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Blow-up and Critical Fujita Exponents in a Degenerate Parabolic Equation

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Abstract: In this paper, we consider the Cauchy problem of degenerate parabolic equation not in divergence form $u_t = u^p \Delta u + u^q$, $p > 1$, $q > 1$, and give the blow-up conditions and the critical Fujita exponents for the existence of global and non-global solutions to the Cauchy problem.

Key words: blow-up, global existence, critical exponent, degenerate parabolic equation.

1. Introduction

We study the Cauchy problem for the nonlinear diffusion equation not in divergence form

$$\begin{cases} u_t = u^p \Delta u + u^q, & x \in \mathbf{R}^n, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1}$$

where $n \geq 1$, $p > 1$, $q > 1$ and $0 < u_0(x) \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$.

It is well known that problem (1) has a bounded positive continuous solution at least locally in time. (See [21, 22, 23].)

We define the blow-up time T^* as

$$T^* = \sup\{T > 0; (1) \text{ possesses a solution in } \mathrm{R}^n \times [0,T)\}. \tag{2}$$

If $T^* = \infty$, the solution is global. On the other hand, if $T^* < \infty$, then the solution is not global in time in the sense that it blows up at $t = T^*$ such as

$$\limsup_{t \to T^*} \| u(\cdot,t) \|_{L^\infty(\mathbf{R}^n)} = \infty. \tag{3}$$

A lot of significant result on the critical exponents for nonlinear parabolic equations have been obtained during the past decades. Fujita [1] considered the Cauchy problem:

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^n, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n. \end{cases} \tag{4}$$

In [1], it is shown that (4) possesses the critical Fujita exponent $p^* = 1 + 2/n$ such that

- if $p \in (1, p^*)$, then the solution $u(x, t)$ blows up in finite time for any $u_0(x)$;
- if $p \in (p^*,\infty)$, then there are both global solutions and non-global solutions corresponding to small and large initial data, respectively.

Hayakawa [6], Kobayashi-Sirao-Tanaka [9] and Weissler [20] have known that $p = p^* = 1 + 2/n$ belongs to the blow-up case. In some situations, the size of initial data required by the global and non-global solutions can be determined via the so-called second critical exponent concerning the decay rates of initial data as $|x| \to \infty$. When $p > p^* = 1 + 2/n$, Lee-Ni [10] established the second critical exponent $a^* = 2/(p-1)$ for (4) with initial data $u_0(x) = \lambda \varphi(x)$, where $\lambda > 0$ and $\varphi(x)$ is a bounded continuous function on $\mathbf{R}^n$, such that

- if $\liminf_{|x| \to \infty} |x|^a \varphi(x) > 0$ for some $a \in (0, a^*)$ and any $\lambda > 0$, then the solution

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$u(x,t)$ blows up in finite time;

- if $\limsup_{|x|\to\infty} |x|^a \varphi(x) < \infty$ for some $a \in (a^*, n)$, and if $\lambda < \lambda_0$ for some λ_0, then the solution $u(x,t)$ is global.

Lee-Ni [10] proved that $a^* = 2/(p-1)$ belongs to the global case.

The degenerate case

$$\begin{cases} u_t = \Delta u^m + u^p, & x \in \mathbf{R}^n, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{5}$$

with $m > 1$ and $\max(0, 1-2/n) < m < 1$ were thoroughly studied with the critical Fujita exponent $p^* = m + 2/n$ by Galaktionov-Kurdyumov-Mikhailov-Samarskii [3], Mochizuki-Mukai [12] and Qi [16]. Furthermore, Galaktionov [2], Mochizuki-Mukai [12], Kawanago [8] and Mochizuki-Suzuki [13] have shown that $p = p^* = m + 2/n$ belongs to the blow-up case. When $p > p^* = m + 2/n$, Mukai-Mochizuki-Huang [14] and Guo-Guo [5] obtained the second critical exponent $a^* = 2/(p-m)$ for (5).

It is mentioned that the degenerate equation (5) can be changed to

$$v_t = v^\alpha \Delta v + v^\beta \tag{6}$$

under the transformation $u(x,t) = a v^m(bx, t)$, $a = m^{m/(p-1)}$, $b = m^{(p-m)/2(p-1)}$, with the special $\alpha = (m-1)/m < 1$ and $\beta = (m+p-1)/m > 1$. Obviously, the equations (5) and (6) are not equivalent to each other for general α. Winkler [22] considered the Cauchy problem (1) for $p \ge 1$, and obtained the following results:

- For $1 \le q < p+1$ (resp. $1 \le q < 3/2$ if $p = 1$), all positive solutions of (1) are global and unbounded, provided that $u_0(x)$ decreases sufficiently fast in space.
- For $q = p+1$, all positive solutions of (1) blow up in finite time.
- For $q > p+1$, there are both global and non-global positive solutions.

Later, Li-Mu [11] and Yang-Zheng-Zhou [23] considered the Cauchy problem (1) for $p > 1$, and obtained the following results when $q > p + 1 + 2/n$:

(i) Let $n \ge 2$. Assume that $u_0(x) = \lambda\varphi(x)$, where $\lambda > 0$ and $\varphi(x)$ is a bounded continuous function in $\mathbf{R}^n$, and that

$$\liminf_{|x|\to\infty} |x|^a \varphi(x) > 0. \tag{7}$$

If

$$a \in \left(0, \frac{2}{q-p-1}\right), \tag{8}$$

or

$$a = \frac{2}{q-p-1} \text{ with large } \lambda > 0 \tag{9}$$

then the solution $u(x,t)$ blows up in finite time.

(ii) Let $n \ge 1$. Assume that $u_0(x) = \lambda\varphi(x)$, and that

$$\limsup_{|x|\to\infty} |x|^a \varphi(x) < \infty. \tag{10}$$

If

$$a \in \left(\frac{2}{q-p-1}, n\right), \tag{11}$$

then there exist $\lambda_0 = \lambda_0(\varphi)$, $C > 0$ such that the solution $u(x,t)$ is global in time and satisfies

$$\| u(x,t) \|_{L^\infty(\mathbf{R}^n)} \le C t^{-a/(ap+2)} \tag{12}$$

for all $t > 0$ whenever $\lambda < \lambda_0$.

The conclusions (i)-(ii) show that the problem (1) admits the second critical exponent

$$a^* = \frac{2}{q-p-1} \tag{13}$$

with $n \ge 2$ and $q > p + 1 + 2/n$.

In this article, we will study the blow-up of solutions $u(x,t)$ to the Cauchy problem (1) when $p+1 < q \le p+1+2/n$ or $0 < a < 2/(q-p-1)$ with $n \ge 1$.

Theorem Let $n \ge 1$. Suppose that one of the following two conditions holds;

(a) $p+1 < q \le p+1+\dfrac{2}{n}$.

(b) $\liminf_{|x|\to\infty} |x|^a u_0(x) > 0$ with $0 < a < \dfrac{2}{q-p-1}$.

Then the solution $u(x,t)$ of (1) blows up in finite time.

Comparing Theorem and the conclusions of Li-Mu [11] and Yang-Zheng-Zhou [23], we see that (1) possesses the critical Fujita exponent

$$q^* = p+1+\frac{2}{n} \tag{14}$$

and the second critical exponent

$$a^* = \frac{2}{q-p-1} \tag{15}$$

with $n \ge 1$ and $q > p+1$, and may be summarized in the following table:

	$p+1<q\le q^*$	$q>q^*$
$a<a^*$	Blow – up	Blow – up
$a=a^*$	Blow – up	Blow – up for large data
$a>a^*$	Blow – up	Global for small data Blow – up for large data

The rest of this paper is organized as follows. In sections 2 and 3, we state the proof of Theorem for the conditions (a) and (b), respectively.

2. Proof of Theorem (a)

In this section, we shall prove the Theorem for the condition (a) by two case of $n \ge 2$ and $n = 1$.

2.1 Case I: $n \ge 2$

We take the same strategy as in Li-Mu [11] and Igarashi-Umeda [7].

Let

$$B_{r,m} = \{x \in \mathbf{R}^n ; |x - x_m| < r|x_m|\} \tag{16}$$

for some constant $r > 0$ and a sequence $\{x_m\}_{m=1}^{\infty}$ satisfying $0 < |x_m| < |x_{m+1}|$ for any $m \in \mathbf{N}$ and $\lim_{m\to\infty} |x_m| = \infty$.

Remark The method using the sequence of balls $B_{r,m}$ in (16) was used in [17, 4, 18, 19] and the other papers.

Let $\lambda_m > 0$ denote the principal eigenvalue of $-\Delta$ with Dirichlet problem in $B_{r,m}$, and let $\varphi_m(x)$ denote the corresponding eigenfunction, normalized by

$$\int_{B_{r,m}} \varphi_m(x)dx = 1. \tag{17}$$

We define

$$F_m(t) = \frac{1}{p-1}\int_{B_{r,m}} u^{1-p}(x,t)\varphi_m(x)dx. \tag{18}$$

Then we have

$$F_m'(t) = -\int_{B_{r,m}} \frac{u_t}{u^p}\varphi_m(x)dx$$

$$= -\int_{B_{r,m}} (\Delta u + u^{q-p})\varphi_m(x)dx. \tag{19}$$

Integrating by parts, using the fact that $\varphi_m = 0$ and $\partial\varphi_m / \partial\nu \le 0$ on $\partial B_{r,m}$, where ν denote the outward unit normal vector to $B_{r,m}$ at $x \in \partial B_{r,m}$, and applying Green's formula, we obtain

$$F_m'(t) \le \lambda_m \int_{B_{r,m}} u\varphi_m(x)dx - \int_{B_{r,m}} u^{q-p}\varphi_m(x)dx. \tag{20}$$

Since $B_{r,m}$ is a n-dimensional ball of radius $r|x_m|$, it follows that λ_m satisfies

$$\lambda_m \le \frac{c}{|x_m|^2}, \tag{21}$$

where $c > 0$ depends only on the dimension n and r. Thus, we have

$$F_m'(t) \le \frac{c}{|x_m|^2}\int_{B_{r,m}} u\varphi_m(x)dx$$

$$-\int_{B_{r,m}} u^{q-p}\varphi_m(x)dx. \tag{22}$$

Since $p>1$, $q-p>1$, and $\int_{B_{r,m}} \varphi_m(x)dx=1$, by Jensen' s and Hölder' s inequalities, we have

$$\int_{B_{r,m}} u^{1-p}\varphi_m(x)dx \ge \left(\int_{B_{r,m}} u\varphi_m(x)dx\right)^{1-p} \quad (23)$$

and

$$\int_{B_{r,m}} u^{q-p}\varphi_m(x)dx \ge \left(\int_{B_{r,m}} u\varphi_m(x)dx\right)^{q-p}. \quad (24)$$

It follows from (23)-(24) that

$$\int_{B_{r,m}} u^{q-p}\varphi_m(x)dx$$

$$\ge \left(\int_{B_{r,m}} u\varphi_m(x)dx\right)^{q-p-1}\int_{B_{r,m}} u\varphi_m(x)dx$$

$$\ge \left(\int_{B_{r,m}} u^{1-p}\varphi_m(x)dx\right)^{-\frac{q-p-1}{p-1}}\int_{B_{r,m}} u\varphi_m(x)dx$$

$$=[(p-1)F_m(t)]^{-(q-p-1)/(p-1)}\int_{B_{r,m}} u\varphi_m(x)dx. \quad (25)$$

Thus, by (22)-(25), we obtain

$$F_m'(t)\le \frac{c}{|x_m|^2}\int_{B_{r,m}} u\varphi_m(x)dx$$

$$-[(p-1)F_m(t)]^{-(q-p-1)/(p-1)}\int_{B_{r,m}} u\varphi_m(x)dx. \quad (26)$$

Here, if $F_m(t)$ satisfies

$$F_m(t)\le \frac{1}{p-1}\left(\frac{|x_m|^2}{2c}\right)^{(p-1)/(q-p-1)} \quad (27)$$

for all $t\in[0,T^*)$, then by (26), it follows that

$$F_m'(t)\le -\frac{c}{|x_m|^2}\int_{B_{r,m}} u\varphi_m(x)dx. \quad (28)$$

Hence, if (28) holds, then by (23)-(28), we have

$$F_m'(t)\le -\frac{c}{|x_m|^2}(p-1)^{\frac{1}{1-p}}F_m^{\frac{1}{1-p}}(t) \quad (29)$$

from which it follows that if $F_m(0)$ satisfies

$$F_m(0)\le \frac{1}{p-1}\left(\frac{|x_m|^2}{2c}\right)^{\frac{p-1}{q-p-1}}, \quad (30)$$

then $F_m(t)$ decreases and

$$F_m(t)< \frac{1}{p-1}\left(\frac{|x_m|^2}{2c}\right)^{\frac{p-1}{q-p-1}} \text{ for all } t\in[0,T^*]; \quad (31)$$

and an integration of (29) shows that

$$F_m(t)\le (F_m^{\frac{p}{p-1}}(0)-C_1t)^{\frac{p-1}{p}} \quad (32)$$

with $C_1=c|x_m|^{-2}\ p(p-1)^{-\frac{p}{p-1}}$.

Therefore, from (32) we obtain that $F_m(t)\to 0$ as $t\to T^*=\frac{1}{C_1}F_m^{\frac{p}{p-1}}(0)$, that is $u(x,t)$ blows up in finite time.

As a result of these arguments, we have the following proposition:

Proposition 1. If $F_m(0)$ satisfies (30) for some $m\in\mathbf{N}$, that is

$$F_m(0)\le A|x_m|^{\frac{2(p-1)}{q-p-1}} \quad (33)$$

with

$$A=(p-1)^{-1}(2c)^{\frac{1-p}{q-p-1}}, \quad (34)$$

then $u(x,t)$ blows up in finite time.

Here, we shall state the rest of the proof of Theorem (a) for $n\ge 2$.

Suppose that $u(x,t)$ be a nontrivial global solution. Thus, by Proposition 1, it follows that for any $m\in\mathbf{N}$

$$F_m(0)> A|x_m|^{\frac{2(p-1)}{q-p-1}}, \quad (35)$$

that is

$$\int_{B_{r,m}} u_0^{1-p}(x)\varphi_m(x)dx>(2c)^{\frac{1-p}{q-p-1}}|x_m|^{\frac{2(p-1)}{q-p-1}}. \quad (36)$$

Let φ_m be radial function, that is $\varphi_m(x)=\varphi_m(\rho)$ $(\rho=|x|)$. Then, $\varphi_m(\rho)$ satisfies

$$(\varphi_m)_{\rho\rho}+\frac{n-1}{\rho}(\varphi_m)_\rho+\lambda_m\varphi_m=0 \quad \text{in } B_{r,m}. \quad (37)$$

Solving the equation (37), it follows that for some constant $a = a(\lambda_m) > 0$

$$\varphi_m(\rho) = a\rho^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\sqrt{\lambda_m}\rho), \tag{38}$$

where $J_\nu(z)$ is the Bessel function:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\Gamma(k+\nu+1)} \tag{39}$$

with the Gamma function

$$\Gamma(k+\nu+1) = \int_0^\infty s^{k+\nu} e^{-s} ds. \tag{40}$$

Then, we have

$$\varphi_m(\rho) = a\rho^{-\frac{n-2}{n}} \left(\frac{\sqrt{\lambda_m}\rho}{2}\right)^{\frac{n-2}{2}}$$

$$\times \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{n-2}{2}+1)} \left(\frac{\sqrt{\lambda_m}\rho}{2}\right)^{2k}$$

$$\le a\left(\frac{\sqrt{c}}{2|x_m|}\right)^{\frac{n-2}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{n}{2})} \left(\frac{\rho\sqrt{c}}{2|x_m|}\right)^{2k} \tag{41}$$

due to (21). Noting that $\rho \le (1+r)|x_m|$ by (16), we obtain

$$\varphi_m(\rho) \le a\left(\frac{\sqrt{c}}{2|x_m|}\right)^{\frac{n-2}{2}}$$

$$\times \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{n}{2})} \left\{\frac{(1+r)\sqrt{c}}{2}\right\}^{2k}. \tag{42}$$

Multiplying both sides of (36) by $|x_m|^{-n(p-1)}$, we have

$$|x_m|^{-n(p-1)} \int_{B_{r,m}} u_0^{1-p} \varphi_m dx$$

$$> (2c)^{\frac{1-p}{q-p-1}} |x_m|^{\left(\frac{2}{q-p-1}-n\right)(p-1)}. \tag{43}$$

Then, it follows from (42) that

$$a\left(\frac{\sqrt{c}}{2}\right)^{\frac{n-2}{2}} |x_m|^{-\frac{n-2}{2}-n(p-1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{n}{2})}$$

$$\times \left\{\frac{(1+r)\sqrt{c}}{2}\right\}^{2k} \int_{B_{r,m}} u_0^{1-p}(x) dx$$

$$> (2c)^{\frac{1-p}{q-p-1}} |x_m|^{\left(\frac{2}{q-p-1}-n\right)(p-1)}. \tag{44}$$

We note that

$$-\frac{n-2}{2} - n(p-1) < 0 \quad \text{if} \quad p > 1 \text{ and } n \ge 2, \tag{45}$$

and

$$\frac{2}{q-p-1} - n \ge 0 \quad \text{if} \quad p+1 < q \le p+1+\frac{2}{n}. \tag{46}$$

Then, if $m \in \mathbf{N}$ is sufficiently large, the right-hand side of (44) is larger than the left-hand side of (44). Thus we arrive at a contradiction. □

2.2 Case II: $n = 1$

The inequality (44) lead to a contradiction for $p > \frac{3}{2}$ when $n = 1$. So, we take the same strategy as in Li-Mu [11] and Pinsky [15] to prove Theorem for $n = 1$ and $p > 1$.

For $r > 0$, let $\lambda_r > 0$ denote the principal eigenvalue of $-\frac{\partial^2}{\partial x^2}$ with Dirichlet problem in $(-r, r)$, and let $\varphi_r(x)$ denote the corresponding eigenfunction, normalized by

$$\int_{-r}^{r} \varphi_r(x) dx = 1. \tag{47}$$

In fact,

$$\varphi_r(x) = \frac{\pi}{4r} \cos\frac{\pi x}{2r} \text{ and } \lambda_r = \frac{\pi^2}{4r^2}. \tag{48}$$

We define

$$J_r(t) = \frac{1}{p-1} \int_{-r}^{r} u^{1-p}(x,t) \varphi_r(x) dx. \tag{49}$$

Then we have

$$J_r'(t) = -\int_{-r}^{r} \frac{u_t}{u^p}\varphi_r(x)dx$$

$$= -\int_{-r}^{r}\left(\frac{\partial^2 u}{\partial x^2} + u^{q-p}\right)\varphi_r(x)dx. \qquad (50)$$

Integrating by parts, using (48), and noting that $\varphi_r'(r) < 0$ and $\varphi_r'(-r) > 0$, we obtain

$$J_r'(t) \le \lambda_r \int_{-r}^{r} u\varphi_r(x)dx - \int_{-r}^{r} u^{q-p}\varphi_r(x)dx$$

$$= \frac{\pi^2}{4r^2}\int_{-r}^{r} u\varphi_r(x)dx - \int_{-r}^{r} u^{q-p}\varphi_r(x)dx. \qquad (51)$$

Since $p > 1$, $q - p > 1$, and $\int_{-r}^{r}\varphi_r(x)dx = 1$, by Jensen' s and Hölder' s inequalities, we have

$$\int_{-r}^{r} u^{1-p}\varphi_r(x)dx \ge \left(\int_{-r}^{r} u\varphi_r(x)dx\right)^{1-p} \qquad (52)$$

and

$$\int_{-r}^{r} u^{q-p}\varphi_r(x)dx \ge \left(\int_{-r}^{r} u\varphi_r(x)\right)^{q-p}. \qquad (53)$$

It follows from (52)-(53) that

$$\int_{-r}^{r} u^{q-p}\varphi_r(x)dx$$

$$\ge \left(\int_{-r}^{r} u\varphi_r(x)dx\right)^{q-p-1}\int_{-r}^{r} u\varphi_r(x)dx$$

$$\ge \left(\int_{-r}^{r} u^{1-p}\varphi_r(x)dx\right)^{-\frac{q-p-1}{p-1}}\int_{-r}^{r} u\varphi_r(x)dx$$

$$= [(p-1)J_r(t)]^{-(q-p-1)/(p-1)}\int_{-r}^{r} u\varphi_r(x)dx. \qquad (54)$$

Thus, by (51)-(54), we obtain

$$J_r'(t) \le \frac{\pi^2}{4r^2}\int_{-r}^{r} u\varphi_r(x)dx$$

$$-[(p-1)J_r(t)]^{-\frac{q-p-1}{p-1}}\int_{-r}^{r} u\varphi_r(x)dx. \qquad (55)$$

Here, if $J_r(t)$ satisfies

$$J_r(t) \le \frac{1}{p-1}\left(\frac{2r^2}{\pi^2}\right)^{(p-1)/(q-p-1)} \qquad (56)$$

for all $t \in [0, T^*)$, then by (55), it follows that

$$J_r'(t) \le -\frac{\pi^2}{4r^2}\int_{-r}^{r} u\varphi_r(x)dx. \qquad (57)$$

Hence, if (57) holds, then by (52)-(57), we have

$$J_r'(t) \le -\frac{\pi^2}{4r^2}(p-1)^{\frac{1}{1-p}} J_r^{\frac{1}{1-p}}(t) \qquad (58)$$

from which it follows that if $J_r(0)$ satisfies

$$J_r(0) \le \frac{1}{p-1}\left(\frac{2r^2}{\pi^2}\right)^{\frac{p-1}{q-p-1}}, \qquad (59)$$

then $J_r(t)$ decreases and

$$J_r(t) < \frac{1}{p-1}\left(\frac{2r^2}{\pi^2}\right)^{\frac{p-1}{q-p-1}} \text{ for all } t \in [0, T^*]; \qquad (60)$$

and an integration of (58) shows that

$$J_r(t) \le (J_r^{\frac{p}{p-1}}(0) - C_2 t)^{\frac{p-1}{p}} \qquad (61)$$

with $C_2 = \frac{\pi^2}{4} r^{-2} p(p-1)^{-\frac{p}{p-1}}$.

Therefore, from (61) we obtain that $J_r(t) \to 0$ as $t \to T^* = \frac{1}{C_2} J_r^{\frac{p}{p-1}}(0)$, that is $u(x,t)$ blows up in finite time.

As a result of these arguments, we have the following proposition:

Proposition 2. If $J_r(t)$ satisfies (59) for some $r > 0$, that is

$$J_r(0) \le B r^{\frac{2(p-1)}{q-p-1}} \qquad (62)$$

with $B = (p-1)^{-1}\left(\frac{\pi^2}{2}\right)^{\frac{1-p}{q-p-1}}$, then $u(x,t)$ blows up in finite time.

Here, we shall state the rest of the proof of Theorem (a) for $n = 1$.

Suppose that $u(x,t)$ be a nontrivial global solution. Thus, by Proposition 2, it follows that for any $r > 0$

$$J_r(0) > Br^{\frac{2(p-1)}{q-p-1}}, \tag{63}$$

that is

$$\int_{-r}^{r} u_0^{1-p}(x)\varphi_r(x)dx > \left(\frac{\pi^2}{2}\right)^{\frac{1-p}{q-p-1}} r^{\frac{2(p-1)}{q-p-1}}. \tag{64}$$

Note that

$$\varphi_r(x) = \frac{\pi}{4r}\cos\frac{\pi x}{2r} \le \frac{\pi}{4r} \tag{65}$$

by (48). Then, it follows that

$$\frac{\pi}{4r}\int_{-r}^{r} u_0^{1-p}(x)dx > \left(\frac{2}{\pi^2}\right)^{\frac{p-1}{q-p-1}} r^{\frac{2(p-1)}{q-p-1}}. \tag{66}$$

Multiplying both sides of (66) by r^{1-p}, we have

$$\frac{\pi}{4r^p}\int_{-r}^{r} u_0^{1-p}(x)dx > \left(\frac{2}{\pi^2}\right)^{\frac{p-1}{q-p-1}} r^{\left(\frac{2}{q-p-1}-1\right)(p-1)}. \tag{67}$$

We note that $p > 1$, and

$$\frac{2}{q-p-1} - 1 \ge 0 \quad \text{if} \quad p+1 < q \le p+3. \tag{68}$$

Then, if $r > 0$ is sufficiently large, the right-hand side of (67) is larger than the left-hand side of (67). Thus we arrive at a contradiction.

3. Proof of Theorem (b)

In this section, we shall prove the Theorem for the condition (b).

Since $\liminf_{|x|\to\infty} |x|^a u_0(x) > 0$ with $0 < a < \frac{2}{q-p-1}$, we have

$$\begin{aligned}
&|x_m|^{-\frac{2(p-1)}{q-p-1}} F_m(0) \\
&= \frac{|x_m|^{-\frac{2(p-1)}{q-p-1}}}{p-1}\int_{B_{r,m}} u_0^{1-p}(x)\varphi_m(x)dx \\
&\le \frac{|x_m|^{-\frac{2(p-1)}{q-p-1}}}{p-1}\int_{B_{r,m}} |x|^{a(p-1)}\varphi_m(x)dx \\
&\le \frac{|x_m|^{-\frac{2(p-1)}{q-p-1}}}{p-1}\int_{B_{r,m}} \{(1+r)|x_m|\}^{a(p-1)}\varphi_m(x)dx \\
&= \frac{(1+r)^{a(p-1)}}{p-1}|x_m|^{(p-1)\left(a-\frac{2}{q-p-1}\right)} \le A
\end{aligned} \tag{69}$$

with A defined in (34) for $m \in \mathbf{N}$ large enough. Thus, $u(x,t)$ blows up in finite time by Proposition 1.

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