

# Hilbert's Foundations Remain Intact

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**Abstract:** Some of the seemingly important contributions made by Russell, Cantor and Gödel regarding the foundations of mathematics lose credibility when addressed at a more fundamental level which assumes language consists of symbols devoid of what they mean. Such an approach can reveal neglected assumptions which are of consequence when made explicit and which, in particular, mean that Hilbert in fact had no need to heed Gödel's revolutionary results regarding the foundations of mathematics.

**Key words:** Paradox; self-reference; foundations of mathematics

## 1. Introduction

This paper seeks to expose the assumptions implicit in the use of a symbolic language by taking as primitive not, as invariably one does, the *meaning of the expressions* of Russell's set of all sets, or of Cantor's set of all numbers, or of Gödel's constructed sentence, but rather *the expressions themselves* regarded as, to use Hilbert's phrase, 'marks on paper'. The transition from such marks to their meaningful use is central to the argument here, and hence a questioning critic is invoked to justify the sequence involved.

In what follows this transition is considered for each of the above authors in turn and shows how their use of self-referential expressions in the context given is, in effect, meaningless. Hence in this respect their conclusions have no validity.

## 2. The Errors

### 2.1 Russell's Paradox

Consider the set  $S$  of all sets which are not members of themselves. Russell asked if  $S$  is a member of  $S$ ? For if it is then it is not and if it is not

then it is.

To answer this question by beginning at the primitive level mentioned we first need to consider the following:

a) The critic is unlikely to object if we start with the assumption that when addressed the paradox consists of marks on paper in an arrangement recognised as the words of a natural language.

b) But obviously the marks themselves – and they are all that is available – tell us nothing about sets or anything else. The words are not pictograms; they themselves contain no information regarding what they mean. They are simply codes for something other than themselves, and hence (as with a formal language) when used require interpretation to provide meaning.

c) So the critic then points out we need an assumption that will allow us to infer knowledge other than words from an expression of natural language. We can do this by assuming it has an author (in this case Russell) who has used the expression to convey the author's knowledge. The critic then asks how the reader can be aware of such knowledge.

d) The answer is a deceptively simple general principle: if that for which the author would use the expression 'A' includes that for which the author would use the expression 'B' then the author can say 'A is B', which allows the reader to infer the author's

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knowledge. Notice however the critic can make the salient point that for the reader, who may not know A is B, this principle means the interpretation of 'A' requires the interpretation of 'B'.

For example, if for an author what is named by the *single* expression 'Paris' includes it being a city, then for an audience to infer the author's knowledge from his saying 'Paris is a city', the interpretation of the subject term 'Paris' *also requires* the interpretation of the predicate 'is a city'.

Of course steps a), b), c) and d) may appear obvious and unduly pedantic; so much so in fact that familiarity with language means that the steps can normally, for everyday use, be safely ignored and not even realised.

If now we return to Russell's question, the critic reminds us that in a) and b) above we have maintained that words are simply codes which mean something other than themselves – just as the meaning of word 'red' should not include the word 'red'.

As a result Russell's question is impossible to decode since, *given d)*, in order to infer the author's knowledge the interpretation of the subject term 'S' for the reader requires the interpretation of 'S' in the predicate.

Thus Russell's 'Paradox' is not a paradox but a syntactically correct, meaningless (in the above context) sentence.

The same observation can be made of other self-referential paradoxes. For example consider the classic liar paradox 'this sentence is false'.

Here we assume the context requires that 'this sentence' is the sentence 'this sentence is false', in which case 'this sentence' cannot be interpreted for its interpretation would require the interpretation of 'this sentence' in 'this sentence is false'.

Of course there are other contexts where the sentence would be valid. I could for example point to some false sentence on the blackboard and use the demonstrative 'this' to say 'this sentence is false'.

Paradoxes like the liar had a role in Gödel's

formulation of his first incompleteness theorem [1] although if the forgoing is correct this was unwarranted.

Furthermore without the handicap of the Russell Paradox, Frege's set axioms were not inconsistent and there is no objection to the use of a property to define a set [2].

## 2.2 Cantor's Theorem

Here as is the case with Russell's Paradox it will be shown that Cantor's theorem is invalid because of overlooked assumptions regarding the use of language – in this case the language of arithmetic.

In simplistic terms Cantor proved there are a larger sets than an infinite set. He did this by showing that the real numbers were not denumerable because if we put an infinite list of the real numbers in a one to one correspondence with the infinite set of natural numbers there will always be a number which will be absent from the original list of [all] real numbers.

Hence we will have a 'larger' set than the infinite set of natural numbers.

To do this Cantor showed that if we assemble a list of all real numbers, we can construct a number that differs from the first number on the list by making its first decimal place different from that of the first listed number. Similarly we can construct a number different from the second listed number by making it different in its second decimal place... and so on. In this way we get a real number that is an infinite decimal and is different from any number on the list.

Cantor's technique can be readily seen when applied to the list, L, of real numbers below.

3.23056...

2.11345...

1.57222...

0.72852...

2.42318...

6.121262... etc.

Here the diagonal, D, **3.17816...** is shown in bold. Each number of D is then changed – say by

subtracting 1 from each number – to give a number,  $D'$ , 2.06705...

$D'$  cannot be the first number on the list since it begins with 2, nor the second number since its second digit is 0, nor the  $n^{\text{th}}$  number on the list since its  $n^{\text{th}}$  digit is not the  $n^{\text{th}}$  digit of the  $n^{\text{th}}$  number on the list.

The above is a brief outline of Cantor's proof, but the error is not with the logic of the proof itself but with the use of so-called numbers.

Just as the words of natural language are marks on paper which tell you nothing of what they mean so are the numerical digits of the language of arithmetic.

Such digits can only represent numbers and the number they represent must be known or assumed to be known before they are used. I say assumed to be known because, as with natural language, familiarity and expertise with mathematical language often mean that digits are conveniently addressed as numbers (which, it will be noticed, I have done here). In this case the assumption of prior number representation remains implicit rather than explicit.

But there is a crucial difference between regarding some digital sequence as representing a particular number and assuming a particular number is represented by a digital sequence. In the former case, for the reader, the digital sequence determines the number represented whilst in the latter, for the author, the number determines the digital sequence.

Failure to acknowledge this difference can result in number properties being wrongly attributed to digital sequences, and this is where Cantor erred.

For example consider the digits 1, 2, 3 arrayed as 123. Depending on the context it would be quite normal practice to address the array as the number 123. But that context should not prohibit the implicit assumption that '123' has been used (by an author) to represent a number.

Note the mild restriction here. Rarely will we go astray by referring to numerals or digits loosely as numbers as long as the context allows such latitude. That is we must always be able to assume there is a

number (and even the unrealised idea of a number is sufficient for the assumption to hold) which has been represented by the digits; without such an assumption we can only address digits and not what they mean.

To make the point quite clear, we could not regard '123' as a number in a context where it was typed at random by, say, a monkey, for it cannot be assumed that '123' represented, as it must, the author's knowledge of a number.

So what has this to do with Cantor's proof?

Well, as indicated above, Cantor constructs a number which he shows cannot be found on a list assumed to be of *all* real numbers. But that constructed 'number' is in fact a sequence of digits,  $D'$ , defined by changes to the digits of the diagonal sequence,  $D$ .

No matter, one might say. We have produced an infinite sequence of digits by Cantor's construction, so let us simply assume, as we invariably do, that the sequence of digits represent a real number, in which case the Cantor construction shows this number will not be found on the list.

But this will not do.

The  $n^{\text{th}}$  digit of  $D'$  is a function of the  $n^{\text{th}}$  digit of  $D$  which in turn is the  $n^{\text{th}}$  digit of the  $n^{\text{th}}$  number,  $L_n$ , of the list  $L$ .

Thus the Cantor construction provides the sequence of digits  $D'$  as:

(not 1<sup>st</sup> digit of  $L_1$ ), (not 2<sup>nd</sup> digit of  $L_2$ ), (not 3<sup>rd</sup> digit of  $L_3$ )... (not  $n^{\text{th}}$  digit of  $L_n$ ) ...

We now need the assumption mentioned earlier which will enable us to remove the commas and so allow  $D'$  to represent a number, and therefore one which will not be found in  $L$ .

But we cannot make such an assumption. Why? Because until the conclusion of the proof any number we introduce must belong to  $L$ , for  $L$  was necessarily (*otherwise we would assume the conclusion*) required to be a list of *all* real numbers.

In which case the all-important assumption (that the constructed sequence represented a number) would

entail the contradiction that the  $n^{\text{th}}$  digit of  $D'$  was necessarily the  $n^{\text{th}}$  digit of  $L_n$  yet, by construction, was not the  $n^{\text{th}}$  digit of  $L_n$ .

So  $D'$  simply has to remain a sequence of digits not representing a number in  $L$ . Of course this is not to say there is no number represented by the  $D'$  sequence. There are an infinite number of other permutations of  $L$  all of which for whom  $D'$  would not be a function of the diagonal and hence there would be no problem in supposing it represented a number.

But whatever the order of  $L$ , the Cantor diagonal method would always be blocked by the necessary requirement (before the end of the proof) that no digital sequence of the changed diagonal can be assumed to represent a number not belonging to  $L$ .

Thus it would seem the diagonal construction cannot show the set of real numbers is 'larger' than the infinite set of natural numbers. This set remains a denumerable infinite set.

### 2.3 The Gödel Sentence

In an outline of Gödel's argument Nagel and Newman [3] observed that Gödel constructed a formula,  $G$ , in a formal language which was uniquely associated with a certain number  $g$  (its Gödel number) and whose meta-mathematical rendering was 'The formula that has Gödel number  $g$  [i.e. Formula  $G$ ] is not demonstrable'.

But again, as with Russell's Paradox, this resort to self-reference results in a meaningless statement.

To see why this should be so consider the following informal representation – call it  $F$  – of the Gödel formula  $G$ :

Formula  $G$  says there is no proof of formula  $G$ .

Here, given d) in 2.1, the reader has to assume that what the author called 'Formula  $G$ ' in the informal representation also included the property called by the author 'says there is no proof of formula  $G$ '. But that cannot be coherent for it would mean the interpretation of the subject 'Formula  $G$ ' would require the interpretation of 'Formula  $G$ ' in the

predicate. Hence the reader must conclude that  $F$  is meaningless.

However, if  $F$  is meaningless but, nevertheless, faithfully translates the formal sentence  $G$ , then correspondingly one would expect Gödel's construction of  $G$  in the formal language to be in error.

And this is the case, since to translate  $G$  as referring to itself, albeit indirectly, Gödel had to adopt [4] the innovative strategy of assuming the proof sequence was not of a formula with Gödel number  $n$ , say, but was of that formula when  $n$  had been substituted for its free variable – that is the proof was of the 'diagonalisation' of the formula with Gödel number  $n$ .

This procedure, however, has a restriction on the values of  $n$  [strictly the numeral for  $n$ ] which can be substituted for the free variable. Gödel numbers are defined by association with the symbols and subsequent formulae of a language of logic. And because of the Gödel strategy  $n$  must be the Gödel number of a formula whose diagonalisation was then proved.

But this requirement cannot be observed when the Gödel number is defined, as is the case here, by a formula constructed *before* diagonalisation since this would mean that the value of  $n$  when substituted for the free variable would not be valid under the self-referential interpretation, for it would simply be the Gödel number of a formula and not of a formula prior to diagonalisation.

This last sentence could well be the source of Gödel's error for it warrants some attention.

Certainly  $n$  can be defined as indicated, and certainly the defining formula can then be diagonalised. So why are we saying  $n$  cannot be defined as the Gödel number of a formula which is subsequently diagonalised? The answer is because it *cannot be known* before  $n$  is substituted for the free variable of the formula that the latter will be diagonalised.

We have the order: constructed formula, its Gödel number ( $n$ ), diagonalisation of constructed formula  $G$ ,

proof of  $G$ . This order means we can only refer *retrospectively*, after diagonalisation, to  $n$  being the Gödel number of a formula which is subsequently diagonalised; and  $n$  cannot be defined retrospectively. Thus Gödel's strategy for self-reference mentioned above is not possible.

Hence the claim at the beginning of this section - that the meaningless of the informal interpretation of  $G$  was mirrored in the formal construction - is justified by revealing the time-ordering invalidity of a formula in the construction of  $G$ .

### 3. Conclusions

This paper has shown that because natural language, including the language of arithmetic, is solely a language of symbols devoid of any meaning, their use must follow and not precede what they represent. And what they represent has been assumed to be an author's knowledge which is inferred by the reader of

the symbols. It is the use of language in a context where the order mentioned cannot occur or is not observed, that has resulted in the claim of meaningless for some of the well-established statements of Russell, Cantor and Gödel which were a source of consternation for Hilbert's programme.

Such a claim, if upheld, means that Hilbert was vindicated in his approach to the foundations of mathematics.

### References

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- [4] *Ibid*, 97.