3. [16]. Let f be a normalized biholomorphic function on U. Then

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right| \le 4. \tag{4}$$

Lemma 4. [23]. If $\rho(z)$ is the Minkowski function of the domain $\Omega_{n,p_2,\ldots,p_n}$, $z \ne 0$, then

$$\frac{\partial \rho}{\partial z_1}(z) = \frac{\overline{z}_1}{\rho(z)\left[2\left|\frac{z_1}{\rho(z)}\right|^2 + \sum_{j=2}^{n} p_j \left|\frac{z_j}{\rho(z)}\right|^{p_j}\right]},$$

$$\frac{\partial \rho}{\partial z_j}(z) = \frac{p_j \overline{z}_j |z_j/\rho(z)|^{p_j-2}}{2\rho(z)\left[2\left|\frac{z_1}{\rho(z)}\right|^2 + \sum_{j=2}^{n} p_j \left|\frac{z_j}{\rho(z)}\right|^{p_j}\right]}. \tag{5}$$

3. Main Results

We begin this section with the main results of this paper.

Theorem 1. Let $0 \le \alpha < 1$ and $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. Suppose that the operator $F(z)$ is defined by (2) and $\Omega_{n,p_2,\ldots,p_n}$ is defined by (3). If $\|P_j\| \le \frac{(1-\alpha)\cos\beta}{4}$, $j = 2,\ldots,n$, then $F \in A\hat{S}_\alpha(\Omega_{n,p_2,\ldots,p_n},\beta)$ if and only if $f \in A\hat{S}_\alpha(U,\beta) = A\hat{S}_\alpha(\beta)$.

Proof. Suppose that $\rho(z)$ is the Minkowski functional of $\Omega_{n,p_2,\ldots,p_n}$. From (1), we may obtain that $\rho(z)$ is a C^1 function except for a lower dimensional manifold in $\overline{\Omega}_{n,p_2,\ldots,p_n}$. By the definition of an almost spirallike mapping of type β and order α, we only need to prove that the following inequality holds

$$Re\left[2e^{-i\beta}\frac{\partial \rho}{\partial z}(z)J_F^{-1}(z)F(z)\right] \ge \rho(z)\alpha\cos\beta, \tag{6}$$

for all $z \in \Omega_{n,p_2,\ldots,p_n}$, $z \ne 0$ and $\|P_j\| \le \frac{(1-\alpha)\cos\beta}{4}$. Now, for $z = (z_1,\hat{z}) \in \overline{\Omega}_{n,p_2,\ldots,p_n} \setminus \{0\}$, we have two cases:

First, if $\hat{z} = 0$, then we can get the conclusion easily.

Second, suppose $\hat{z} \ne 0$. Obviously, the mapping F is holomorphic in a neighborhood of each $z = (z_1,\hat{z}) \in \overline{\Omega}_{n,p_2,\ldots,p_n} \setminus \{0\}$. Let us write $z = \lambda u = |\lambda| e^{i\theta} u$ for $u \in \partial\Omega_{n,p_2,\ldots,p_n}$ such that $\hat{u} \ne 0$ and $\lambda \in \overline{U} \setminus \{0\}$, then from (1) we have

$$Re\left[\frac{2}{\rho(z)}e^{-i\beta}\frac{\partial \rho}{\partial z}(z)J_F^{-1}(z)F(z)\right] \ge \alpha\cos\beta$$

$$\Longleftrightarrow Re\left[\frac{2}{\rho(|\lambda| e^{i\theta}u)}e^{-i\beta}\frac{\partial \rho}{\partial z}(|\lambda| e^{i\theta}u)J_F^{-1}(|\lambda| e^{i\theta}u)F(|\lambda| e^{i\theta}u)\right] \ge \alpha\cos\beta \tag{7}$$

$$\Longleftrightarrow Re\left[\frac{2}{|\lambda|}e^{-i\beta}\frac{e^{-i\theta}\partial\rho}{\partial z}(u)J_F^{-1}(|\lambda|e^{i\theta}u)F(|\lambda|e^{i\theta}u)\right]\geq\alpha\cos\beta$$

$$\Longleftrightarrow Re\left[2e^{-i\beta}\frac{\partial\rho}{\partial z}(u)\frac{J_F^{-1}(\lambda u)F(\lambda u)}{\lambda}\right]\geq\alpha\cos\beta.$$

For the fixed u, the expression

$$Re\left[2e^{-i\beta}\frac{\partial\rho}{\partial z}(u)\frac{J_F^{-1}(\lambda u)F(\lambda u)}{\lambda}-\alpha\cos\beta\right]$$

is the real part of an analytic function of the complex variable λ, and hence is harmonic. Due to the minimum principle for harmonic functions, we know that it attains it's minimum on $|\lambda|=1$, so we need only to prove for all $z=(z_1,\hat{z})\in\partial\Omega_{n,p_2,\ldots,p_n}\setminus\{0\}$ such that $\hat{z}\neq 0$. Hence, $\rho(z)=1$ and inequality (6) becomes

$$Re\left[2e^{-i\beta}\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)\right]$$
$$\geq\alpha\cos\beta, z\in\partial\Omega_{n,p_2,\ldots,p_n}\setminus\{0\},\ \hat{z}\neq 0.$$

Since

$$F(z)=\begin{pmatrix}f(z_1)+f'(z_1)\sum_{j=2}^{n}P_j(z_j),\\ (f'(z_1))^{\frac{1}{p_2}}z_2,\ldots,(f'(z_1))^{\frac{1}{p_n}}z_n\end{pmatrix},$$

hence

$$J_F(z)=\begin{bmatrix}\lambda_1 & \alpha_2 & \cdots & \alpha_n\\ \beta_2 & \lambda_2 I & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \beta_n & 0 & \cdots & \lambda_n I\end{bmatrix}$$

where I is the identity operator on C^{n-1} and

$$\lambda_1=f'(z_1)+f''(z_1)\sum_{j=2}^{n}P_j(z_j),$$
$$\lambda_j=(f'(z_1))^{\frac{1}{p_j}},\quad j=2,\ldots,n,$$
$$\alpha_j=f'(z_1)\nabla P_j(z_j),$$
$$\beta_j=\frac{1}{p_j}(f'(z_1))^{\frac{1}{p_j}-1}f''(z_1)z_j,\quad j=2,\ldots,n.$$

We denote $J_F^{-1}(z)F(z)=A=(x_1,x_2,\ldots,x_n)\in C^n$, then we have

$$x_1=\frac{f(z_1)}{f'(z_1)}-\sum_{j=2}^{n}(p_j-1)P_j(z_j),$$

$$x_2=\left(1-\frac{f(z_1)f''(z_1)}{p_2(f'(z_1))^2}+\frac{f''(z_1)}{p_2f'(z_1)}\sum_{j=2}^{n}(p_j-1)P_j(z_j)\right)z_2,$$

$$\vdots$$

$$x_n=\left(1-\frac{f(z_1)f''(z_1)}{p_n(f'(z_1))^2}+\frac{f''(z_1)}{p_nf'(z_1)}\sum_{j=2}^{n}(p_j-1)P_j(z_j)\right)z_n.$$

Consequently

$$\frac{\partial\rho(z)}{\partial z}J_F^{-1}(z)F(z)=\frac{f(z_1)}{z_1f'(z_1)}\frac{\partial\rho(z)}{\partial z_1}z_1-\sum_{j=2}^{n}(p_j-1)P_j(z_j)\frac{\partial\rho(z)}{\partial z_1}$$
$$+\sum_{j=2}^{n}\begin{pmatrix}1-\frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}+\\ \frac{f''(z_1)}{p_jf'(z_1)}\sum_{k=2}^{n}(p_k-1)P_k(z_k)\end{pmatrix}\frac{\partial\rho(z)}{\partial z_j}z_j.\ (8)$$

Now, from Lemma 4, we obtain

$$\frac{\partial\rho}{\partial z_1}(z)=\frac{\overline{z}_1}{2|z_1|^2+\sum_{j=2}^{n}p_j|z_j|^{p_j}},$$

$$\frac{\partial\rho}{\partial z_j}(z)=\frac{p_j\overline{z}_j|z_j|^{p_j-2}}{2(2|z_1|^2+\sum_{j=2}^{n}p_j|z_j|^{p_j})}.\quad (9)$$

In terms of (8) and (9), we obtain

$$2e^{-i\beta}\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)=\frac{G(z)}{2|z_1|^2+\sum_{j=2}^{n}p_j|z_j|^{p_j}},\ (10)$$

where

$$G(z)=e^{-i\beta}\sum_{j=2}^{n}p_j|z_j|^{p_j}\left(\begin{array}{l}1-\dfrac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}\\ +\dfrac{f''(z_1)}{p_jf'(z_1)}\sum_{k=2}^{n}(p_k-1)P_j(z_j)\end{array}\right)+2e^{-i\beta}\overline{z}_1\left(\frac{f(z_1)}{f'(z_1)}-\sum_{j=2}^{n}(p_j-1)P_j(z_j)\right)$$

$$=2e^{-i\beta}|z_1|^2\frac{f(z_1)}{z_1f'(z_1)}+e^{-i\beta}\sum_{j=2}^{n}p_j|z_j|^{p_j}\left(1-\frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}\right)+e^{-i\beta}\sum_{k=2}^{n}(p_k-1)P_k(z_k)\left(\frac{f''(z_1)}{f'(z_1)}\sum_{j=2}^{n}|z_j|^{p_j}-2\overline{z}_1\right).$$

Since $z\in\partial\Omega_{n,p_2,\ldots,p_n}$, i.e., $|z_1|^2+\sum_{j=2}^{n}|z_j|^{p_j}=1$, so

$$G(z)=2e^{-i\beta}|z_1|^2\frac{f(z_1)}{z_1f'(z_1)}+e^{-i\beta}\sum_{j=2}^{n}p_j|z_j|^{p_j}\left(1-\frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}\right)$$

$$+e^{-i\beta}\sum_{j=2}^{n}(p_j-1)P_j(z_j)\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\overline{z}_1\right).\tag{11}$$

Since $f\in A\hat{S}_\alpha(U,\beta)$, we set

$$p(z_1)=\frac{e^{-i\beta}\frac{f(z_1)}{z_1f'(z_1)}+i\sin\beta-\alpha\cos\beta}{(1-\alpha)\cos\beta},\tag{12}$$

then $p(z_1)$ is analytic on U such that $Re\ p(z_1)>0$ for $z_1\in U$ with $p(0)=1$ and

$$e^{-i\beta}\left(1-\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}\right)=(1-\alpha)\cos\beta(p(z_1)+z_1p'(z_1))+\alpha\cos\beta-i\sin\beta.\tag{13}$$

Substituting (12) and (13) into (11), we get

$$G(z)=2|z_1|^2((1-\alpha)(\cos\beta)p(z_1)+\alpha\cos\beta-i\sin\beta)$$

$$+e^{-i\beta}\sum_{j=2}^{n}p_j|z_j|^{p_j}\left(1-\frac{1}{p_j}\right)+\sum_{j=2}^{n}|z_j|^{p_j}((1-\alpha)(\cos\beta)(p(z_1)+z_1p'(z_1)))$$

$$+\sum_{j=2}^{n}|z_j|^{p_j}(\alpha\cos\beta-i\sin\beta)+e^{-i\beta}\sum_{j=2}^{n}(p_j-1)P_j(z_j)\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\overline{z}_1\right)$$

$$=(1-\alpha)p(z_1)\cos\beta\left(2|z_1|^2+\sum_{j=2}^{n}|z_j|^{p_j}\right)+2|z_1|^2\alpha\cos\beta-2i|z_1|^2\sin\beta$$

$$+\sum_{j=2}^{n}|z_j|^{p_j}(e^{-i\beta}(p_j-1)+\alpha\cos\beta-i\sin\beta)+(1-\alpha)\cos\beta z_1p'(z_1)\sum_{j=2}^{n}|z_j|^{p_j}$$

$$+e^{-i\beta}\sum_{j=2}^{n}(p_j-1)P_j(z_j)\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\overline{z}_1\right).$$

Hence

$$Re\, G(z) \geq (1-\alpha)\cos\beta\left(2|z_1|^2 + \sum_{j=2}^{n}|z_j|^{p_j}\right)Re p(z_1) + 2|z_1|^2\,\alpha\cos\beta$$
$$+\sum_{j=2}^{n}(\cos\beta(p_j-1)+\alpha\cos\beta)|z_j|^{p_j} - (1-\alpha)\cos\beta\,|z_1 p'(z_1)|(1-|z_1|^2)$$
$$-|e^{-i\beta}|\sum_{j=2}^{n}(p_j-1)|P_j(z_j)|\left|\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\overline{z}_1\right|.$$

By Lemma 1 and 3, we can get that

$$Re\, G(z) \geq (1-\alpha)\cos\beta\left(1+|z_1|^2\right)Re\; p(z_1) + 2|z_1|^2\,\alpha\cos\beta + \sum_{j=2}^{n}((p_j-1)\cos\beta+\alpha\cos\beta)|z_j|^{p_j}$$
$$-(1-\alpha)\cos\beta(1-|z_1|^2)\frac{2|z_1|\,Re\; p(z_1)}{1-|z_1|^2} - 4\sum_{j=2}^{n}\|P_j\|(p_j-1)|z_j|^{p_j} = (1-\alpha)\cos\beta\left(1-|z_1|\right)^2 Re\; p(z_1) + 2|z_1|^2\alpha\cos\beta$$
$$+\sum_{j=2}^{n}((p_j-1)(\cos\beta-4\|P_j\|)+\alpha\cos\beta)|z_j|^{p_j}.$$

Therefore, when $\|P_j\| \leq \frac{(1-\alpha)\cos\beta}{4}$, $j=2,\ldots,n$, we have

$$Re\, G(z) \geq (1-\alpha)(1-|z_1|)^2 Re p(z_1)\cos\beta + 2|z_1|^2\,\alpha\cos\beta + \alpha\cos\beta\sum_{j=2}^{n}p_j|z_j|^{p_j}$$
$$\geq \alpha\cos\beta\left(2|z_1|^2 + \sum_{j=2}^{n}p_j|z_j|^{p_j}\right). \tag{14}$$

In the terms of (10) and (14), we obtain

$$Re\left(2e^{-i\beta}\frac{\partial\rho(z)}{\partial z}J_F^{-1}(z)F(z)\right) \geq \alpha\cos\beta.$$

Hence $F \in A\hat{S}_\alpha(\Omega_{n,p_2,\ldots,p_n},\beta)$.

Conversely, if

$$F(z) = \left(f(z_1)+f'(z_1)\sum_{j=2}^{n}P_j(z_j), (f'(z_1))^{\frac{1}{p_2}}z_2,\ldots,(f'(z_1))^{\frac{1}{p_n}}z_n\right) \in A\hat{S}_\alpha(\Omega_{n,p_2,\ldots,p_n},\beta),$$

then we prove that $f \in A\hat{S}_\alpha(U,\beta)$. In fact $z=(z_1,0,\ldots,0)\in\Omega_{n,p_2,\ldots,p_n}$ with $z_1 \neq 0$, from (8) and (9), we have

$$Re\left(e^{-i\beta}\frac{f(z_1)}{z_1 f'(z_1)}\right) = \frac{2}{\rho(z)}Re\left(e^{-i\beta}\frac{\partial\rho(z)}{\partial z}J_F^{-1}(z)]F(z)\right) \geq \alpha\cos\beta,$$

for $0<|z_1|<1$. This completes the proof. □

If we take $\alpha=0$, in Theorem 1, we obtain:

Corollary 2. Let $-\frac{\pi}{2}<\beta<\frac{\pi}{2}$. Suppose that the operator $F(z)$ is defined by (2). If $\|P_j\|\le\frac{\cos\beta}{4}$, $j=2,\dots,n$, then $F\in A\hat{S}(\Omega_{n,p_2,\dots,p_n},\beta)$ if and only if $f\in A\hat{S}(\beta)$.

Set $\beta=0$, in Theorem 1, then we get the following corollary due to [8]:

Corollary 3. Let $0\le\alpha<1$. Suppose that the operator $F(z)$ is defined by (2). If $\|P_j\|\le\frac{1-\alpha}{4}$, $j=2,\dots,n$, then $F\in AS_\alpha^*(\Omega_{n,p_2,\dots,p_n})$ if and only if $f\in AS_\alpha^*$.

If we take $\alpha=0$, in Corollary 3, we obtain following corollary:

Corollary 4. Suppose that the operator $F(z)$ is defined by (2). If $\|P_j\|\le\frac{1}{4}$, $j=2,\dots,n$, then $F\in S^*(\Omega_{n,p_2,\dots,p_n})$ if and only if $f\in S^*$.

Theorem 5. Let $0\le\alpha<1$, $-\frac{\pi}{2}<\beta<\frac{\pi}{2}$. Suppose that the operator $F(z)$ is defined by (2) and $\Omega_{n,p_2,\dots,p_n}$ is defined by (3). If $\|P_j\|\le\frac{1-|2\alpha-1|}{8\alpha}\cos\beta$, $j=2,\dots,n$, then $F\in\hat{S}_\alpha(\Omega_{n,p_2,\dots,p_n},\beta)$ if and only if $f\in\hat{S}_\alpha(U,\beta)$.

Proof. We first prove that $F\in\hat{S}_\alpha(\Omega_{n,p_2,\dots,p_n},\beta)$ when $f\in\hat{S}_\alpha(U,\beta)$. By the definition of almost spirallike mapping of type β and order α, we need to prove that the following inequality

$$\left|4\alpha e^{-i\beta}\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)-\rho(z)(\cos\beta-2i\alpha\sin\beta)\right|\le\rho(z)\cos\beta, \tag{15}$$

Similar to the theorem 1 we need only to prove that (15) holds for $\rho(z)=1$ and $\hat{z}\ne0$, according to the maximum modulus theorem for analytic functions. So, it is sufficient to show that

$$\left|4\alpha(1-i\tan\beta)\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)-1+i2\alpha\tan\beta\right|\le1,\qquad z\in\partial\Omega_{n,p_2,\dots,p_n}\setminus\{0\},\ \hat{z}\ne0.$$

Case 1. When $\alpha=0$, noting that $\hat{S}_0(U,\beta)=A\hat{S}_0(U,\beta)=\hat{S}(U,\beta)$, from theorem 1, we obtain that $F\in A\hat{S}_0(\Omega_{n,p_2,\dots,p_n},\beta)=\hat{S}_0(\Omega_{n,p_2,\dots,p_n},\beta)$ if and only if $f\in\hat{S}(U,\beta)$.

Case 2. When $0<\alpha<1$, we set

$$q(z_1)=2\alpha(1-i\tan\beta)\frac{f(z_1)}{z_1f'(z_1)}-1+i2\alpha\tan\beta, \tag{16}$$

then $q(z_1)\in H(U)$ and $|q(z_1)|<1$ for $z_1\in U$ and

$$1-\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}=\frac{1-i2\alpha\tan\beta+q(z_1)+z_1q'(z_1)}{2\alpha(1-i\tan\beta)}. \tag{17}$$

From (8) and (9), we obtain

$$4\alpha(1-i\tan\beta)\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)-(1-i2\alpha\tan\beta)=\frac{H(z)}{2|z_1|^2+\sum_{j=2}^n p_j|z_j|^{p_j}}, \tag{18}$$

where

$$H(z)=2\alpha(1-i\tan\beta)\left(2|z_1|^2\frac{f(z_1)}{z_1f'(z_1)}+\sum_{j=2}^{n}p_j|z_j|^{p_j}\left(1-\frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}\right)\right)$$
$$+2\alpha(1-i\tan\beta)\left(\sum_{j=2}^{n}(p_j-1)P_j(z_j)\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\overline{z}_1\right)\right)$$
$$-(1-i2\alpha\tan\beta)\left(2|z_1|^2+\sum_{j=2}^{n}p_j|z_j|^{p_j}\right). \tag{19}$$

Substituting (16) and (17) into (19), we get

$$H(z)=2|z_1|^2q(z_1)+(2\alpha-1)\sum_{j=2}^{n}(p_j-1)|z_j|^{p_j}+q(z_1)\sum_{j=2}^{n}|z_j|^{p_j}+z_1q'(z_1)\sum_{j=2}^{n}|z_j|^{p_j}$$
$$+2\alpha(1-i\tan\beta)\sum_{j=2}^{n}(p_j-1)P_j(z_j)\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\overline{z}_1\right)$$
$$=(1+|z_1|^2)q(z_1)+(2\alpha-1)\sum_{j=2}^{n}(p_j-1)|z_j|^{p_j}+z_1q'(z_1)\sum_{j=2}^{n}|z_j|^{p_j}$$
$$+2\alpha(1-i\tan\beta)\sum_{j=2}^{n}(p_j-1)P_j(z_j)\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\overline{z}_1\right)$$

By Lemma 2 and 3, we can get that

$$|H(z)|\le(1+|z_1|^2)|q(z_1)|+|2\alpha-1|\sum_{j=2}^{n}(p_j-1)|z_j|^{p_j}+|z_1q'(z_1)|\sum_{j=2}^{n}|z_j|^{p_j}$$
$$+2\alpha|1-i\tan\beta|\sum_{j=2}^{n}|P_j(z_j)|(p_j-1)\left|\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\overline{z}_1\right|$$
$$\le(1+|z_1|^2)|q(z_1)|+|2\alpha-1|\sum_{j=2}^{n}(p_j-1)|z_j|^{p_j}$$
$$+|z_1|\frac{1-|q(z_1)|^2}{1-|z_1|^2}(1-|z_1|^2)+\frac{8\alpha}{\cos\beta}\sum_{j=2}^{n}|P_j|(p_j-1)|z_j|^{p_j}$$
$$\le(1+|z_1|^2)(|q(z_1)|-1)+1+|z_1|^2+2|z_1|(1-|q(z_1)|)$$
$$+|2\alpha-1|\sum_{j=2}^{n}(p_j-1)|z_j|^{p_j}+\frac{8\alpha}{\cos\beta}\sum_{j=2}^{n}|P_j|(p_j-1)|z_j|^{p_j}$$
$$=(1+|z_1|^2)+(1-|z_1|)^2(|q(z_1)|-1)+\sum_{j=2}^{n}\left(|2\alpha-1|+\frac{8\alpha}{\cos\beta}|P_j|\right)(p_j-1)|z_j|^{p_j}.$$

If $|P_j|\le\frac{1-|2\alpha-1|}{8\alpha}\cos\beta$, then we obtain

$$|H(z)|\le 1+|z_1|^2+\sum_{j=2}^{n}(p_j-1)|z_j|^{p_j}$$

$$=2|z_1|^2+\sum_{j=2}^{n}p_j|z_j|^{p_j}. \tag{20}$$

The equality (18) and (20) show that

$$\left|4\alpha(1-i\tan\beta)\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)-(1-i2\alpha\tan\beta)\right|\le 1.$$

then $F\in\widehat{S}_\alpha(\Omega_{n,p_2,\ldots,p_n},\beta)$.

Conversely, if

$$F(z)=\left(f(z_1)+f'(z_1)\sum_{j=2}^{n}P_j(z_j),(f'(z_1))^{\frac{1}{p_2}}z_2,\ldots,(f'(z_1))^{\frac{1}{p_n}}z_n\right)\in\widehat{S}_\alpha(\Omega_{n,p_2,\ldots,p_n},\beta),$$

then we prove that $f\in\widehat{S}_\alpha(U,\beta)$. In fact, letting $z=(z_1,0,\ldots,0)\in\Omega_{n,p_2,\ldots,p_n}$ with $z_1\ne 0$. From (3.3) and (9), we have

$$Re[e^{-i\beta}\frac{f(z_1)}{z_1f'(z_1)}]=\frac{2}{\rho(z)}Re\left[e^{-i\beta}\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)\right]_{z=\hat{z}}\ge 0,$$

for $0<|z_1|<1$ and $\alpha=0$, and

$$\left|2\alpha(1-i\tan\beta)\frac{f(z_1)}{z_1f'(z_1)}-1+i2\alpha\tan\beta\right|$$
$$=\left|\frac{4\alpha(1-i\tan\beta)}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)-(1-i2\alpha\tan\beta)\right|_{z=\hat{z}}\le 1$$

for $0<|z_1|<1$ and $0<\alpha<1$. This completes the proof. □

Set $\beta=0$, in Theorem 5, then we get the following corollary due to [8]:

Corollary 6. Let $0\le\alpha<1$. Suppose that the operator $F(z)$ is defined by (2). If $\|P_j\|\le\frac{1-|2\alpha-1|}{8\alpha}$, $j=2,\ldots,n$, then $F\in S_\alpha^*(\Omega_{n,p_2,\ldots,p_n})$ if and only if $f\in S_\alpha^*$.

References

[1] S. X. FENG, T. S. LIU, *Modified Roper-Suffridge operator for some holomorphic mappings*, Front. Math. Chin., (2011), No. 3, 411-426.

[2] S. GONG, *Convex and starlike mappings in several complex variables*, New York, Kluwer Academic Publishers, (1998).

[3] S. GONG, T. S. LIU, *On the Roper-Suffridge extension operator*, J. Anal. Math., (2002), 397-404.

[4] S. GONG, T. S. LIU, *The generalized Roper-Suffridge extension operator*, J. Math. Anal. Appl., (2003), No. 2, 425-434.

[5] I. GRAHAM, G. KOHR, *Univalent mappings associated with the Roper-Suffridge extension operator*, J. Anal. Math., (2000), 331-342.

[6] I. GRAHAM, G. KOHR, *Geometric function theory in one and higher dimensions*, Marcel Dekker, New York, (2003).

[7] G. KOHR, *Loewner chains and a modification of the Roper-Suffridge extension operator*, Mathematica(Cluj), (2006), No. 1, 41-48.

[8] H. LI, S. FENG, *Roper-Suffridge extension operator on a Reinhardt domain*, Acta. Math. Sci., B (2014), No. 6,

1761-1774.

[9] T. S. LIU, S. GONG, *The family of starlike mappings. I*, Chin. Annal. Math. Ser. A., (2002), No. 3, 273-282.

[10] X. S. LIU, T. S. LIU, *The generalized Roper-Suffridge extension operator on a Reinhardt domain and the unit ball in a complex Hilbert space*, Chin. Annal. Math. Ser. A., (2005), No. 5, 721-730.

[11] T. S. LIU, G. B. REN, *The growth theorem for starlike mappings on bounded starlike circular domains*, Chin. Annal. Math. Ser. B., (1998), No. 4, 401-408.

[12] J. R. MUIR, *A class of the Loewner chain preserving extention operators*, J. Math. Anal. Appl., (2005), No. 1, 237-251.

[13] J. R. MUIR, *A Modification of the Roper-Suffridge extension operator*, Comput. Meth. Func. Theo., (2008), 862-879.

[14] J. R. MUIR, T. J. SUFFRIDGE, *Extreme points for convex mappings of* B^n, J. Anal. Math., (2006), No. 2, 169-182.

[15] J. R. MUIR, T. J. SUFFRIDGE, *A generalization of half-plane mappings to the ball in* $\mathbb{C}^n$, Trans. Amer. Math. Sco., (2007), No. 4, 1485-1498.

[16] C. POMMERENKE, *Univalent functions*, Vandenhoek and Ruprecht, Göttingen, (1975).

[17] S. RAHROVI, A. EBADIAN AND S. SHAMS, *G-Loewner chains and parabolic starlike mappings in several complex variables*, General Math., (2012) No. 2-3, 59-73.

[18] S. RAHROVI, A. EBADIAN AND S. SHAMS, *Applications of the Roper-Suffridge Extension Operator to the Spirallike mappings of type β*, Acta Uni. Apul., (2014), 171-183.

[19] K. A. ROPER, T. J. SUFFRIDGE, *Convex mappings on the unit ball* $\mathbb{C}^n$, J. Anal. Math., (1995), 333-347.

[20] J. WANG, C. GAO, *A new Roper-Suffridge extension operator on a Reinhardt domain*, Abst. Appl. Anal., (2011), 1-14.

[21] J. F. WANG, T. S. LIU, *A modified Roper-Suffridge extension operator for some holomorphic mappings*, Chin. Annal. Math. Ser. A., (2010), No. 4, 487-496.

[22] Y. C. ZHU, M. S. LIU, *The generalized Roper-Suffridge extension operator on a Reinhardt domain* D_p, Taiwanese J. Math., (2010), No. 2, 359-372.

[23] W. J. ZHANG, T. S. LIU, *On decomposition theorem of normalized biholomorphic convex mappings in Reinhardt domains*, Sci. Chin., (2003), No. 1, 94-106.