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Multivariate *q*-Bernstein-Schurer-Kantorovich Operators

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Abstract: The purpose of this paper is to construct a multivariate generalization of a new kind of Kantorovich type q-Bernstein-Schurer operators. First, we establish the moments of the operators and then prove the rate of convergence by using the modulus of continuity. Finally, we obtain the degree of approximation by means of Lipschitz type class.

Keywords: q-Bernstein-Schurer-Kantorovich operators, rate of convergence, modulus of continuity, Lipschitz type class, multivariate operators.

1. Introduction

In recent years, one of the most interesting area of research in approximation theory is the application of q-calculus. Lupas and Phillips [16] was the first person who initiated the q-type generalization of linear positive operators. He introduced the q-analogue of well known Bernstein polynomials and obtained the rate of convergence and Voronovskaja type theorem for these operators. After that q-parametric operators is an active area of research in the field of approximation theory.

In [13], Muraru constructed the q-Bernstein-Schurer operators defined by

$$B_{n,p}(f;x)=\sum_{k=0}^{n+p} b_{n+p,k}(q;x) f\left(\frac{[k]_q}{[n]_q}\right),\ x\in[0,1] \tag{1}$$

where $b_{n+p,k}(q;x)=\binom{n+p}{k}_q x^k(1-x)_q^{n+p-k}$ and

$$(1-x)_q^m=\begin{cases} 1, & m=0,\\ (1-x)(1-qx)...(1-q^{m-1}x), & m\in N\end{cases}$$

for $x\in[0,1]$ and $0<q<1$. For $p=0$, the sequence of operators (1) reduces to the q-Bernstein polynomials (see [16]).

Dalmanoglu [4] studied some approximation properties of Kantorovich type generalization of q-Bernstein operators. In [17], Radu has obtained the statistical convergence of q-Bernstein-Kantorovich polynomials. Also, the Kantorovich type generalizations of the linear positive operators based on q-integers were studied by some authors (see [8], [10], [15], etc.). Recently, Agrawal, Finta and Sathish Kumar [1] introduced a new Kantorovich type generalization of the q-Bernstein-Schurer operators defined in [13] as follows

$$K_{n,p}(f;q,x)$$
$$=[n+1]_q\sum_{k=0}^{n+p} b_{n+p,k}^q(x)q^{-k}\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} f(t)d_q^R t,$$
$$x\in[0,1].$$

In 2014, Same authors introduced Bivariate q-Bernstein-Schurer operators defined as

$$K_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y)$$

$$=[n_1+1]_{q_{n_1}}[n_2+1]_{q_{n_2}}\sum_{k_1=0}^{n_1+p}\sum_{k_2=0}^{n_2+p} b_{n_1+p,n_2+p,k_1,k_2}^{q_{n_1},q_{n_2}}(x,y)q_{n_1}^{-k_1}q_{n_2}^{-k_2}$$

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$$\times\int_{[k_2]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}^{[k_2+1]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}\int_{[k_1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}^{[k_1+1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}} f(t,s)d_{q_{n_1}}^R(t)d_{q_{n_2}}^R(s), \quad (2)$$

where $(x,y)\in[0,1]\times[0,1]$ and

$$b_{n_1+p,n_2+p,k_1,k_2}^{q_{n_1},q_{n_2}}(x,y)=\binom{n_1+p}{k_1}_{q_{n_1}}\binom{n_2+p}{k_2}_{q_{n_2}} x^{k_1}y^{k_2}(1-x)_{q_{n_1}}^{n_1+p-k_1}(1-y)_{q_{n_2}}^{n_2+p-k_2}.$$

Stancu [19], was the first person who introduced the linear positive operators in two and several variables. Later on, Bărbosu [3] considered the bivariate q-Bernstein polynomials and studied some Korovkin type approximation results. In [6], Erkuş and Duman proved the Korovkin-type approximation theorem for the bivariate linear positive operators to the functions in space H_{ω_2}. Doğru and Gupta [5] studied the rate of convergence of the bivariate generalization of q-MKZ operators. In 2009, Ersan and Doğru [7] obtained the statistical approximation properties of q-Bleimann, Butzer and Hahn operators. Recently, Örkcü [14] constructed the bivariate generalization of the q-Szász-Mirakjan-Kantorovich operators by using the q-integral and obtained a weighted A-statistical convergence and the rate of pointwise approximation in terms of the modulus of continuity.

Now, we recall some basic definitions and notations of q-calculus [9]. For any fixed real number q satisfying the conditions $0<q<1$, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ for $k\in\{0,1,2...\}$ are defined as

$$[k]_q=\begin{cases}(1-q^k)(1-q), & \text{if } q\neq 1\\ k, & \text{if } q=1\end{cases}$$

and

$$[k]_q!=\begin{cases}[k]_q[k-1]_q\dots[1]_q, & \text{if } k\geq 1\\ 1, & \text{if } k=0\end{cases}$$

respectively. For any integers n,k satisfying $0\leq k\leq n$, the q-binomial coefficient is given by

$$\binom{n}{k}_q=\frac{[n]_q!}{[n-k]_q![k]_q!}.$$

Let $0\leq a<b.$ and $0<q<1$. Following [20], we consider the Riemann type q-integral is defined as

$$\int_a^b f(t)d_q^R t=(1-q)(b-a)\sum_{i=0}^{\infty} f(a+(b-a)q^i)q^i.$$

This Riemann type q-integral is appropriate to derive the q-analogues of some well-known integral inequalities. Then Riemann type q-integral for a multivariate function is given by

$$\int_{a_m}^{b_m}\int_{a_{m-1}}^{b_{m-1}}\dots\int_{a_1}^{b_1} f(t_1,t_2,...,t_m)d_{q_1}^R t_1 d_{q_2}^R t_2\dots d_{q_m}^R t_m$$
$$=(1-q_1)(1-q_2)...(1-q_m)(b_1-a_1)(b_2-a_2)...(b_m-a_m)\sum_{i_m=0}^{\infty}\dots\sum_{i_2=0}^{\infty}\sum_{i_1=0}^{\infty}$$
$$f(a_1+(b_1-a_1)q_1^{i_1},a_2+(b_2-a_2)q_2^{i_2},...,a_m+(b_m-a_m)q_m^{i_m})q_1^{i_1}q_2^{i_2}...q_m^{i_m}. \quad (3)$$

where $0\leq a_i<b_i$ and $0<q_i<1, i=1,2,...,m$.

2. Construction of Operators

In this study, we construct a multivariate case of new Kantorovich type generalization of the q-Bernstein-Schurer operators defined in [13]. In what follows, let $I=[0,1+p],$ where $p\in\mathbf{N}^0=\mathbf{N}\cup\{0\}$ and $J=[0,1]$. For $I^m=I\times I...\times I,$ let $C(I^m)$ denote the space of all real valued continuous functions on I^m endowed with the norm

$$\|f\|_I=\sup_{(x_1,x_2,...,x_m)\in I^m}|f(x_1,x_2,...,x_m)|.$$

For $f\in C(I^m)$ and $0<q_{n_i}<1, i=1,2,...,m$, then we define the multivariate case of the Kantorovich type q-Bernstein-Schurer operators as follows:

$$K_{n_1,n_2,...,n_m,p}(f;q_{n_1},q_{n_2},...,q_{n_m},x_1,x_2,...,x_m)$$
$$=[n_1+1]_{q_{n_1}}[n_2+1]_{q_{n_2}}...[n_m+1]_{q_{n_m}}\sum_{k_1=0}^{n_1+p}\sum_{k_2=0}^{n_2+p}...\sum_{k_m=0}^{n_m+p}b_{n_1+p,n_2+p,...,n_m+p,k_1,k_2,...,k_m}^{q_{n_1},q_{n_2},...,q_{n_m}}(x_1,x_2,...,x_m)$$
$$\times q_{n_1}^{-k_1}q_{n_2}^{-k_2}...q_{n_m}^{-k_m}\int_{[k_m]_{q_{n_m}}/[n_m+1]_{q_{n_m}}}^{[k_m+1]_{q_{n_m}}/[n_m+1]_{q_{n_m}}}\int_{[k_{m-1}]_{q_{n_{m-1}}}/[n_{m-1}+1]_{q_{n_{m-1}}}}^{[k_{m-1}+1]_{q_{n_{m-1}}}/[n_{m-1}+1]_{q_{n_{m-1}}}}...$$
$$\int_{[k_1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}^{[k_1+1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}f(t_1,t_2,...,t_m)d_{q_{n_1}}^R t_1 d_{q_{n_2}}^R t_2...d_{q_{n_m}}^R t_m, \tag{4}$$

where

$$b_{n_1+p,n_2+p,...,n_m+p,k_1,k_2,...,k_m}^{q_{n_1},q_{n_2},...,q_{n_m}}(x_1,x_2,...,x_m)$$
$$=\binom{n_1+p}{k_1}_{q_{n_1}}\binom{n_2+p}{k_2}_{q_{n_2}}...\binom{n_m+p}{k_m}_{q_{n_m}}x_1^{k_1}x_2^{k_2}...x_m^{k_m}$$
$$(1-x_1)_{q_{n_1}}^{n_1+p-k_1}(1-x_2)_{q_{n_2}}^{n_2+p-k_2}...(1-x_m)_{q_{n_m}}^{n_m+p-k_m}.$$

The purpose of this study is to obtain some approximation properties of a Kantorovich type multivariate generalization of *q*-Bernstein-Schurer operators. We obtain the rate of convergecne by using the modulus of continuity and degree of approximation in terms of Lipschitz type space.

3. Basic Results

In what follows, for $i=1,2,...,m,$ let (q_{n_i}) be a sequence in $(0,1)$ satisfying $q_{n_i}\to 1$ and $q_{n_i}^{n_i}\to 0$ as $n\to\infty$.

Lemma 1. Let $e_{i_1i_2...i_m}(x_1,x_2,...,x_m)=x_1^{i_1}x_2^{i_2}...x_m^{i_m}$, $(x_1,x_2,...,x_m)\in I^m$, $(i_1,i_2,...,i_m)\in N^0\times N^0\times...\times N^0$ with $i_1+i_2+...+i_m\le m$ be the m-dimensional test functions. Then the following equalities hold for the operators given by (4).

$$K_{n_1,n_2,...,n_m,p}(e_{00...0};q_{n_1},q_{n_2},...,q_{n_m},x_1,x_2,...,x_m)=1;$$
$$K_{n_1,n_2,..,n_i,..,n_m,p}(e_{00..1.0};q_{n_1},q_{n_2},..,q_{n_i},..,q_{n_m},x_1,x_2,..,x_i,..,x_m)=\frac{[n_i+p]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}}\frac{2q_{n_i}}{[2]_{q_{n_i}}}x_i$$
$$+\frac{1}{[2]_{q_{n_i}}[n_i+1]_{q_{n_i}}};$$
$$K_{n_1,n_2,..,n_i,..,n_m,p}(e_{00..2.0};q_{n_1},q_{n_2},..,q_{n_i},..,q_{n_m},x_1,x_2,..,x_i,..,x_m)=\frac{1}{[n_i+1]_{q_{n_i}}^2[3]_{q_{n_i}}}$$
$$+\frac{q_{n_i}(3+5q_{n_i}+4q_{n_i}^2)}{[2]_{q_{n_i}}[3]_{q_{n_i}}}\frac{[n_i+p]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}^2}x_i+\frac{q_{n_i}^2(1+q_{n_i}+4q_{n_i}^2)}{[2]_{q_{n_i}}[3]_{q_{n_i}}}\frac{[n_i+p]_{q_{n_i}}[n_i+p-1]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}^2}x_i^2.$$

Proof. Using (3), we obtain,

$$\int_{[k_m]_{q_{n_m}}/[n_m+1]_{q_{n_m}}}^{[k_m+1]_{q_{n_m}}/[n_m+1]_{q_{n_m}}}...\int_{[k_2]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}^{[k_2+1]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}\int_{[k_1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}^{[k_1+1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}1d_{q_{n_1}}^R(t_1)d_{q_{n_2}}^R(t_2)...d_{q_{n_m}}^R(t_m)=$$
$$\frac{q_{n_1}^{k_1}q_{n_2}^{k_2}...q_{n_m}^{k_m}}{[n_1+1]_{q_{n_1}}[n_2+1]_{q_{n_2}}...[n_m+1]_{q_{n_m}}}.$$

Now, using the above equation in (4)and taking into account that $K_{n,p}(1;q,x)=1$ (see[[1], Lemma4]), we get

$$K_{n_1,n_2,...,n_m,p}(e_{00...0};q_{n_1},q_{n_2},...,q_{n_m},x_1,x_2,...,x_m)$$
$$=\sum_{k_1=0}^{n_1+p}\sum_{k_2=0}^{n_2+p}...\sum_{k_m=0}^{n_m+p}b_{n_1+p,n_2+p,...,n_m+p,k_1,k_2,...,k_m}^{q_{n_1},q_{n_2},...,q_{n_m}}(x_1,x_2,...,x_m)$$
$$=1.$$

Now, applying (3) we have

$$\int_{[k_m]_{q_{n_m}}/[n_m+1]_{q_{n_m}}}^{[k_m+1]_{q_{n_m}}/[n_m+1]_{q_{n_m}}}...\int_{[k_2]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}^{[k_2+1]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}\int_{[k_1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}^{[k_1+1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}t_i d_{q_{n_1}}^R(t_1)d_{q_{n_2}}^R(t_2)...d_{q_{n_m}}^R(t_m)$$
$$=\frac{q_{n_1}^{k_1}\cdots q_{n_{i-1}}^{k_{i-1}}}{[n_1+1]_{q_{n_1}}...[n_{i-1}+1]_{q_{n_{i-1}}}}\frac{q_{n_i}^{k_i}}{[n_i+1]_{q_{n_i}}}\left(\frac{[k_i]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}}+\frac{q_{n_i}^{k_i}}{[n_i+1]_{q_{n_i}}[2]_{q_{n_i}}}\right)\frac{q_{n_{i+1}}^{k_{i+1}}...q_{n_m}^{k_m}}{[n_{i+1}+1]_{q_{n_{i+1}}}...[n_m+1]_{q_{n_m}}}.$$

In view of (4) and

$$\sum_{k_i=0}^{n_i+p}b_{n_i+p,k_i}^{q_{n_i}}(x_i)q_{n_i}^{k_i}=1-(1-q_{n_i})[n_i+p]_{q_{n_i}}x_i, \tag{5}$$

we get

$$K_{n_1,n_2,..,n_i,..,n_m,p}(e_{00.1.0};q_{n_1},q_{n_2},..,q_{n_i},..,q_{n_m},x_1,x_2,..,x_i,..,x_m)$$
$$=\sum_{k_1=0}^{n_1+p}\sum_{k_2=0}^{n_2+p}...\sum_{k_m=0}^{n_m+p}\begin{pmatrix}n_1+p\\k_1\end{pmatrix}_{q_{n_1}}\begin{pmatrix}n_2+p\\k_2\end{pmatrix}_{q_{n_2}}...\begin{pmatrix}n_m+p\\k_m\end{pmatrix}_{q_{n_m}}x_1^{k_1}x_2^{k_2}...x_m^{k_m}(1-x_1)_{q_{n_1}}^{n_1+p-k_1}$$
$$\times(1-x_2)_{q_{n_2}}^{n_2+p-k_2}...(1-x_m)_{q_{n_m}}^{n_m+p-k_m}\left(\frac{[k_i]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}}+\frac{q_{n_i}^{k_i}}{[n_i+1]_{q_{n_i}}[2]_{q_{n_i}}}\right)$$
$$=\sum_{k_i=0}^{n_i+p}\begin{pmatrix}n_i+p\\k_i\end{pmatrix}_{q_{n_i}}x_i^{k_i}(1-x_i)_{q_{n_i}}^{n_i+p-k_i}\left(\frac{[k_i]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}}+\frac{q_{n_i}^{k_i}}{[n_i+1]_{q_{n_i}}[2]_{q_{n_i}}}\right)$$
$$=\frac{1}{[n_i+1]_{q_{n_i}}}\sum_{k_i=0}^{n_i+p}\begin{pmatrix}n_i+p\\k_i\end{pmatrix}_{q_{n_i}}x_i^{k_i}(1-x_i)_{q_{n_i}}^{n_i+p-k_i}[k_i]_{q_{n_i}}$$
$$+\frac{1}{[n_i+1]_{q_{n_i}}[2]_{q_{n_i}}}\sum_{k_i=0}^{n_i+p}\begin{pmatrix}n_i+p\\k_i\end{pmatrix}_{q_{n_i}}x_i^{k_i}(1-x_i)_{q_{n_i}}^{n_i+p-k_i}q_{n_i}^{k_i}$$
$$=\frac{[n_i+p]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}}\sum_{k_i=0}^{n_i+p-1}\begin{pmatrix}n_i+p-1\\k_i\end{pmatrix}_{q_{n_i}}x_i^{k_i+1}(1-x_i)_{q_{n_i}}^{n_i+p-k_i-1}$$
$$+\frac{1}{[n_i+1]_{q_{n_i}}[2]_{q_{n_i}}}\sum_{k_i=0}^{n_i+p}\begin{pmatrix}n_i+p\\k_i\end{pmatrix}_{q_{n_i}}x_i^{k_i}(1-x_i)_{q_{n_i}}^{n_i+p-k_i}q_{n_i}^{k_i}$$
$$=\frac{[n_i+p]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}}x_i+\frac{1}{[2]_{q_{n_i}}[n_i+1]_{q_{n_i}}}(1-(1-q_{n_i})[n_i+p]_{q_{n_i}}x_i)$$
$$=\frac{[n_i+p]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}}\frac{2q_{n_i}}{[2]_{q_{n_i}}}x_i+\frac{1}{[2]_{q_{n_i}}[n_i+1]_{q_{n_i}}}.$$

Again using (3), we find

$$\int_{[k_m]_{q_{n_m}}/[n_m+1]_{q_{n_m}}}^{[k_m+1]_{q_{n_m}}/[n_m+1]_{q_{n_m}}} \ldots \int_{[k_2]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}^{[k_2+1]_{q_{n_2}}/[n_2+1]_{q_{n_2}}} \int_{[k_1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}^{[k_1+1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}} t_i^{\,2} d_{q_{n_1}}^R(t_1) d_{q_{n_2}}^R(t_2) \ldots d_{q_{n_m}}^R(t_m)$$
$$= \frac{q_{n_1}^{k_1} \ldots q_{n_i}^{k_i}}{[n_1+1]_{q_{n_1}} \ldots [n_i+1]_{q_{n_i}}} \left(\frac{[k_i]_{q_{n_i}}^2}{[n_i+1]_{q_{n_i}}^2} + \frac{q_{n_i}^{2k_i}}{[n_i+1]_{q_{n_i}}[3]_{q_{n_i}}} + \frac{2q_{n_i}^{k_i}[k_i]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}^2} \right)$$
$$\times \frac{q_{n_{i+1}}^{k_{i+1}} \ldots q_{n_m}^{k_m}}{[n_{i+1}+1]_{q_{n_{i+1}}} \ldots [n_m+1]_{q_{n_m}}}.$$

From (2),(5) and

$$\sum_{k_i=0}^{n_i+p} b_{n_i+p,k_i}^{q_{n_i}}(x_i) q_{n_i}^{2k_i} = 1-(1-q_{n_i}^2)[n_i+p]_{q_{n_i}} x_i + q_{n_i}(1-q_{n_i})^2 [n_i+p]_{q_{n_i}} [n_i+p-1]_{q_{n_i}} x_i^{\,2},$$

we get

$$K_{n_1,n_2,.,n_i,.,n_m,p}(e_{00.2.0}; q_{n_1}, q_{n_2}, ., q_{n_i}, ., q_{n_m}, x_1, x_2, ., x_i, ., x_m)$$
$$= \sum_{k_1=0}^{n_1+p} \sum_{k_2=0}^{n_2+p} \ldots \sum_{k_2=0}^{n_2+p} \begin{pmatrix} n_1+p \\ k_1 \end{pmatrix}_{q_{n_1}} \begin{pmatrix} n_2+p \\ k_2 \end{pmatrix}_{q_{n_2}} \ldots \begin{pmatrix} n_m+p \\ k_m \end{pmatrix}_{q_{n_m}} x_1^{\,k_1} x_2^{\,k_2} \ldots x_m^{\,k_m} (1-x_1)_{q_{n_1}}^{n_1+p-k_1}$$
$$(1-x_2)_{q_{n_2}}^{n_2+p-k_2} \ldots (1-x_m)_{q_{n_m}}^{n_m+p-k_m} \left(\frac{[k_i]_{q_{n_i}}^2}{[n_i+1]_{q_{n_i}}^2} + \frac{q_{n_i}^{2k_i}}{[n_i+1]_{q_{n_i}}^2 [3]_{q_{n_i}}} + \frac{2q_{n_i}^{k_i}[k_i]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}^2 [2]_{q_{n_i}}} \right)$$
$$= \sum_{k_i=0}^{n_i+p} \begin{pmatrix} n_i+p \\ k_i \end{pmatrix}_{q_{n_i}} x_i^{\,k_i} (1-x_i)_{q_{n_i}}^{n_i+p-k_i}$$
$$\times \left(\frac{[k_i]_{q_{n_i}}^2}{[n_i+1]_{q_{n_i}}^2} + \frac{q_{n_i}^{2k_i}}{[n_i+1]_{q_{n_i}}^2 [3]_{q_{n_i}}} + \frac{2q_{n_i}^{k_i}[k_i]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}^2 [2]_{q_{n_i}}} \right)$$
$$= \sum_{k_i=0}^{n_i+p} \begin{pmatrix} n_i+p \\ k_i \end{pmatrix}_{q_{n_i}} x_i^{\,k_i} (1-x_i)_{q_{n_i}}^{n_i+p-k_i} \frac{[k_i]_{q_{n_i}}^2}{[n_i+1]_{q_{n_i}}^2} + \sum_{k_i=0}^{n_i+p} \begin{pmatrix} n_i+p \\ k_i \end{pmatrix}_{q_{n_1}} x_i^{\,k_i} (1-x_i)_{q_{n_i}}^{n_i+p-k_i}$$
$$\frac{2q_{n_i}^{k_i}[k_i]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}^2 [2]_{q_{n_i}}} + \sum_{k_i=0}^{n_i+p} \begin{pmatrix} n_i+p \\ k_i \end{pmatrix}_{q_{n_i}} x_i^{\,k_i} (1-x_i)_{q_{n_i}}^{n_i+p-k_i} \frac{q_{n_i}^{2k_i}}{[n_i+1]_{q_{n_i}}^2 [3]_{q_{n_i}}}$$
$$= \frac{[n_i+p]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}} \left(\frac{1}{[n_i+1]_{q_{n_i}}} + \frac{q_{n_i} x_i [n_i+p-1]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}} \right)$$
$$+ \frac{2[n_i+p]_{q_{n_i}} [n_i+1]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}^2 [2]_{q_{n_i}}} x_i (1-(1-q_{n_i})[n_i+p-1]_{q_{n_i}} x_i)$$
$$+ \frac{1}{[n_i+1]_{q_{n_i}}^2 [3]_{q_{n_i}}} (1-(1-q_{n_i}^2)[n_i+p]_{q_{n_i}} x_i + q_{n_i}(1-q_{n_1})^2 [n_i+p]_{q_{n_i}} [n_i+p-1]_{q_{n_i}} x_i^{\,2})$$
$$= \frac{1}{[n_i+1]_{q_{n_i}}^2 [3]_{q_{n_i}}} + \frac{q_{n_i}(3+5q_{n_i}+4q_{n_i}^{\,2})}{[2]_{q_{n_i}}[3]_{q_{n_i}}} \frac{[n_i+p]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}^2} x_i$$
$$+ \frac{q_{n_i}^2(1+q_{n_i}+4q_{n_i}^{\,2})}{[2]_{q_{n_i}}[3]_{q_{n_i}}} \frac{[n_i+p]_{q_{n_i}}[n_i+p-1]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}^2} x_i^{\,2}.$$

Hence, the proof is completed.

Remark 1. From Lemma 1, we get

$$K_{n_1,n_2,\dots,n_m,p}((t_i-x_i);q_{n_1},q_{n_2},\dots,q_{n_m},x_1,x_2,\dots,x_m)=\frac{1}{[2]_{q_{n_i}}[n_i+1]_{q_{n_i}}}$$
$$+x_i\left(\frac{[n_i+p]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}}\frac{2q_{n_i}}{[2]_{q_{n_i}}}-1\right);$$

$$K_{n_1,n_2,\dots,n_m,p}((t_i-x_i)^2;q_{n_1},q_{n_2},\dots,q_{n_m},x_1,x_2,\dots,x_m)=\left(\frac{q_{n_i}^2(1+q_{n_i}+4q_{n_i}{}^2)}{[2]_{q_{n_i}}[3]_{q_{n_i}}}\right.$$
$$\left.\times\frac{[n_i+p]_{q_{n_i}}[n_i+p-1]_{q_{n_i}}}{[n_i+1]^2_{q_{n_i}}}-\frac{[n_i+p]_{q_{n_i}}}{[n_i+1]_{q_{n_i}}}\frac{4q_{n_i}}{[2]_{q_{n_i}}}+1\right)x_i{}^2+\left(\frac{q_{n_i}(3+5q_{n_i}+4q_{n_i}{}^2)}{[2]_{q_{n_i}}[3]_{q_{n_i}}}\frac{[n_i+p]_{q_{n_i}}}{[n_i+1]^2_{q_{n_i}}}-\frac{2}{[n_i+1]_{q_{n_i}}[2]_{q_{n_i}}}\right)x_i+\frac{1}{[n_i+1]^2_{q_{n_i}}[3]_{q_{n_i}}};$$

4. Main Results

Theorem 1. For any $f\in C(I^m)$, we have

$$\lim_{n_1,n_2,\dots,n_m\to\infty}\|K_{n_1,n_2,\dots,n_m,p}(f)-f\|_I=0.$$

Proof. To proof our theorem, we need some notions concerning j th projection and Korovkin subset according to [2]. For locally compact Hausdorff spaces $X_1,X_2,\dots,X_m$ we denote pr_j : $X_1\times X_2\times\dots\times X_m\rightarrow X_j$ ($j=1,2,\dots,m$) the j th projection which is defined by $pr_j=x_j$ for every $x=(x_1,x_2,\dots,x_m)\in X_1\times X_2\times\dots\times X_m$. Let X and Y be two locally compact Hausdorff spaces and $T:C(X)\to C(Y)$ be a positive linear operator. A subset H of C(X) is called Korovkin subset for T with respect to positive linear operator if it Satisfies the following property: if $L_{i\, i\in I}^{\le}$ is an arbitrary net of of positive linear operators from C(X) into C(y) such that $\sup_{i\in I}\|L_{i\|<\infty}$ and if $\lim_{i\in I\le}L_i(h)=T(h)$ for all $h\in H$, then $\lim_{i\in I\le}L_i(f)=T(f)$ for every $f\in C(X)$. □

For $f\in C(I^m)$, the modulus of continuity for multivariate case is defined as follows:

$$\omega(f;\delta_1,\delta_2,\dots,\delta_m)=\sup\{|f(t_1,t_2,\dots,t_m)-f(x_1,x_2,\dots,x_m)|:(t_1,t_2,\dots,t_m),(x_1,x_2,\dots,x_m)\in I^m$$
$$and\,|t_i-x_i|\le\delta_i,\ for\ i=1,2,\dots m\},$$

where $\omega(f;\delta_1,\delta_2,\dots,\delta_m)$ satisfies the following properties:

$$\omega(f;\delta_1,\delta_2,\dots,\delta_m)\to 0 \text{ if } \delta_i\to 0 \text{ for } i=1,2,\dots,m;$$

$$|f(t_1,t_2,\dots,t_m)-f(x_1,x_2,\dots,x_m)|\le$$
$$\omega(f,\delta_1,\delta_2,\dots,\delta_m)(1+|t_1-x_2|/\delta_1)(1+|t_2-x_2|/\delta_2)$$
$$\dots(1+|t_m-x_m|/\delta_m).$$

Now, we give the estimate of the rate of convergence of the multivariate operators defined in (4).

Theorem 2. Let $f\in C(I^m)$ and $0<q_{n_i}<1$ for $i=1,2,\dots,m$. Then for all $(x_1,x_2,\dots,x_m)\in J^m$, we have

$$|K_{n_1,n_2,\dots n_m,p}(f;q_{n_1},q_{n_2}\dots q_{n_m},x_1,x_2,\dots,x_m)$$
$$-f(x_1,x_2,\dots,x_m)|$$
$$\le 4\omega(f;\sqrt{\delta_{n_1}(x_1)},\sqrt{\delta_{n_2}(x_2)},\dots,\sqrt{\delta_{n_m}(x_m)}),$$

where $\delta_{n_i}(x_i)=K_{n_i,p}((t_i-x_i)^2;q_{n_i},x_i)$ for $i=1,2,\dots,m$.

Proof. Using linearity and positivity of the operator $K_{n_1,n_2,\dots,n_m,p}(f;q_{n_1},q_{n_2},\dots,q_{n_m},x_1,x_2,\dots,x_m)$, we have

$$|K_{n_1,n_2,\dots,n_m,p}(f;q_{n_1},q_{n_2},\dots,q_{n_m},x_1,x_2,\dots,x_m)-f(x_1,x_2,\dots,x_m)|$$
$$\le K_{n_1,n_2,\dots,n_m,p}(|f(t_1,t_2,\dots,t_m)-f(x_1,x_2,\dots,x_m)|;q_{n_1},q_{n_2},\dots,q_{n_m},x_1,x_2,\dots,x_m|)$$
$$\le \omega(f;\sqrt{\delta_{n_1}(x_1)},\sqrt{\delta_{n_2}(x_2)}\dots\sqrt{\delta_{n_m}(x_m)})(K_{n_1,p}(1;q_{n_1},x_1)+\frac{1}{\sqrt{\delta_{n_1}(x)}}K_{n_1,p}(|t_1-x_1|;q_{n_1},x_1))$$
$$\times(K_{n_2,p}(1;q_{n_2},x_2)+\frac{1}{\sqrt{\delta_{n_2}(y)}}K_{n_2,p}(|t_2-x_2|;q_{n_2},x_2))$$
$$\dots\times(K_{n_m,p}(1;q_{n_m},x_m)+\frac{1}{\sqrt{\delta_{n_m}(x_m)}}K_{n_m,p}(|t_m-x_m|;q_{n_m},x_m)).$$

Now, applying the Cauchy-Schwarz inequality we have

$$K_{n_i,p}(|t_i-x_i|;q_{n_i},x_i)\le K_{n_i,p}((t_i-x_i)^2;q_{n_i},x_i)^{1/2}(K_{n_i,p}(1;q_{n_i},x_i)^{1/2}\ for\ i=1,2,\dots,m.$$

Using the above, we get the desired result. □

4.1 Degree of approximation

Now, we study the degree of approximation for the multivariate operators (4) by means of the Lipschitz class.

For $0<\alpha_i<1$ and $i=1,2,\dots,m,$ we define the Lipschitz class $Lip_M(\alpha_1,\alpha_2,\dots,\alpha_m)$ for the multivariate case as follows:

$$|f(t_1,t_2,\dots,t_m)-f(x_1,x_2,\dots,x_m)|\le$$
$$M|t_1-x_1|^{\alpha_1}|t_2-x_2|^{\alpha_2}\dots|t_m-x_m|^{\alpha_m}.$$

Theorem 3. Let $f\in Lip_M(\alpha_1,\alpha_2,\dots\alpha_m,)$. Then, we have

$$|K_{n_1,n_2,\dots,n_m,p}(f;q_{n_1},q_{n_2},\dots,q_{n_m},x_1,x_2,\dots,x_m)$$
$$-f(x_1,x_2,\dots,x_m)|$$
$$\le M\delta_{n_1}^{\alpha_1/2}(x_1)\delta_{n_2}^{\alpha_2/2}(x_2)\dots\delta_{n_m}^{\alpha_m/2}(x_m),$$

where $\delta_{n_i}(x_i)$ for $i=1,2,\dots,m$ are defined as in Theorem 2.

Proof. By our hypothesis, we may write

$$|K_{n_1,n_2,\dots,n_m,p}(f;q_{n_1},q_{n_2},\dots,q_{n_m},x_1,x_2,\dots,x_m)-f(x_1,x_2,\dots,x_m)|$$
$$\le K_{n_1,n_2,\dots,n_m,p}(|f(t_1,t_2,\dots,t_m)-f(x_1,x_2,\dots,x_m)|;q_{n_1},q_{n_2},\dots,q_{n_m},x_1,x_2,\dots,x_m)$$
$$\le MK_{n_1,n_2,\dots,n_m,p}(|t_1-x_1|^{\alpha_1}|t_2-x_2|^{\alpha_2}\dots|t_m-x_m|^{\alpha_m};q_{n_1},q_{n_2},\dots,q_{n_m},x_1,x_2,\dots,x_m)$$
$$=MK_{n_1,p}(|t_1-x_1|^{\alpha_1};q_{n_1},x_1)K_{n_2,p}(|t_2-x_2|^{\alpha_2};q_{n_2},x_2)\dots K_{n_m,p}(|t_m-x_m|^{\alpha_m};q_{n_m},x_m).$$

Now, using the Hölder's inequality with $u_1=2/\alpha_1$, $v_1=2/(2-\alpha_1)$, $u_2=2/\alpha_2$, $v_2=2/(2-\alpha_2)$, …, $u_m=2/\alpha_m$, $v_m=2/(2-\alpha_m)$, we have

$$|K_{n_1,n_2,\dots,n_m,p}(f;q_{n_1},q_{n_2},\dots,q_{n_m},x_1,x_2,\dots,x_m)$$
$$-f(x_1,x_2,\dots,x_m)|$$
$$\le MK_{n_1,p}((t_1-x_1)^2;q_{n_1},x_1)^{\frac{\alpha_1}{2}}K_{n_1,p}(1;q_{n_1},x_1)^{\frac{2-\alpha_1}{2}}$$
$$\times K_{n_2,p}((t_2-x_2)^2;q_{n_2},x_2)^{\frac{\alpha_2}{2}}K_{n_2,p}(1;q_{n_2},x_2)^{\frac{2-\alpha_2}{2}}$$
$$\dots\times K_{n_m,p}((t_m-x_m)^2;q_{n_m},x_m)^{\frac{\alpha_m}{2}}K_{n_m,p}(1;q_{n_m},x_m)^{\frac{2-\alpha_m}{2}}$$
$$\le M\delta_{n_1}^{\frac{\alpha_1}{2}}(x_1)\delta_{n_2}^{\frac{\alpha_2}{2}}(x_2)\dots\delta_{n_m}^{\frac{\alpha_m}{2}}(x_m).$$

Hence, the proof is completed.

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