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Numerical Computation for Time-fractional Gas Dynamics Equations by Fractional Reduced Differential Transforms Method

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Abstract: The present article is concerned with the implementation of a recent semi-analytical method referred to as fractional reduced differential transform method (FRDTM) for computation of approximate solution of time-fractional gas dynamics equation (TFGDE) arising in shock fronts. In this approach, the fractional derivative is described in the Caputo sense. Four numeric experiments have been carried out to confirm the validity and the efficiency of the method. It is found that the exact or a closed approximate analytical solution of a fractional nonlinear differential equations arising in allied science and engineering can be obtained easily. Moreover, due to its small size of calculation contrary to the other analytical approaches while dealing with a complex and tedious physical problems arising in various branches of natural sciences and engineering, it is very easy to implement.

Key words: Gas Dynamics equation, Caputo time-fractional derivatives, Mittag-Leffler function, reduced differential transform method, Analytic solution

1. Introduction

In the recent years, several physical phenomena arising in engineering as well as in allied sciences can be explained successfully by developing models with the help of the fractional calculus theory. The fractional order equations response ultimately converges to the integer order equations, and so, it has achieved a special attention. The fractional differentiations are very effective and find its wide range of applications for the description of the mathematical modeling of real world problems, e.g. in the earthquake modeling, the traffic flow model with fractional derivatives, diffusion models, measurement of viscoelastic material properties, control, relaxation processes and so on [1-11]. In the beginning of twentieth century, a great deal of effort has been expanded in trying to find the robust and stable analytical approaches for the exact (approximate) solution of fractional differential equations of physical interest. However, several analytical schemes, for instance, Adomian decomposition methods [12-13], differential transform methods [14-16], homotopy perturbation methods [17-20], homotopy analysis methods [21-22], etc. have been developed for the analytical solutions of fractional differential equations but the major disadvantage of these schemes is their complicacy and huge calculation. To overcome from such type of the drawbacks, a semi-analytical approach so called the fractional reduced differential transform method (FRDTM) has been developed by Keskin and Oturanc [23]. It is demonstrated that the proposed FRDTM is the most easily implemented analytical method which provides the exact solution for both the linear and nonlinear differential equations. It is very effective, reliable and efficient, and very powerful analytical approach, see [24-28].

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The gas dynamics equation is the mathematical expressions of conservation laws, e.g. in the conservation of mass, momentum, and energy etc.

This paper is concerned with approximate analytical solution of the time fractional nonlinear gas dynamics equations (1) below, using FRDTM.

$$D_t^{\alpha}u+uD_x^1-u\left(1-u\right)=0,\ x\in\mathbb{R},\ 0<\alpha\le 1, \quad (1)$$

subject to the initial condition $u\left(x,0\right)=f\left(x\right)$, where $D_t^{\alpha}u=\dfrac{\partial^{\alpha}u}{\partial t^{\alpha}}, D_x^1u=\dfrac{\partial u}{\partial x}$, α is a parameter describing the order of the time fractional derivatives. For $\alpha=1$, Eq. (1) reduced into the classical gas dynamics equation. In the recent articles, the approximate analytical solutions of the few different types of gas dynamics equations arising in physics, in terms of infinite series, have been obtained using several analytical and numerical approaches [29-36]. The homogenous and non-homogenous nonlinear gas dynamics equations were solved by implementing the differential transform method (DTM) [37-38] and Fractional homotopy analysis transform method (FHATM) [39]. In this paper our main aim is to reconsider the analytical approximate solution of time-fractional gas dynamics equations of order α $(0<\alpha\le 1)$ in series form converges to the exact solution rapidly, applying directly the FRDTM. It is demonstrated that the obtained FRDTM approximate results are much better approximations and convergence much faster than those given by using DTM and FHATM [38-39].

The rest of the paper is organized as follows: in Section 2, basic preliminaries and notations on fractional calculus theory are reviewed, which is used for further study. Section 3 represents the basic of FRDTM which we use to find the exact solution of the time-fractional gas dynamics equation. In Section 4, exact solutions of four test problems time-fractional homogenous and non-homogeneous fractional gas dynamics equations are presented and compared with the exact solutions available, in the literature. The concluding remarks are presented in Section 5.

2. Fractional Calculus Theory

In this section, the basic definitions and notations are revisited that will be used for further ongoing study. Fractional calculus theory is more than twenty decades years' old theory present in the literature. In fractional integrals and derivatives, several definitions are proposed but the first major contribution to give a proper and most meaningful definition goes to Liouville [2].

Definition 2.1 A real valued function $f\left(\mathrm{x}\right)\in\mathbb{R}$, x>0 is said to be in the space C_{μ}, $\mu\in\mathbb{R}$ if there exists a real number $\mathrm{q}\ \left(>\mu\right)$ such that $f\left(\mathrm{x}\right)=\mathrm{x}^{\mathrm{q}}\mathrm{g}\left(\mathrm{x}\right)$, where $\mathrm{g}\left(\mathrm{x}\right)\in C[0,\infty)$, and is said to be in the space C_{μ}^{m} if $f^{(m)}\in C_{\mu}$, $\mathrm{m}\in\mathbb{N}$.

Definition 2.2 For any given function $f\in\mathbb{R}$, Riemann-Liouville fractional integral operator [3] of order $\alpha\ge 0$ is defined by

$$\begin{cases} J_x^{\alpha}f\left(x\right)=\dfrac{1}{\Gamma\left(\alpha\right)}\displaystyle\int_0^{\mathrm{x}}\left(\mathrm{x-t}\right)^{\alpha-1}f\left(t\right)\mathrm{dt},\ \alpha>0, \\ J_x^0 f\left(x\right)=f\left(x\right). \end{cases} \quad (2)$$

In his work, Caputo and Mainardi [3] proposed a modified fractional differentiation operator D_x^{α} on the theory of visco-elasticity to overcome the discrepancy of Riemann-Liouville derivative [2] while modeling the real world problems using the fractional differential equations. They further, demonstrated that their proposed Caputo fractional derivative allow the utilization of initial and boundary conditions involving integer order derivatives, a straightforward physical interpretations.

Definition 2.3 The fractional derivative of $f\left(x\right)\in\mathbb{R}$, in Caputo sense [3] is defined as

$$\begin{aligned} D_x^{\alpha}f\left(x\right)&=J_x^{m-\alpha}D_x^m f\left(x\right) \\ &=\frac{1}{\Gamma\left(m-\alpha\right)}\int_0^{\mathrm{x}}\left(\mathrm{x-t}\right)^{m-\alpha-1}f^{(m)}\left(t\right)\mathrm{dt}, \end{aligned} \quad (3)$$

For $m-1<\alpha\le m,\ m\in\mathbb{N}, x>0, f\in C_{-1}^m$. The basic properties of the Caputo fractional derivative can be given by the following

Lemma 2.1 If $m-1<\alpha\le m,\ m\in N$ and $f\in C_{\mu}^{m},\ \mu\ge -1,$ then

$$\begin{cases} D_x^{\alpha}J_x^{\alpha}f(x)=f(x),\ \text{x>0}, \\ J_x^{\alpha}D_x^{\alpha}f(x)=f(x)-\sum_{k=0}^{m}f^{(k)}\left(0^{+}\right)\frac{x^{k}}{k!},\ \text{x>0}, \end{cases} \tag{4}$$

In the present work, the Caputo fractional derivative is considered because it allows the traditional initial and boundary conditions to be included in the formulation of the physical problems, for further important characteristics of fractional derivatives, see [1-11].

3. The Basic Idea of FRDTM

In this section, the basic properties of the fractional reduced differential transform method are described. Let $w(x,t)$ be a function of two variables, which can be represented as a product of two single-variable functions, that is $w(x,t)=F(x)G(t)$. Using the properties of the one-dimensional differential transform (DT) method, $w(x,t)$ can be written as

$$w(x,t)=\sum_{i=0}^{\infty}F(i)x^{i}\sum_{j=0}^{\infty}G(j)t^{j}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}W(i,j)x^{i}t^{j}, \tag{5}$$

where $W(i,j)=F(i)G(j)$ is referred to as the spectrum of $w(x,t)$. Assume that R_D and R_D^{-1} denotes operators for fractional reduced differential transform (FRDT) and inverse FRDT, respectively. The basic definition and properties of the FRDTM is described below.

Definition 3.1 If $w(x,t)$ is analytic and continuously differentiable with respect to space variable x and time variable t in the domain of interest, then the t-dimensional spectrum function

$$W_k(x)=\frac{1}{\Gamma(k\alpha+1)}\left[D_t^{\alpha k}\left(w(x,t)\right)\right]_{t=t_0} \tag{6}$$

is referred to as the FRDT function of $w(x,t)$, where α is a parameter, which describes the order of time-fractional derivative. Throughout the paper, $w(x,t)$ (lowercase) is used for the original function and $W_k(x)$ (uppercase) stands for the fractional reduced transformed function.

The inverse FRDT of $W_k(x)$ is defined by

$$w(x,t)=\sum_{k=0}^{\infty}W_k(x)(t-t_0)^{k\alpha}. \tag{7}$$

From Eq. (6) and Eq. (7), it can be found that

$$w(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[D_t^{\alpha k}w(x,t)\right]_{t=t_0}(t-t_0)^{k\alpha}. \tag{8}$$

In particular, for $t_0=0$, Eq. (8) reduces to

$$w(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[D_t^{\alpha k}w(x,t)\right]_{t=0}t^{k\alpha}. \tag{9}$$

This shows that FRDTM is advanced form of power series expansion. Let

$$u(x,t)=R_D^{-1}\left[U_k(x)\right],\ v(x,t)=R_D^{-1}\left[V_k(x)\right]$$

and the convolution $\otimes$ denotes the FRDTM version of the multiplication, then the fundamental operations of the FRDT are illustrated in Table I, where Γ denotes *Gamma function*, defined by $\Gamma(z):=\int_0^{\infty}e^{-t}t^{z-1}dt, z\in\mathbb{C}$, is the continuous extension to the factorial function, where $\delta(k):=\begin{cases}1 & \text{if } k=0\\ 0 & \text{otherwise}\end{cases}$.

Definition 3.2 The Mittag-Leffler function $E_\alpha(z)$ where $\alpha>0$ is defined by the following series representation, is valid in the whole complex plane [40]

$$\mathrm{E}_\alpha(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)} \tag{10}$$

It is an advanced form of $\exp(z)$. In particular, $\exp(z)=\lim_{\alpha\to 1}\mathrm{E}_\alpha(z)$.

Table 1 Basic properties of FRDT method.

$\left(w\left(x,t\right)\right)$	$R_D\left\{w(x,t)\right\}=W_k\left(x\right)$
$u\left(x,t\right)v\left(x,t\right)$	$U_k\left(x\right)\otimes V_k\left(x\right)=\sum_{r=0}^{k}U_r\left(x\right)V_{k-r}\left(x\right)$
$a_1u\left(x,t\right)\pm a_2v\left(x,t\right)$	$a_1U_k\left(x\right)\pm a_2V_k\left(x\right)$
$x^mt^nu\left(x,t\right)$	$\begin{cases}x^mU_{k-n}\left(x\right), & \text{if } k\ge n\\ 0, & else.\end{cases}$
$D_x^lu\left(x,t\right)$	$D_x^lU_k\left(x\right)$
$D_t^{N\alpha}\left(u\left(x,t\right)\right)$	$\dfrac{\Gamma\left(1+(k+N)\alpha\right)}{\Gamma\left(1+k\alpha\right)}U_{k+N}\left(x\right)$
x^m	$x^m\delta(k)$
$e^{\lambda t}$	$\lambda^k/k!$
$\sin\left(wt+c\right)$	$\dfrac{w^k}{k!}\sin\left(\dfrac{\pi k}{2!}+c\right)$
$\cos\left(wt+c\right)$	$\dfrac{w^k}{k!}\cos\left(\dfrac{\pi k}{2!}+c\right)$

4. Numerical Experiments and Discussions

This section describes FRDTM as explained in Section 3 by taking four examples of the TFGDE to validate its efficiency and reliability.

Example 4.1 Consider a homogenous nonlinear TFGDE as

$$D_t^{\alpha}u+uD_x^1u-u\left(1-u\right)=0,\ 0<\alpha\le 1, \tag{11}$$

subject to the initial condition

$$u\left(x,0\right)=e^{-x}. \tag{12}$$

Applying FRDTM on Eq. (11), we obtain the following recurrence relation

$$\frac{\Gamma\left(k\alpha+\alpha+1\right)}{\Gamma\left(k\alpha+1\right)}U_{k+1}(x)=-\sum_{r=0}^{k}U_r(x)D_x^1(U_{k-r}(x))+U_k(x)-\sum_{r=0}^{k}U_r(x)U_{k-r}(x). \tag{13}$$

Using FRDTM to Eq. (12), we have

$$U_0\left(x\right)=e^{-x}. \tag{14}$$

Utilizing Eq. (14) into the recurrence relation (13), $U_k\left(x\right)$ values are given

$$U_1\left(x\right)=\frac{1}{\Gamma\left(1+\alpha\right)}e^{-x},\ U_2\left(x\right)=\frac{1}{\Gamma\left(1+2\alpha\right)}e^{-x},\cdots$$

$$U_k\left(x\right)=\frac{1}{\Gamma\left(1+k\alpha\right)}e^{-x},\ldots \tag{15}$$

Therefore, from Eq. (7) the analytical approximate solution of the nonlinear homogenous equation (11) can be derived as

$$\begin{aligned}u\left(x,t\right)&=\sum_{k=0}^{\infty}U_k\left(x\right)t^{k\alpha}=U_0\left(x\right)+U_1\left(x\right)t^{\alpha}+\ldots\\&=e^{-x}\left(1+\frac{t^{\alpha}}{\Gamma\left(1+\alpha\right)}+\ldots+\frac{t^{k\alpha}}{\Gamma\left(1+k\alpha\right)}+\ldots\right) \qquad (16)\\&=e^{-x}\sum_{k=0}^{\infty}\frac{\left(t^{\alpha}\right)^k}{\Gamma\left(1+k\alpha\right)}=e^{-x}E_{\alpha}\left(t^{\alpha}\right),\end{aligned}$$

Where $E_{\alpha}(t)$ is well known Mittag-Leffler function as defined in Eq. (10). As $\alpha=1$, the exact solution (16) has the closed form $u(x,t)=e^{-x+t}$ which is an exact solution of the given gas dynamics equation (11) for standard value $\alpha=1$ and the same solution is obtained by Kumar et al. [39] using FHATM. It is observe that the solution $u(x,t)=e^{t-x}$ grows exponentially with time t.

Fig.1 shows the behavior of the approximate solution for different fraction Brownian motion $\alpha=0.6, 0.7, 0.8, 0.9$ and for standard motion ($\alpha=1$) in Ex. 4.1. Fig. 2 shows the comparison between the well-known exact solution (line) and solution (star) obtained by FRDTM for different values of x at $t=1$ (and also different values of t at $x=1$). Fig. 2 shows that the FRDTM solution is identical with the exact solution. Fig. 3 depicts the physical solution profile obtained by FRDTM for $\alpha=1$.

Example 4.2 Consider the following homogenous nonlinear time fractional gas dynamics equation as given by:

$$D_t^{\alpha}u+uD_x^{1}u-u\left(1-u\right)\log a=0, \tag{17}$$

where $0<\alpha\le 1, a>0$ together with the initial condition

$$u\left(x,0\right)=a^{-x}. \tag{18}$$

Applying the aforesaid FRDTM to Eq. (17), we obtain the following recurrence relation:

$$\begin{aligned}\frac{\Gamma\left(k\alpha+\alpha+1\right)}{\Gamma\left(k\alpha+1\right)}U_{k+1}(x)&=-\sum_{r=0}^{k}U_r(x)D_x^1U_{k-r}(x)\\&+\left(\log a\right)U_k(x)\text{-}\left(\log a\right)\sum_{r=0}^{k}U_r(x)U_{k-r}(x).\end{aligned} \tag{19}$$

Using the aforesaid FRDTM to the initial condition (18), we obtain

$$U_0\left(x\right)=a^{-x}. \tag{20}$$

Utilizing the transformed initial conditions (20) into Eq. (19), the $U_k\left(\mathrm{x}\right)$ values are given as

$$\begin{aligned}&U_1\left(x\right)=\frac{\log a}{\Gamma\left(1+\alpha\right)}a^{-x},\ U_2\left(x\right)=\frac{\left(\log a\right)^2}{\Gamma\left(1+2\alpha\right)}a^{-x},\\&\ldots,U_k\left(x\right)=\frac{\left(\log a\right)^k}{\Gamma\left(1+k\alpha\right)}a^{-x}\end{aligned} \tag{21}$$

Using the differential inverse reduced transform of $U_k\left(\mathrm{x}\right)$, we have

$$\begin{aligned}u\left(x,t\right)&=\sum_{k=0}^{\infty}U_k\left(x\right)t^{k\alpha}\\&=U_0\left(x\right)+U_1\left(x\right)t^{\alpha}+U_2\left(x\right)t^{2\alpha}+\ldots\\&=a^{-x}\left(1+\frac{\left(t^{\alpha}\log a\right)}{\Gamma\left(1+\alpha\right)}+\frac{\left(t^{\alpha}\log a\right)^2}{\Gamma\left(1+2\alpha\right)}+\ldots+\frac{\left(t^{\alpha}\log a\right)^k}{\Gamma\left(1+k\alpha\right)}\ldots\right)\\&=a^{-x}\sum_{k=0}^{\infty}\frac{\left(t^{\alpha}\log a\right)^k}{\Gamma\left(1+k\alpha\right)}=a^{-x}E_{\alpha}\left(t^{\alpha}\log a\right).\end{aligned} \tag{22}$$

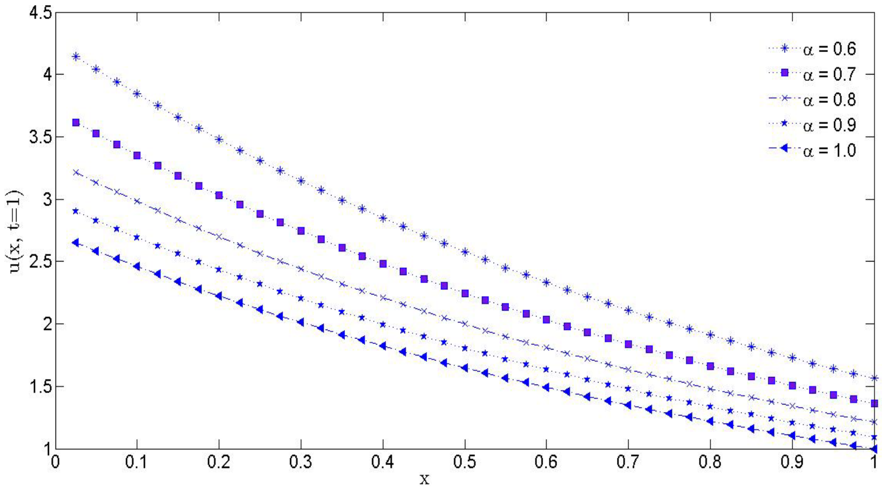


Fig. 1 Plot of FRDTM solutions $u(x,t)$ for different value of α.

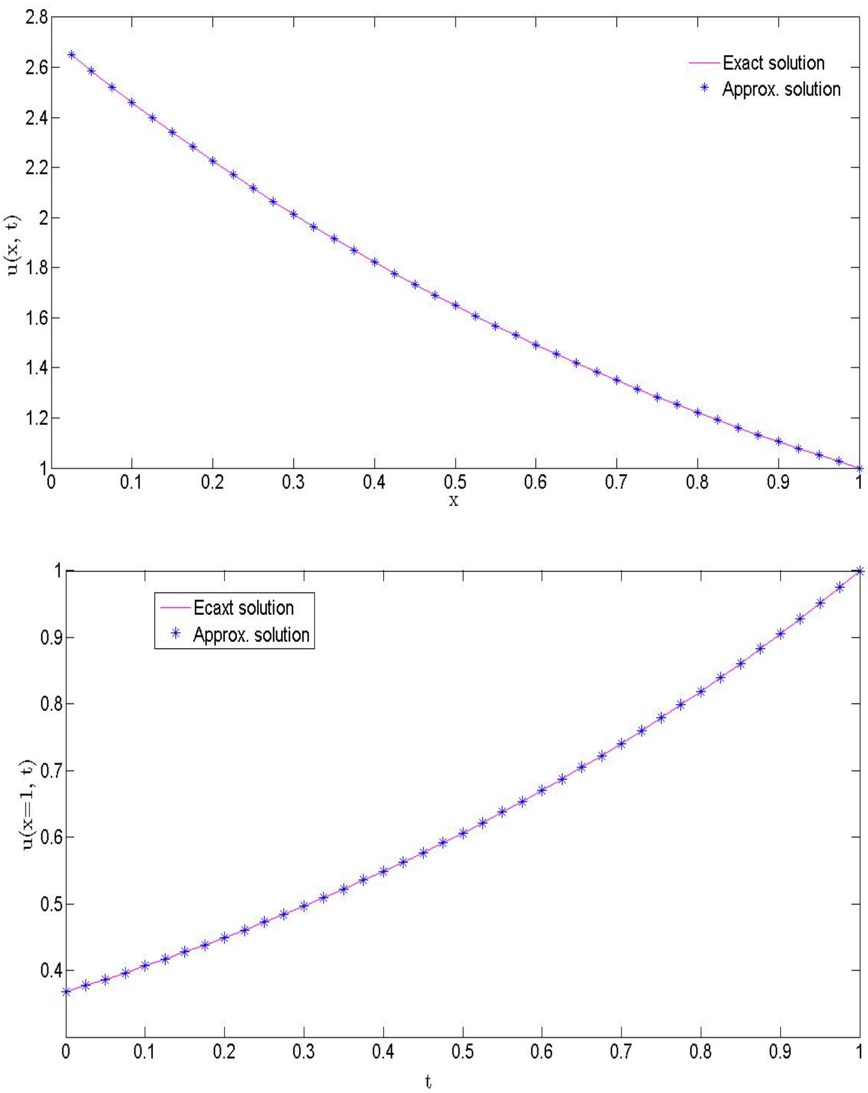


Fig. 2 Comparison between exact solution (line) and FRDTM solution (star)

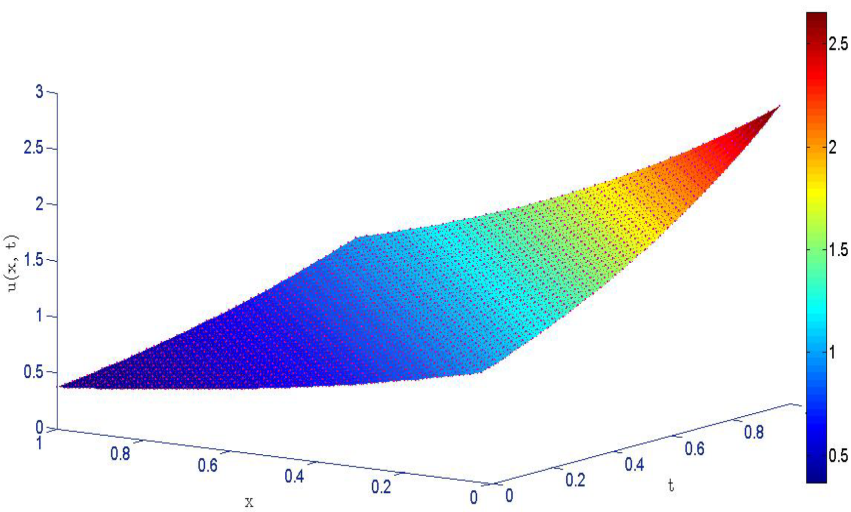


Fig. 3 Surface solution by FRDTM for $\alpha = 1$.

The same solution was obtained by Kumar et al. [39] using FHATM. When $\alpha \to 1$ in Eq.(22), we obtain

$$u(x,t) = a^{-x}\sum_{k=0}^{\infty}\frac{(t\log a)^k}{\Gamma(1+k)} = a^{-x+t}. \quad (23)$$

Eq. (23) is same as the exact solution for the classical gas dynamics equation (17) with $\alpha = 1$ .The above result is complete agreement with Kumar et al. [39]. Fig. 4 shows the comparison of the exact and approximate FRDTM solution for different values of $a = 10, 20, 30, 40$ with standard Brownian motion ($\alpha = 1$) in Ex. 4.2. Fig. 4 confirms that exact and approximate solutions are in good agreement for different values of a. Fig. 5 shows the physical solution profile obtained by FRDTM for $a = 10, \alpha = 1$.

Example 4.3 Consider the following inhomogeneous time fractional gas dynamics equation:

$$D_t^{\alpha}u + uD_x^1u + (1+t)^2u^2 = x^2,\ 0 < \alpha \le 1, \quad (24)$$

subject to the initial condition

$$u(x,0) = x. \quad (25)$$

Applying the FRDTM to Eq. (24), we obtain the following iteration formula

$$\begin{aligned}\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x) &= x^2\delta(k) - \\ &\sum_{r=0}^{k}U_r(x)D_x^1U_{k-r}(x) - R_D\left[(1+t)^2u^2\right].\end{aligned} \quad (26)$$

The non-linear term $R_D\left[(1+t)^2u^2\right]$ treated as

$$\begin{aligned}R_D\left[(1+t)^2u^2\right] &= R_D\left[v + 2tv + t^2v\right] \\ &= V_k + 2V_{k-1} + V_{k-2},\end{aligned} \quad (27)$$

where $v(x,t) = u(x,t)u(x,t)$, and so,

$$V_k(x) = \sum_{r=0}^{k}U_r(x)U_{k-r}(x).$$

Using FRDTM to Eq. (28), we get

$$U_0(x) = x. \quad (28)$$

Using Eq. (28) in Eq. (26) and Eq. (27), the following $U_k(x)$ values are obtained successively

$$\begin{aligned}
U_1(x) &= \frac{-x}{\Gamma(1+\alpha)}, \\
U_2(x) &= \frac{2x}{\Gamma(1+2\alpha)} + 2x^2\left(\frac{1}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right), \\
U_3(x) &= -x\left(\frac{4}{\Gamma(1+3\alpha)} + \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)(\Gamma(1+\alpha))^2}\right) \\
&-x^2\left(\begin{array}{l}\frac{10}{\Gamma(1+3\alpha)} - \frac{6\Gamma(1+\alpha)}{\Gamma(1+3\alpha)} + \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)(\Gamma(1+\alpha))^2} \\ -\frac{4\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\Gamma(1+\alpha)} + \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\end{array}\right) \\
&-4x^3\left(\frac{1}{\Gamma(1+3\alpha)} - \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\right), \ldots
\end{aligned} \quad (29)$$

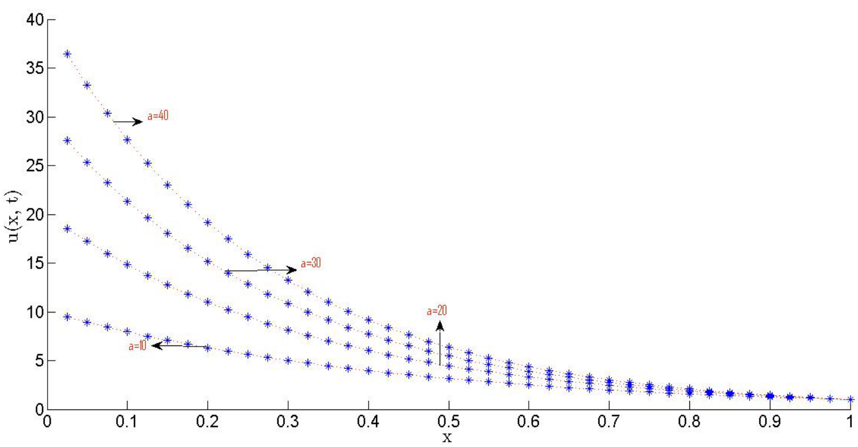


Fig. 4 Comparison between exact solution (dotted line) and FRDTM solution (stars) for $a = 10, 20, 30, 40$ and $t = 1$.

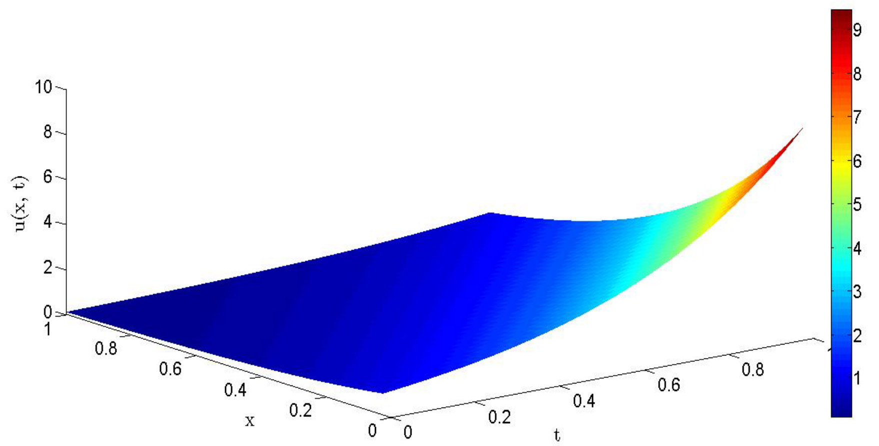


Fig. 5 Physical solution profile using FRDTM for $a=10, \alpha=1$.

Continuing this process, $U_k(x)$ for $k \geq 4$ can be obtained and then by using the differential inverse reduced transform of $U_k(\mathrm{x})$, we get

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} \\ &= U_0(x) + U_1(x) t^{\alpha} + U_2(x) t^{2\alpha} + \ldots \end{aligned} \tag{30}$$

$$\begin{aligned} u(x,t) &= x - \frac{x}{\Gamma(1+\alpha)} t^{\alpha} \\ &+ \left\{ \frac{2x}{\Gamma(1+2\alpha)} + 2x^2 \left(\frac{1}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \right) \right\} t^{2\alpha} \\ &- \left\{ x \left(\frac{4}{\Gamma(1+3\alpha)} + \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)(\Gamma(1+\alpha))^2} \right) \right. \\ &+ x^2 \left(\begin{array}{l} \dfrac{10}{\Gamma(1+3\alpha)} - \dfrac{6\Gamma(1+\alpha)}{\Gamma(1+3\alpha)} + \dfrac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)(\Gamma(1+\alpha))^2} \\ - \dfrac{4\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\Gamma(1+\alpha)} + \dfrac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \end{array} \right) \\ &\left. + 4x^3 \left(\frac{1}{\Gamma(1+3\alpha)} - \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)} \right) \right\} + \ldots \end{aligned} \tag{31}$$

As $\alpha \to 1$ in Eq. (31), we have

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(\mathrm{x}) t^{k\alpha} \\ &= x - xt + xt^2 - xt^3 + \ldots = \frac{x}{1+t}. \end{aligned} \tag{32}$$

which is the same exact solution as obtained by Das and Kumar [38]. Fig. 6(a) shows the comparison of the exact solution $u(x,t)$ and approximate solution $\tilde{u}_4(x,t)$ obtained by FRDTM for the standard Brownian motion and $\alpha=1$ for Example 4.3, depicting the good agreement between the exact and approximate solution. The absolute error is given in Table II. From Table II, it is observed that at a fixed value of α, absolute error increases with time t increase. Fig. 6(b) shows the physical solution profile of the standard Brownian motion obtained by FRDTM.

Example 4.4 Consider the following homogenous non-linear time fractional-order gas dynamics equation:

$$D_t^{\alpha} u + u D_x^1 u - u(1-u) = -e^{-x+t}, \; 0 < \alpha \leq 1, \tag{33}$$

subject to the initial condition

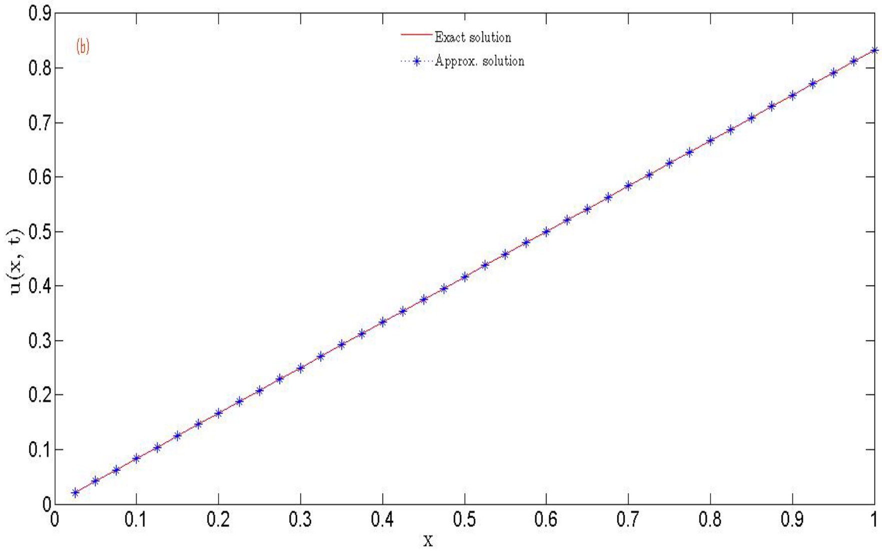
$$u(x,0) = 1 - e^{-x}. \tag{34}$$

Applying the FRDTM to Eq. (33), we obtain

$$\begin{aligned} \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)} U_{k+1}(x) &= -\sum_{r=0}^{k} U_r(x) D_x^1 U_{k\text{-}r}(x) \\ &+ U_k(x) - \sum_{r=0}^{k} U_r(x) U_{k\text{-}r}(x) - \frac{e^{-x}}{k!}. \end{aligned} \tag{35}$$

Using the FRDTM to Eq. (34), we get

$$U_0 = 1 - e^{-x}. \tag{36}$$



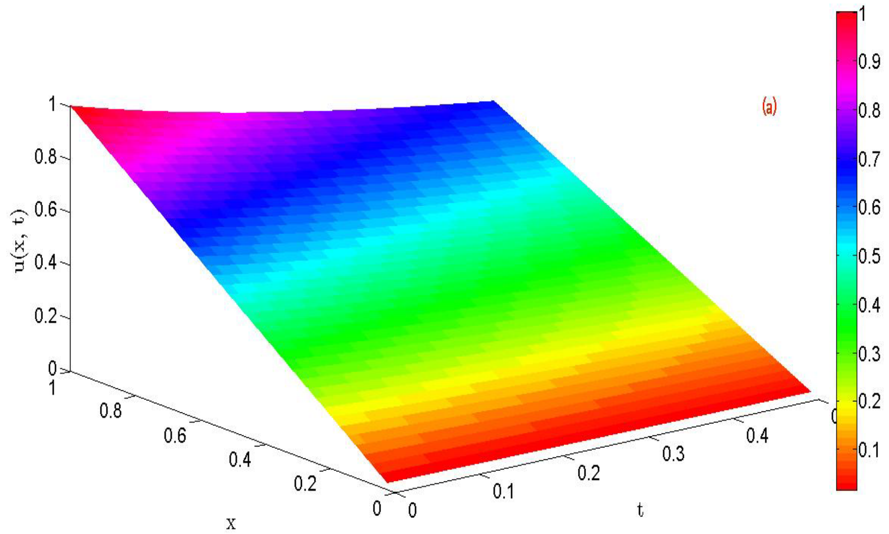


Fig.6 (a) Comparison between exact solution and solution obtained by FRDTM for $\alpha = 1$ and $t = 0.2$; (b) Physical solution profile obtained by FRDTM for $\alpha = 1$ and $t \leq 0.5$.

Table 2 Error between exact and approximate solution for different values of x and t in Example 4.3 with $\alpha = 1$

x	$t = 0.2$	$t = 0.4$	$t = 0.8$	x
0.2	0.0533E-003	0.0015	0.0364	0.100
0.4	0.1067 E-003	0.0029	0.0728	0.200
0.6	0.1600E-003	0.0044	0.1092	0.300
0.8	0.2133E-003	0.0059	0.1456	0.400
1.0	0.2667E- 003	0.0073	0.1820	0.500

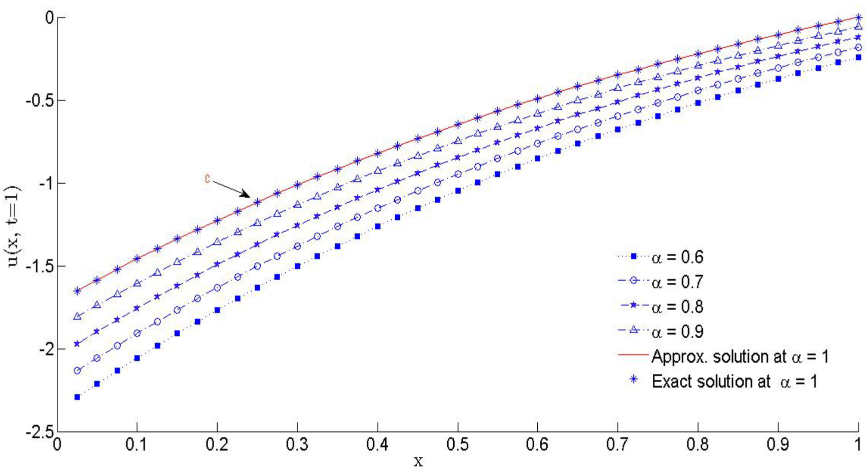


Fig.8 Comparison between exact solution and solution (curves in c) obtained by FRDTM for $\alpha = 1$ and $t = 1$; and FRDTM solution at $t = 1$ for different values of $\alpha \le 0.9$.

Using Eq. (36) in Eq. (35), the following values are obtained successively

$$
\begin{aligned}
U_1(x) &= -\frac{e^{-x}}{0!}\frac{1}{\Gamma(1+\alpha)},\\
U_2(x) &= -\frac{e^{-x}}{1!}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)},\\
U_3(x) &= -\frac{e^{-x}}{2!}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)},\\
&\vdots\\
U_k(x) &= -\frac{e^{-x}}{(k-1)!}\frac{\Gamma(1+(k-1)\alpha)}{\Gamma(1+k\alpha)},\ k \ge 1
\end{aligned}
\tag{37}
$$

Using the differential inverse reduced transform of $U_k(x)$, we get

$$
\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} = U_0(x) + U_1(x)\, t^{\alpha} + \ldots\\
&= 1 - e^{-x} - \frac{e^{-x}}{0!}\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{e^{-x}}{1!}\frac{\Gamma(1+\alpha)t^{2\alpha}}{\Gamma(1+2\alpha)} -\\
&\quad \ldots - \frac{e^{-x}}{(k-1)!}\frac{\Gamma(1+(k-1)\alpha)t^{k\alpha}}{\Gamma(1+k\alpha)} - \ldots\\
&= 1 - e^{-x}\left(1 + \sum_{k=1}^{\infty}\frac{\Gamma(1+(k-1)\alpha)t^{k\alpha}}{(k-1)!\ \Gamma(1+k\alpha)}\right)\\
&= 1 - e^{-x} E_{\alpha}\left(K t^{\alpha}\right),
\end{aligned}
\tag{38}
$$

where

$$
K^n = \begin{cases} 1, & n = 0,\\ \dfrac{\Gamma((n-1)\alpha+1)}{(n-1)!} & n > 0. \end{cases}
$$

When $\alpha \to 1$ in Eq. (38), we obtain

$$
u(x,t) = 1 - e^{-x}\sum_{k=0}^{\infty}\frac{t^k}{k!} = 1 - e^{-x+t},
$$

which is an exact solution of the given classical gas dynamics equation (33) for standard value $\alpha = 1$ and this result is complete agreement with Das and Kumar [38].

Curve C in Fig. 8 depicts the comparison of the exact solution (dotted line) and approximate solution (stars) obtained by FRDTM for the standard Brownian motion $\alpha = 1$ of Ex. 4.4 and the behavior of the approximate FRDTM solutions for different fraction Brownian motion $\alpha = 0.6, 0.7, 0.8, .09$ is also depicted in Fig. 8.

5. Conclusions

In this paper, FRDTM has been implemented for the Caputo time-fractional order gas dynamics equation arising in shock fronts. The proposed approximated solutions of gas dynamics equations with an appropriate initial condition are obtained in terms of a power series, without using any kind of discretization, perturbation, or restrictive conditions, etc.

Four examples are illustrated to study the effectiveness and accurateness of FRDTM. It is found that FRDTM solutions are in excellent agreement with those obtained using DTM and FHATM. However, computations show that the FRDTM is very easy to implement and needs small size of computation contrary to DTM and FHATM. This shows that FRDTM is very effective and efficient powerful mathematical tool, which is easily applicable in finding out the approximate analytic solutions of a wide range of real world problems arising in engineering and allied sciences.

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