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Convergence Rates for the Stratified Periodic Homogenization Problems

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Abstract: In this paper, we study the convergence rates of homogenization problems for composites with general stratified periodic structure. After introduced auxiliary function, we get the representation formula satisfied by oscillatory solution and homogenized solution. Then we utilize the formula in combination with the asymptotic estimates of Green functions to obtain convergence rates in L^p of solutions for any $1 < p \le \infty$.

1. Introduction

Composite material have nonhomogeneous structures which fascinates people for a long time. Analysis of macroscopic properties of composites was investigated by the physicists Maxwell. Around 1970, the problem of found the physical properties of material structure was reformulated in such a way that this field became interesting from a purely mathematical point of view. This formulation initiated a new mathematical discipline called homogenization theory.

Homogenization in partial differential equation is a well studied topic which deals with the asymptotic analysis of physics in heterogeneous with a periodic structure. We refer to some classical books [1, 2, 3, 4, 5, 6, 7] for background and overview. The concept of stratified periodic homogenization was introduced by Bensoussan, Lions and Papanicolaou [1] in 1978 and developed by Briane [8] with all necessary proofs and some interesting biomechanics and engineering applications. Generalized multilayered materials, such as stratified locally periodic materials or wavy fiber composites are found in many branches of engineering and biomechanics. For instance, nano-shells, jackets of fiber reinforced polymers (FRP), repair of civil engineering structures, modeling of the human heart tissue and so on.

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The main purpose of this paper is to study the convergence rates of solutions for the stratified periodic homogenization operators of the form,

$$L_\varepsilon = -div(A(\rho(x)/\varepsilon)) = -\frac{\partial}{\partial x_i}\left(a_{ij}(\rho(x)/\varepsilon)\frac{\partial}{\partial x_j}\right), \quad \varepsilon > 0,$$

where

$$\rho(x) \in C^2(\overline{\Omega}, \mathbb{R}^m), \quad (m \le n),$$

and satisfies

$$\lambda|\xi|^2 \le \sum_{i=1}^{n}\left(\sum_{k=1}^{m}\frac{\partial\rho_k(x)}{\partial x_i}\xi_k\right)^2 \le \frac{1}{\lambda}|\xi|^2,$$

$$\textit{for any } \xi = (\xi_i) \in \mathbb{R}^m, \tag{1}$$

where $\lambda > 0$.

Throughout this paper, the summation convention is used. We assume that the matrix $A(y) = (a_{ij}(y))$ with $1 \le i, j \le n$ is real symmetric and satisfies the ellipticity condition,

$$a_{ij}(y) = a_{ji}(y),\ \lambda|\xi|^2 \le a_{ij}(y)\xi_i\xi_j \le \frac{1}{\lambda}|\xi|^2,$$
$$for\ y \in \mathbb{R}^m\ \ and\ \ \xi = (\xi_i) \in \mathbb{R}^m, \quad (2)$$

the periodicity condition

$$A(y+l) = A(y),\ \ for\ y \in \mathbb{R}^m\ \ and\ \ l \in \mathbb{Z}^m. \quad (3)$$

We also impose the smoothness condition

$$\| A(y) \|_{C^{0,\alpha}(\mathbb{R}^m)} \le \Lambda,\ for\ some\ \ \alpha, \Lambda > 0. \quad (4)$$

The proof of error estimates is a one of the main questions in homogenization theory. There are many papers about results of convergence rates for elliptic homogenization problems. Assume that all of functions are smooth enough, the $O(\varepsilon)$ error estimate in L^∞ was presented by Bensoussan, Lions and Papanicolaou [1]. In 1987, Avelcaneda and Lin [9] proved L^p convergence for the elliptic homogenization problems by the method of maximum principle. After that, they [10] obtained L^∞ error estimate when f is less regular than Bensoussan, Lions and Papanicolaou's. Griso [12, 13] obtained interior error estimates by using the periodic unfolding method. In 2010, Kenig, Lin and Shen [14] studied rates of convergence of solutions in L^2 and $H^{\frac{1}{2}}$ in Lipschitz domains with Dirichlet or Neumann problems. Recently, they [15] have also studied the asymptotic behavior of the Green and Neumann functions and obtained some error estimates for solutions.

As the author know, there are very few and incomplete results of convergence rates for stratified periodic homogenization problems. It is more difficult to deal with stratified periodic homogenization problems than homogenization problems, since the main characteristics of a material are the coefficients of a partial differential equation, but now the vector value function $\rho(x)$ is a nonlinear transformation generally which will cause new difficulties in the estimation of the representation formula satisfied by oscillatory solution and homogenized solution (see Proposition 2). In this paper, we overcome this problem after introduced auxiliary function.

The procedure we used for obtaining convergence rates estimates is somewhat analogous to the process Kenig, Lin and Shen [15] used for the most classical homogenization problems. The main purpose of this paper is to extend their [15] results to the general form of homogenization operators. This would be more interesting and technical. In particular, if $\rho(x) = x$, we have the usual classical form of homogenization operators. Furthermore, if $\rho(x) = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, we have a periodic structure in spheres. If $\rho(x) = (x_1^2 + x_2^2)^{1/2}$, we have a cylindrical structure with axis in the x_3 direction.

The rest of the paper is organized as follows. Section 2 contains some basic formulas and useful propositions which play important roles to get error estimates. In section 3, we show that the solutions of Dirichlet problems convergence in $L^p(\Omega)$ for any $1 < p \le \infty$ to the solutions of the corresponding homogenized problems based on obtaining estimates for the Green functions of the operators.

2. Preliminaries

In this section we will establish some useful estimates and propositions.

Let u_ε be a solution of the following elliptic equation Dirichlet boundary value problem,

$$\begin{aligned} L_\varepsilon u_\varepsilon = -\frac{\partial}{\partial x_i}\left(a_{ij}(\rho(x)/\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j} \right) &= f(x)\ \ in\ \ \Omega, \\ u_\varepsilon &= 0 \quad on\ \partial\Omega, \end{aligned} \quad (5)$$

where Ω is a bounded $C^{1,1}$ domain contained in $\mathbb{R}^n$. Associated with (5) is the homogenized problem

$$\begin{aligned} L_0 u_0 = -\frac{\partial}{\partial x_i}\left(q_{ij}(x)\frac{\partial u_0}{\partial x_j} \right) &= f(x)\ \ in\ \ \Omega, \\ u_0 &= 0 \quad on\ \partial\Omega, \end{aligned}$$

where L_0 is the homogenized operator of L_ε. The homogenized matrix is given by

$$q_{ij}(x)=\int_{Y_m}\left[a_{ij}(y)-a_{ik}(y)\frac{\partial\chi_j}{\partial y_l}(x,y)\frac{\partial\rho_l}{\partial x_k}(x)\right]dy, \tag{6}$$

where $Y_m=[0,1)^m\simeq\mathbb{R}^m/\mathbb{Z}^m$. Functions $\chi(x,y)=\chi^j(x,y)$ is defined by the following cell problem,

$$\begin{cases}\dfrac{\partial}{\partial y_k}\left[a_{im}(y)\dfrac{\partial\chi_j}{\partial y_l}(x,y)\dfrac{\partial\rho_k}{\partial x_i}(x)\dfrac{\partial\rho_l}{\partial x_m}(x)\right]\\ =\dfrac{\partial}{\partial y_k}a_{ij}(y)\dfrac{\partial ho_k}{\partial x_i}(x)\quad in\ Y_m,\\ \chi_j(x,y+l)=\chi_j(x,y)\, for\ y\in\mathbb{R}^m\quad l\in\mathbb{Z}_m,\\ \int_{Y_m}\chi_j(x,y)dy=0,\end{cases} \tag{7}$$

for each $1\le j\le n$.

Let $G_\varepsilon(x,y)$ denote the Green functions for operators L_ε in a bounded domain Ω. It follows essentially the same steps as [10, 11], we can obtain the estimates of Green functions which are more or less standard. More precisely, if Ω is a bounded $C^{1,\eta}$ domain for some $0<\eta<1$, then for any $x,y\in\Omega$,

$$\begin{cases}|G_\varepsilon(x,y)|\le\dfrac{C}{|x-y|^{n-2}},(8)\\ |_xG_\varepsilon(x,y)|\le\dfrac{C}{|x-y|^{n-1}},(9)\\ |_yG_\varepsilon(x,y)|\le\dfrac{C}{|x-y|^{n-1}},\end{cases} \tag{10}$$

where constant C depends only on n,λ,Λ,α and Ω.

Remark 1. If $\rho(x)$ satisfies (1), we can construct a transformation

$$\overline{\rho}(x)=(\rho_1(x),\cdots,\rho_m(x),\overline{\rho_{m+1}}(x),\cdots,\overline{\rho_n}(x)),$$

such that $\overline{\rho}(x):R^n\to R^n$ is an one-to-one non-degenerate transformation and satisfies

$$\lambda|\xi|^2\le\xi^\tau\overline{\rho}\xi\le\frac{1}{\lambda}|\xi|^2,\quad for\ any\ \ \xi\in\mathbb{R}^n,$$

where λ is a strictly positive real number and satisfies $0<\lambda<\infty$.

The next proposition had been proved by Kenig, Lin and Shen [14].

Proposition 1. Let $B_{ij}(y)\in L^2(Y)$ with $1\le i,j\le n$, where $Y=[0,1)^n\simeq\mathbb{R}^n/\mathbb{Z}^n$. Suppose that $\int_Y B_{ij}(y)dy=0$ and $\dfrac{\partial}{\partial y_i}(B_{ij}(y))=0$. Then there exists $\Phi_{kij}\in H^1(Y)$ such that $B_{ij}=\dfrac{\partial\Phi_{kij}}{\partial y_k}$ and $\Phi_{kij}=-\Phi_{ikj}$.

Remark 2. Let

$$B_{ij}(x,y)=q_{ij}(x)-a_{ij}(y)+ a_{im}(y)\frac{\partial\chi_j}{\partial y_l}(x,y)\frac{\partial\rho_l}{\partial x_m}(x). \tag{11}$$

In view of (6) and (7), we find that $Y_m\int\left(B_{ij}(x,y)\dfrac{\partial\rho_k}{\partial x_i}(x)\right)dy=0$ and $\dfrac{\partial}{\partial y_k}\left(B_{ij}(x,y)\dfrac{\partial\rho_k}{\partial x_i}(x)\right)=0$. It follows from Proposition 1 that there exists Φ_{lkj} such that $\Phi_{lkj}=-\Phi_{klj}$ and

$$B_{ij}\frac{\partial\rho_k}{\partial x_i}=\frac{\partial\Phi_{lkj}}{\partial y_l}. \tag{12}$$

From Remark 1, we obtain $B_{ij}=\dfrac{\partial\Phi_{lkj}}{\partial y_l}\left(\dfrac{\partial\rho_k}{\partial x_i}\right)^{-1}$, where $\left(\dfrac{\partial\rho_k}{\partial x_i}\right)^{-1}$ is the inverse of matrix $\left(\dfrac{\partial\rho_k}{\partial x_i}\right)$.

In view of (4) and (7), we obtain $\chi\in C^{0,\alpha}(\mathbb{R}^m)$ (Theorem 8.14 in reference [16]). This implies that $\Phi\in C^{0,\alpha}(\mathbb{R}^m)$. In particular,

$$\|\chi\|_{W^{1,\infty}(R^m)}+\|\Phi_{kij}\|_{L^\infty(R^m)}\le C, \tag{13}$$

where constant C depends only on Λ,m,α,λ.

Proposition 2. Suppose that $u_\varepsilon\in H^1(\Omega)$, $u_0\in H^2(\Omega)$ and $L_\varepsilon(u_\varepsilon)=L_0(u_0)$ in Ω. Let

$$\omega_\varepsilon(x) = u_\varepsilon(x) - u_0(x) + \varepsilon\chi_j(x, \rho(x)/\varepsilon)\frac{\partial u_0}{\partial x_j}(x).$$

Then

$$L_\varepsilon(\omega_\varepsilon) = -\varepsilon\frac{\partial}{\partial x_i}\left(a_{ij}\chi^k\frac{\partial^2 u_0}{\partial x_j\partial x_k} + a_{ij}\frac{\partial\chi^k}{\partial x_j}\frac{\partial u_0}{\partial x_k}\right) + \varepsilon\frac{\partial}{\partial x_i}\left[\Phi_{lkj}\left(\frac{\partial\rho_l}{\partial x_m}\right)^{-1}\left(\frac{\partial\rho_k}{\partial x_i}\right)^{-1}\frac{\partial^2 u_0}{\partial x_j\partial x_m}\right]$$
$$+\varepsilon\frac{\partial}{\partial x_i}\left[\Phi_{lkj}\frac{\partial\left(\frac{\partial\rho_l}{\partial x_m}\right)^{-1}}{\partial x_m}\left(\frac{\partial\rho_k}{\partial x_i}\right)^{-1}\frac{\partial u_0}{\partial x_j}\right] + \varepsilon\frac{\partial}{\partial x_i}\left[\Phi_{lkj}\frac{\partial\left(\frac{\partial\rho_k}{\partial x_i}\right)^{-1}}{\partial x_m}\left(\frac{\partial\rho_l}{\partial x_m}\right)^{-1}\frac{\partial u_0}{\partial x_j}\right] \tag{14}$$

Proof. Note that

$$a_{ij}\frac{\partial\omega_\varepsilon}{\partial x_j} = a_{ij}\frac{\partial u_\varepsilon}{\partial x_j} - a_{ij}\frac{\partial u_0}{\partial x_j} + \varepsilon a_{ij}\chi_k\frac{\partial^2 u_0}{\partial x_k\partial x_j} + \varepsilon a_{ij}\frac{\partial\chi_k}{\partial x_j}\frac{\partial u_0}{\partial x_k} + a_{ik}\frac{\partial\chi_j}{rtialy_m}\frac{\partial\rho_m}{\partial x_k}\frac{\partial u_0}{\partial x_j}.$$

This together with $L_\varepsilon(u_\varepsilon) = L_0(u_0)$, gives

$$L_\varepsilon(\omega_\varepsilon) = -\frac{\partial}{\partial x_i}\left[\left(q_{ij} - a_{ij} + a_{ik}\frac{\partial\chi^j}{\partial y_m}\frac{\partial\rho_m}{\partial x_k}\right)\frac{\partial u_0}{\partial x_j}\right] - \varepsilon\frac{\partial}{\partial x_i}\left[a_{ij}\chi^k\frac{\partial^2 u_0}{\partial x_j\partial x_k} + a_{ij}\frac{\partial\chi^k}{\partial x_j}\frac{\partial u_0}{\partial x_k}\right]$$
$$= -\frac{\partial}{\partial x_i}\left[\varepsilon\frac{\partial\Phi_{lkj}}{\partial x_m}\left(\frac{\partial\rho_l}{\partial x_m}\right)^{-1}\left(\frac{\partial\rho_k}{\partial x_i}\right)^{-1}\frac{\partial u_0}{\partial x_j}\right] - \varepsilon\frac{\partial}{\partial x_i}\left[a_{ij}\chi^k\frac{\partial^2 u_0}{\partial x_j\partial x_k} + a_{ij}\frac{\partial\chi^k}{\partial x_j}\frac{\partial u_0}{\partial x_k}\right]$$
$$= -\varepsilon\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_m}\left[\Phi_{lkj}\left(\frac{\partial\rho_l}{\partial x_m}\right)^{-1}\left(\frac{\partial\rho_k}{\partial x_i}\right)^{-1}\frac{\partial u_0}{\partial x_j}\right] - \varepsilon\frac{\partial}{\partial x_i}\left[a_{ij}\chi^k\frac{\partial^2 u_0}{\partial x_j\partial x_k} + a_{ij}\frac{\partial\chi^k}{\partial x_j}\frac{\partial u_0}{\partial x_k}\right]$$
$$= +\varepsilon\frac{\partial}{\partial x_i}\left[\Phi_{lkj}\left(\frac{\partial\rho_l}{\partial x_m}\right)^{-1}\left(\frac{\partial\rho_k}{\partial x_i}\right)^{-1}\frac{\partial^2 u_0}{\partial x_j\partial x_m}\right] + \varepsilon\frac{\partial}{\partial x_i}\left[\Phi_{lkj}\frac{\partial\left(\frac{\partial\rho_l}{\partial x_m}\right)^{-1}}{\partial x_m}\left(\frac{\partial\rho_k}{\partial x_i}\right)^{-1}\frac{\partial u_0}{\partial x_j}\right]$$
$$+\varepsilon\frac{\partial}{\partial x_i}\left[\Phi_{lkj}\frac{\partial\left(\frac{\partial\rho_k}{\partial x_i}\right)^{-1}}{\partial x_m}\left(\frac{\partial\rho_l}{\partial x_m}\right)^{-1}\frac{\partial u_0}{\partial x_j}\right]$$

where we have used (11) and (12). From the antisymmetry of Φ_{lkj}, we obtain (14). This completes the proof.

3. L^p Convergence estimates

The goal of this section is to establish L^p convergence rates of solutions. Firstly, we want to obtain an L^∞ estimate for local solutions. Then we obtain the asymptotic behavior of Green functions. By the Green functions representation of solutions, we obtain error estimates of $\|u_\varepsilon - u_0\|_{L^p(\Omega)}$ for any $1 < p \le \infty$.

In the rest of this paper, we set

$$D_r = D_r(x_0) = B_r(x_0) \cap \Omega,$$
$$\Gamma_r = \Gamma_r(x_0) = B_r(x_0) \cap \partial\Omega$$

for some $x_0 \in \overline{\Omega}$ and $0 < r < r_0$, where $B_r(x_0)$ is the open ball of radius r centered at x_0 and r_0 is a constant.

Lemma 1. Let u_ε satisfy

$$L_\varepsilon(u_\varepsilon)=0 \ \ in\ D_{2r} \quad and \quad u_\varepsilon = g \ \ on\ \Gamma_{2r}.$$

Then for any $1<q<\infty$, we have

$$\| u_\varepsilon \|_{L^\infty(D_r)} \le Cr^{n(1-q)/q} \| u_\varepsilon \|_{L^{q/(q-1)}(D_{2r})} + C\| g \|_{L^\infty(\Gamma_{2r})}$$

where C depends on $n, q, \lambda, \Lambda, \alpha$ and Ω.

Proof. This estimate follows from the maximum principle and De Giorgi-Nash estimate [16].

Lemma 2. Assume that $u_\varepsilon \in H^1(D_{3r})$ and $u_0 \in W^{2,p}(D_{3r})$ for any $n<p\le\infty$. Suppose that

$$L_\varepsilon(u_\varepsilon)=L_0(u_0) \ \ in\ D_{3r} \quad and \quad u_\varepsilon = u_0 \ \ on\ \Gamma_{3r}.$$

Then

$$\| u_\varepsilon - u_0 \|_{L^\infty(D_r)} \le \begin{aligned} &C\varepsilon \| \nabla u_0 \|_{L^\infty(D_{3r})} \\ &+C\varepsilon r^{1-n/p} \| \nabla^2 u_0 \|_{L^p(D_{3r})} \\ &+Cr^{n(1-q)/q} \| u_\varepsilon - u_0 \|_{L^{q/(q-1)}(D_{3r})}, \end{aligned}$$

where C depends only on $n, p, q, \alpha, \Lambda, \lambda$.

Proof. Note that if $L_\varepsilon(u_\varepsilon)=f$, then $L_{\varepsilon/r}(v)=\widetilde{f}$, where $v(x)=r^{-2}u_\varepsilon(rx)$ and $\widetilde{f}=f(rx)$. Thus by scaling we may assume that $r=1$. Consider

$$\omega_\varepsilon = u_\varepsilon - u_0 + \varepsilon\chi_j \frac{\partial u_0}{\partial x_j} \doteq \omega_\varepsilon^{(1)} + \omega_\varepsilon^{(2)} \ \ in\ \ D_2, \quad (15)$$

where

$$L_\varepsilon(\omega_\varepsilon^{(1)}) = L_\varepsilon(\omega_\varepsilon) \ \ in\ D_2 \quad and \quad \omega_\varepsilon^{(1)}=0 \ \ on\ \partial D_2,$$

$$L_\varepsilon(\omega_\varepsilon^{(2)}) = 0 \ \ in\ D_2 \quad and \quad \omega_\varepsilon^{(2)}=\omega_\varepsilon \ \ on\ \partial D_2.$$

Since $\omega_\varepsilon^{(2)} = \varepsilon\chi^j \frac{\partial u_0}{\partial x_j}$ on Γ_2, in view of Lemma 1 and (13), we obtain

$$\begin{aligned} &\| \omega_\varepsilon^{(2)} \|_{L^\infty(D_1)} \le C \| \varepsilon\chi^j \frac{\partial u_0}{\partial x_j} \|_{L^\infty(\Gamma_2)} + \\ &C\| \omega_\varepsilon^{(2)} \|_{L^{q/(q-1)}(D_2)} \le C\varepsilon \| \nabla u_0 \|_{L^\infty(D_3)} + \\ &C\| \omega_\varepsilon^{(1)} \|_{L^\infty(D_2)} + C\| u_\varepsilon - u_0 \|_{L^{q/(q-1)}(D_3)}. \end{aligned}$$

It follows from (15) that

$$\| u_\varepsilon - u_0 \|_{L^\infty(D_1)} \le \begin{aligned} &C\varepsilon \| \nabla u_0 \|_{L^\infty(D_3)} \\ &+C\| \omega_\varepsilon^{(1)} \|_{L^\infty(D_2)} \\ &+C\| u_\varepsilon - u_0 \|_{L^{q/(q-1)}(D_3)}. \end{aligned} \quad (16)$$

To estimate $\| \omega_\varepsilon^{(1)} \|_{L^\infty(D_2)}$, we use the Green function representation

$$\omega_\varepsilon^{(1)}(x) = \int_{D_2} \widetilde{G}_\varepsilon(x,y) L_\varepsilon(\omega_\varepsilon)(y)dy,$$

where $\widetilde{G}_\varepsilon(x,y)$ denotes the Green function for operator L_ε in D_2. It follows from Proposition 2 that

$$\begin{aligned} |\omega_\varepsilon^{(1)}(x)| \ &\le C\varepsilon \int_{D_2} (|\nabla_y \widetilde{G}_\varepsilon(x,y)||\nabla^2 u_0(y)| \\ &\quad +|\nabla_y \widetilde{G}_\varepsilon(x,y)||\nabla u_0(y)|)dy \\ &\le C\varepsilon(\| \nabla^2 u_0 \|_{L^p(D_3)} \\ &\quad + \| \nabla u_0 \|_{L^p(D_3)}) \| \nabla_y \widetilde{G}_\varepsilon(x,y) \|_{L^{p/(p-1)}(D_2)} \\ &\le C\varepsilon(\| \nabla^2 u_0 \|_{L^p(D_3)} + \| \nabla u_0 \|_{L^p(D_3)}) \end{aligned}$$

if $p>n$, where we have used *Hölder's* inequality $\|_y \widetilde{G}_\varepsilon(x,y) \|_{L^{p/(p-1)}(D_2)} \le C$.

Hence from (16) and imbedding theorem, we obtain the desired result. This completes the proof.

Lemma 3. Assume that u_ε and u_0 satisfy the same condition as in Lemma 2. Suppose that

$$L_\varepsilon(u_\varepsilon)=f \ \ in\ \Omega \quad and \quad u_\varepsilon = 0 \ \ on\ \partial\Omega,$$

and $f \in C_0^\infty(D_r)$. Then

$$\| u_\varepsilon - u_0 \|_{L^\infty(\Omega)} \le C\varepsilon r^{1-n/p} \| f \|_{L^p(D_r)}, \quad (17)$$

if $p>n$, where C depends only on $\Lambda, n, \alpha, \lambda$.

Proof. The proof is the same as that of Lemma 2. Consider

$$\omega_\varepsilon = u_\varepsilon - u_0 + \varepsilon\chi_j \frac{\partial u_0}{\partial x_j} \doteq \omega_\varepsilon^{(1)} + \omega_\varepsilon^{(2)} \ \ in\ \ \Omega, \quad (18)$$

where

$$L_\varepsilon(\omega_\varepsilon^{(1)}) = L_\varepsilon(\omega_\varepsilon) \ \ in\, \Omega \quad and \quad \omega_\varepsilon^{(1)} = 0 \ \ on\, \partial\Omega, \tag{19}$$

$$L_\varepsilon(\omega_\varepsilon^{(2)}) = 0 \ \ in\, \Omega \quad and \quad \omega_\varepsilon^{(2)} = \varepsilon\chi_j \frac{\partial u_0}{\partial x_j} \ \ on\, \partial\Omega. \tag{20}$$

In view of maximum principle, we obtain

$$\| \omega_\varepsilon^{(2)} \|_{L^\infty(\Omega)} \le C\varepsilon \| \nabla u_0 \|_{L^\infty(\Omega)} . \tag{21}$$

It follows from (18) and (21) that

$$\| u_\varepsilon - u_0 \|_{L^\infty(\Omega)} \le C\varepsilon \| \nabla u_0 \|_{L^\infty(\Omega)} + C \| \omega_\varepsilon^{(1)} \|_{L^\infty(\Omega)} . \tag{22}$$

From the Green function representation, in view of (14) and (19), we have

$$|\omega_\varepsilon^{(1)}(x)| \le C\varepsilon \int_\Omega |\nabla_y G_\varepsilon(x,y)| (|\nabla^2 u_0(y)| + |\nabla u_0(y)|) dy,$$

where $G_\varepsilon(x,y)$ denotes the Green function for L_ε in Ω. It follows from (8) and $H\ddot{o}lder's$ inequality, we obtain

$$\| \omega_\varepsilon^{(1)}(x) \|_{L^\infty(\Omega)} \le C\varepsilon (\| \nabla^2 u_0 \|_{L^p(\Omega)} + \| \nabla u_0 \|_{L^p(\Omega)})$$

for any $p > n$.

This together with (22) gives

$$\| u_\varepsilon - u_0 \|_{L^\infty(\Omega)} \le C\varepsilon \| \nabla u_0 \|_{L^\infty(\Omega)} + C\varepsilon \| \nabla^2 u_0 \|_{L^p(\Omega)}, \quad for \ \ any \ \ p > n.$$

Then (17) follows from the following inequality and Sobolev imbedding theorem [16],

$$\begin{cases} \| \nabla^2 u_0 \|_{L^p(\Omega)} \le C \| f \|_{L^p(D_r)}, \ for\ any\ 1 < p < \infty, \\ \| \nabla u_0 \|_{L^\infty(\Omega)} \le C r^{1-n/p} \| f \|_{L^p(D_r)}, \ for\ any\ p > n. \end{cases}$$

This completes the proof.

Now we obtain the growth rate of Green function from the following theorem.

Theorem 1. Assume that $G_\varepsilon(x,y)$, $G_0(x,y)$ denote the Green function for operators L_ε, L_o in Ω respectively. Let $f \in L^2(\Omega)$. Suppose that

$$L_\varepsilon(u_\varepsilon) = f \ \ in\, \Omega \quad and \quad u_\varepsilon = 0 \ \ on\, \partial\Omega.$$

Then for any $x, y \in \Omega$,

$$|G_\varepsilon(x,y) - G_0(x,y)| \le \frac{C\varepsilon}{|x-y|^{n-1}}, \tag{23}$$

where C depends on $\Lambda, n, \alpha, \lambda$ and Ω.

Proof. Firstly, we fix $x_0, y_0 \in \Omega$ and let $r = |x_0 - y_0|/4$. Let $f \in C_0^\infty(D_r(y_0))$. From the Green function representation, we have

$$u_\varepsilon(x) = \int_{D_r(y_0)} G_\varepsilon(x,y) f(y) dy$$

and

$$u_0(x) = \int_{D_r(y_0)} G_0(x,y) f(y) dy \tag{24}$$

It then follows by (17) and duality that

$$\| G_\varepsilon(x_0,y) - G_0(x_0,y) \|_{L^{p/(p-1)}(D_r(y_0))} \le C\varepsilon r^{1-n/p},$$

where $p > n$.

Since $L_\varepsilon(G_\varepsilon(x_0,y)) = L_0(G_0(x_0,y)) = 0$ in $D_r(y_0)$ and $G_\varepsilon(x_0,y) = G_0(x_0,y) = 0$ on $\Gamma_r(y_0)$, we invoke Lemma 2 to conclude that

$$\begin{aligned} &|G_\varepsilon(x_0,y) - G_0(x_0,y)| \\ &\le C\varepsilon \| \nabla_y G_0(x_0,y) \|_{L^\infty(D_r(y_0))} \\ &+ C\varepsilon r^{1-n/p} \| \nabla_y^2 G_0(x_0,y) \|_{L^p(D_r(y_0))} \\ &+ C r^{n/p-n} \| G_\varepsilon(x_0,y) - G_0(x_0,y) \|_{L^{p/(p-1)}(D_r(y_0))} \\ &\le C\varepsilon r^{1-n}, \end{aligned}$$

where we have used (8). This completes the proof.

As an application of Theorem 1, we obtain error estimates of $\| u_\varepsilon - u_0 \|_{L^p(\Omega)}$ for any $1 < p \le \infty$.

Theorem 2. Suppose that $u_\varepsilon \in H^1(\Omega)$ and $f \in L^q(\Omega)$. Let u_ε be the solution of Dirichlet problem

$$L_\varepsilon(u_\varepsilon) = f \ \ in\, \Omega \quad and \quad u_\varepsilon = 0 \ \ on\, \partial\Omega.$$

Then these estimates

$$\begin{cases} \| u_\varepsilon - u_0 \|_{L^\infty(\Omega)} \le C\varepsilon[\ln(\tilde{d}/\varepsilon+2)]^{1-1/n} \| f \|_{L^n(\Omega)}, \; where \;\; \tilde{d} = diam(\Omega), (25) \\ \| u_\varepsilon - u_0 \|_{L^\infty(\Omega)} \le C\varepsilon^\beta \| f \|_{L^{n-\delta}(\Omega)}, \; for \; n/(n-\delta)+\beta = 2, if \;\; 0<\delta<n/2, (26) \\ \| u_\varepsilon - u_0 \|_{L^\infty(\Omega)} \le C\varepsilon \| f \|_{L^q(\Omega)}, \qquad\qquad if \;\; q>n, (27) \\ \| u_\varepsilon - u_0 \|_{L^p(\Omega)} \le C\varepsilon \| f \|_{L^q(\Omega)}, \quad for \; 1/p = 1/q - 1/n, \;\; if \;\; 1<q<n \end{cases} \tag{28}$$

hold, where C depends on $\Lambda, n, \alpha, \lambda$ and Ω.

Proof. In view of (8) and (23), we obtain the convergence rates for Green functions

$$|G_\varepsilon(x,y) - G_0(x,y)| \le \frac{C}{|x-y|^{n-2}}$$

and

$$|G_\varepsilon(x,y) - G_0(x,y)| \le \frac{C\varepsilon}{|x-y|^{n-1}}$$

By the Green function representation and *Hölder's* inequality, it gives

$$\begin{aligned} |u_\varepsilon(x) - u_0(x)| \quad & \le C\int_{D_\varepsilon(x)} \frac{|f(y)|dy}{|x-y|^{n-2}} + C\varepsilon\int_{\Omega\setminus D_\varepsilon(x)} \frac{|f(y)|dy}{|x-y|^{n-1}} \\ & \le C\varepsilon \| f \|_{L^n(\Omega)} + C\varepsilon[\ln(\tilde{d}/\varepsilon+2)]^{1-1/n} \| f \|_{L^n(\Omega)} \\ & \le C\varepsilon[\ln(\tilde{d}/\varepsilon+2)]^{1-1/n} \| f \|_{L^n(\Omega)}, \end{aligned}$$

where $\tilde{d} = diam(\Omega)$, which gets the first estimate.

The second estimate is the same as the first estimate,

$$\begin{aligned} |u_\varepsilon(x) - u_0(x)| \quad & \le C\int_{D_\varepsilon(x)} \frac{|f(y)|}{|x-y|^{n-2}}dy + C\varepsilon\int_{\Omega\setminus D_\varepsilon(x)} \frac{|f(y)|}{|x-y|^{n-1}}dy \\ & \le C\varepsilon^{(n-2\delta)/(n-\delta)} \| f \|_{L^{n-\delta}(\Omega)} + C\varepsilon^{(n-2\delta)/(n-\delta)} \| f \|_{L^{n-\delta}(\Omega)} \\ & \le C\varepsilon^\beta \| f \|_{L^{n-\delta}(\Omega)}, \end{aligned}$$

where $n/(n-\delta)+\beta = 2$ and $0<\delta<n/2$.

The third estimate follows from *Hölder's* inequality directly.

The last inequality follows from (23) and Hardy-Littlewood-Sobolev theorem of fractional integration (Chapter 5, Theorem 1 in reference [19]). This completes the proof.

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