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Mechanical Equations on Riemann Almost Contact Model of a Cartan Space of order 2

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Abstract: In this paper, we present Euler-Lagrange and Hamilton mechanical equations introduced on Riemann almost contact space of a Cartan space of order two. In the conclusion we discussed differential geometric and physical results about related mechanical equations.

Key words: Cartan Manifold, Riemann Manifold, Lagrangian and Hamiltonian Mechanics

1. Introduction

Modern differential geometry has the fundamental elements as vector field, 1-form, curve, tangent and cotangent bundles, manifold, etc. Tangent and cotangent bundles can be modeled as the phase-space of velocities and momentum of a given configuration manifold. So modern differential geometry has an important role for Lagrangian and Hamiltonian dynamic formalism and mechanical equations. It can explain and find Euler-Lagrange and Hamilton mechanical equations using a suitable vector field on tangent and cotangent bundles. These dynamic formalisms and equations are illustrated in [1], [2]

Let 3 is m-dimensional con.guration manifold with coordinates $(x^i), 1 \leq i \leq m$, its tangent and cotangent bundle TM and T^*M with coordinates (x^i, p^i) and (x^i, p_i), $1 \leq i \leq m$ respectively. If the functions $L: TM \to \mathbb{R}$ and $H: T^*M \to \mathbb{R}$ are regular Lagrange and Hamilton functions then there is a unique vector field $\xi\ (X_H)$ on $TM(T^*M)$ such that Lagrangian and Hamiltonian dynamic formalisms are given:

$$i_\xi \phi_L = dE_L \qquad (1)$$

$$i_{X_H} \phi_H = dE_L \qquad (2)$$

respectively, where ϕ_L, ϕ_H are symplectic forms and E_L is the energy associated to L. Further, the geodesies of the manifold M can be found by solving Euler-Lagrange and Hamilton equations shown by

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \qquad (3)$$

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \qquad (4)$$

respectively for $i = 1,2,\dots,m$. Furthermore the triple (TM, ϕ_L, ξ) is called Lagrangian system on TM and the triple (T^*M, ϕ_H, X_H) is called Hamiltonian system on T^*M. It is possible to find many studies about Lagrangian (Hamiltonian) mechanics, dynamics and Euler-Lagrange (Hamilton) equations in articles [3], [4] and books [5], [6]

Hamilton Jacobi formalism is used to solve various mechanical problems in classical mechanics. Also Hamilton-Jacobi theory gets to reach at the frequencies of periodic systems, without finding the full solution of the dynamic problem. Hamilton-Jacobi formulation can be extended to quantum mechanics.

Schrödinger equation for the eigenfunctions and eigenvalues is important equation in quantum mechanics. Because energy eigenstates of a particle in a potential well are usually found out by solving this.

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Quantum Hamilton Jacobi formalism allows calculation of the bound state energies of the system without requiring the determination of the wave-functions. The basis of this formalism is the quantum Hamilton Jacobi equation, which along with the physical boundary conditions. So this equation is equivalent to the Schrödinger equation or Heisenberg.s equation of motion [7]. Also Hamilton.Jacobi equation is a special case of the Hamilton-Jacobi-Bellman equation, which is used to solve optimal control problems [8].

2. Cartan Space of Order-2 ($C^{(2)n}$) and Riemann Almost Contact Structure of $C^{(2)n}$

In this section, we remind of some structures given in [6]. Let M is a manifold of real dimension m and $T^{*2}M$ is 2-osculator cotangent bundle of M with local coordinates (x^i, y^i, p_i) $i = 1,2,..m$ and $K: T^{*2}M \to [0,\infty]$ is a real function where $T^{*2}M = TM \oplus T^*M$. Then the pair $(M, K(x,y,p))$ is a Cartan space of order 2 if the following axioms hold:

(i) K is differentiable on $\widetilde{T^{*2}M} = T^{*2}M/\{0\}$ for $\{0\} = \{(x,0,0) \mid x \in M\}$

(ii) K is positively 1-homogeneous on the fibres of cotangent bundle $T^{*2}M$ that is $K(x,y,\lambda p) = \lambda K(x,y,p)$ for all $\lambda > 0$

(iii) The Hessian of K^2 with elements

$$g^{ij}(x,y,p) = \frac{1}{2}\frac{\partial K}{\partial p_i}\frac{\partial K}{\partial p_j} \tag{5}$$

is positively defined on $\widetilde{T^{*2}M}$. Here g^{ij} is symmetric, nonsingular distinguished tensor field (d-tensor field) and contravariant of order two on the manifold $\widetilde{T^{*2}M}$. The function $K(x,y,p)$ is called fundamental function and the d-tensor field g^{ij} is called fundamental or metric tensor of the space $C^{(2)n} = (M, K(x,y,p))$. A Cartan space can be thought as Riemann almost contact space on the manifold $\widetilde{T^{*2}M}$. By means of a nonsingular connection with coefficients (N_i^j, N_{ij}) on $T^{*2}M$ that is regular distribution supplementary to the vertical distribution $V = W_1 \oplus W_2$ with $N: u \in T^{*2}M \to N_u \subset T_uT^{*2}M$ then we can write $T_uT^{*2}M = N(u) \oplus W_1(u) + W_2(u)$ for all $u \in T^{*2}M$. Afterwards it follows that

$$\left(\frac{\partial}{\partial x^i}\right)^H = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j\frac{\partial}{\partial y^j} + N_{ij}\frac{\partial}{\partial p_j} \tag{6}$$

Moreover $\left(\frac{\delta}{\delta x^i}\right), i = 1,2,\dots,m$ is a local base in HTM , $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i}\right)$ $i = 1,2,..,m$ is local base adapted to the horizontal and vertical distributions. And also $(dx^i, \delta y^i, \delta p_i)$ $i = 1,2,\dots,m$ is the dual base of $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i}\right)$ $i = 1,2,..,m$ where $\delta y^i = dy^i + N_i^j dx^j$ and $\delta p_i = dp_i - N_{ij} dx^j \in HT^*M$. Now, Let g_{ij} be covariant tensor of g^{ij} then the N-lift to $T^{*2}M$ of the fundamental tensor by $\boldsymbol{G^{ij}}(x,y,p)$ by:

$$G = g_{ij}\, dx^i \otimes dx^j + g_{ij}\, \delta y^i \otimes \delta y^i + g^{ij}\, \delta p_i \otimes \delta p_j \tag{7}$$

Theorem 1. *i)* The pair $(\widetilde{T^{*2}M}, \boldsymbol{G})$ is a Riemann space *ii)* The distributions N, W_1 and W_2 are orthogonal with respect to G.

Now suppose that the d-tensor metric $g^{ij}(x,y,p)$ and the symmetric nonlinear connection N are given then we can define almost contact structure F as follow:

$$\boldsymbol{F}\left(\frac{\delta}{\delta x^i}\right) = -g_{ij}\frac{\partial}{\partial p_i}, \boldsymbol{F}\left(\frac{\partial}{\partial y^i}\right) = 0$$

$$F\left(\frac{\partial}{\partial p_i}\right) = g^{ij}\frac{\delta}{\delta x^i} \tag{8}$$

The affect of F on the base $(dx^i, \delta y^i, \delta p_i)$ is as follow:

$$F^*(dx^i) = -g_{ij}\,\delta p_i \;\; F^*(\delta y^i) = 0$$
$$F^*(\delta p_i) = -g^{ij}\, dx^i \tag{9}$$

Theorem 2. *i)* F is a tensor field of type $(1,1)$ on $\widetilde{T^{*2}M}$

(ii) The pair $(\boldsymbol{G}, \boldsymbol{F})$ is Riemann almost contact structure on $\widetilde{T^{*2}M}$

(ii) The almost symplectic 2-form of the structure $(\boldsymbol{G}, \boldsymbol{F})$ is

$$\theta = \delta p_i \wedge dx^i \tag{10}$$

Definition. The space $(\widetilde{T^{*2}M}, \boldsymbol{G}, \boldsymbol{F})$ is called Riemann almost contact model of the space $C^{(2)n}$.

3. Euler-Lagrange Equations

In this section we present Euler-Lagrange equations on almost Riemann model $H^{2m} = \left(\widetilde{T^{*2}M}, \boldsymbol{G}, \boldsymbol{F}\right)$ of the space $C^{(2)n}$. First let F take an almost contact structure on the space $C^{(2)n}$. Let semispray be the vector field ξ given as:

$$\xi = X^i \frac{\delta}{\delta x^i} + Y^i \frac{\partial}{\partial y^i} + P^i \frac{\partial}{\partial p_i} \quad (11)$$

The vector field denoted by

$$V = F(\xi) = -g_{ij} X^i \frac{\partial}{\partial p_i} + g^{ij} P^i \frac{\delta}{\delta x^i} \quad (12)$$

is called Liouville vector field. We call the function $E_L = V(L) - L$ is energy function associated to the map $L: M \longrightarrow \mathbb{R}$ that satisfies $L = T - P$ is a Lagrangian function where T and P are kinetic energy potential energy of the system respectively. So E_L can be obtained as

$$E_L = -g_{ij} X^i \frac{\partial L}{\partial p_i} + g^{ij} P^i \frac{\delta L}{\delta x^i} - L \quad (13)$$

and

$$\begin{aligned} dE_L &= \left(\frac{\delta}{\delta x^i} dx^i + \frac{\partial}{\partial y^i} \delta y^i + \frac{\partial}{\partial p_i} \delta p^i\right)\left(-g_{\boldsymbol{ij}} X^i \frac{\partial L}{\partial p_i} + g^{ij} P^i \frac{\delta L}{\delta x^i} - L\right) \\ &= -g_{ij} X^i \frac{\delta}{\delta x^i}\left(\frac{\partial L}{\partial p_j}\right) dx^i - g_{ij} X^i \frac{\partial}{\partial y^i}\left(\frac{\partial L}{\partial p_j}\right) \delta y^i \\ &\quad -g_{ij} X^i \frac{\partial}{\partial p_i}\left(\frac{\partial L}{\partial p_j}\right) \delta p^i + g^{ij} P^i \frac{\delta}{\delta x^i}\left(\frac{\delta L}{\delta x^j}\right) dx^i \\ &\quad +g^{ij} P^i \frac{\partial}{\partial y^i}\left(\frac{\delta L}{\delta x^j}\right) \delta y^i + g^{ij} P^i \frac{\partial}{\partial p_i}\left(\frac{\delta L}{\delta x^j}\right) \delta p^i \\ &\quad -\frac{\delta L}{\delta x^i} dx^i - \frac{\partial L}{\partial y^i} \delta y^i - \frac{\partial L}{\partial p^j} \delta p^i \end{aligned} \quad (14)$$

For structure F, the closed 2-form given by $\phi_L = -dd_{\boldsymbol{F}} L$ such that

$$\begin{aligned} dd_{\boldsymbol{F}} L &= F\left(\frac{\delta}{\delta x^i} dx^i + \frac{\partial}{\partial y^i} \delta y^i + \frac{\partial}{\partial p_i} \delta p^i\right) L \\ &= -g_{ij} \frac{\partial L}{\partial p_i} dx^i + g^{ij} \frac{\delta L}{\delta x^i} \delta p^i \end{aligned} \quad (15)$$

then, ϕ_L can be found as

$$\begin{aligned} \phi_L &= g_{ij} \frac{\delta}{\delta x^j}\left(\frac{\partial L}{\partial p_i}\right) dx^j \wedge dx^i \\ &\quad - g^{ij} \frac{\delta}{\delta x^j}\left(\frac{\delta L}{\delta x^i}\right) dx^j \wedge \delta p_i \\ &\quad + g_{ij} \frac{\partial}{\partial y^j}\left(\frac{\partial L}{\partial p_i}\right) \delta y^j \wedge dx^i - g^{ij} \frac{\partial}{\partial y^j}\left(\frac{\delta L}{\delta x^i}\right) \delta y^j \wedge \delta p_i \\ &\quad + g_{ij} \frac{\partial}{\partial p_j}\left(\frac{\partial L}{\partial p_i}\right) \delta p_j \wedge dx^i - g^{ij} \frac{\partial}{\partial p_j}\left(\frac{\delta L}{\delta x^i}\right) \delta p_j \wedge \delta p_i \end{aligned} \quad (16)$$

The left hand side of Lagrangian formalism is calculated as

$$\begin{aligned} i_\xi \phi_L &= \phi_L(\xi) = \phi_L\left(X^i \frac{\delta}{\delta x^i} + Y^i \frac{\partial}{\partial y^i} + P^i \frac{\partial}{\partial p_i}\right) \\ &= -g^{ij} X^i \frac{\delta}{\delta x^i}\left(\frac{\delta L}{\delta x^i}\right) \delta p^i - g_{ij} X^i \frac{\partial}{\partial y^i}\left(\frac{\partial L}{\partial p^i}\right) \delta y^i \\ &\quad -g_{ij} X^i \frac{\partial}{\partial p^i}\left(\frac{\partial L}{\partial p^i}\right) \delta p^i + g_{ij} Y^i \frac{\partial}{\partial y^i}\left(\frac{\partial L}{\partial p^i}\right) \delta p^i \\ &\quad -g^{ij} Y^i \frac{\partial}{\partial y^i}\left(\frac{\delta L}{\delta x^i}\right) \delta p^i + g^{ij} P^i \frac{\delta}{\delta x^i}\left(\frac{\delta L}{\delta x^i}\right) dx^i \\ &\quad +g^{ij} P^i \frac{\partial}{\partial y^i}\left(\frac{\delta L}{\delta x^i}\right) \delta y^i + g_{ij} P^i \frac{\partial}{\partial p^i}\left(\frac{\partial L}{\partial p^i}\right) dx^i \end{aligned} \quad (17)$$

So from (1) we get the following as:

$$g_{ij} \xi \frac{\partial L}{\partial p_i} + \frac{\delta L}{\delta x^i} = 0 \,, g^{ij} \xi \frac{\delta L}{\delta x^i} - \frac{\partial L}{\partial p^i} = 0 \quad (18)$$

Now let the curve α be the integral curve of the vector field ξ. So Euler-Lagrange equations can be found as below:

$$g_{ij} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial p_i}\right) + \frac{\delta L}{\delta x^i} = 0, g^{ij} \frac{\partial}{\partial t}\left(\frac{\delta L}{\delta x^i}\right) - \frac{\partial L}{\partial p^i} = 0 \quad (19)$$

Hence, the equations introduced in (19) are called *Euler-Lagrange equations* on the Riemann almost contact model of space $C^{(2)n}$. Furthermore the triple (H^{2m}, ϕ_L, ξ) is said to be a *Lagrangian system* on the Riemann almost contact model H^{2m} of space $C^{(2)n}$.

4. Hamilton-Jacobi Equations

Here, we present Hamlton-Jacobi equations for quantum and classical mechanics constructed on the Riemann almost contact model H^{2m} of space $C^{(2)n}$

Theorem 3. There exists a unique vector field $X_{K^2} \in \widetilde{T^{*2}M}$ with the property

$$i_{X_{K^2}}\phi = -dE_L \quad (20)$$

for any Cartan space. This equation is called Hamilton-Jacobi formalism.

Now firstly set a 1-form

$$\omega = -\frac{1}{2}\left(g^{ij}x^i dx^i + y^i\delta y^i + g_{ij}p_i\delta p_i\right) \quad (21)$$

Then we have the Liouville form

$$\lambda = F^*(\omega) = -\frac{1}{2}\left(-x^i\delta p_i + p_i dx^i\right) \quad (22)$$

and the closed form

$$\phi = -d\lambda = \delta p_j \wedge dx^j \quad (23)$$

Take the Hamiltonian vector field

$$X_{K^2} = X^i\frac{\delta}{\delta x^i} + Y^i\frac{\partial}{\partial y^i} + P^i\frac{\partial}{\partial p_i} \quad (24)$$

Then we find

$$i_{X_{K^2}}\phi = \phi(X_{K^2})$$

$$= \delta p_j \wedge dx^j\left(X^i\frac{\delta}{\delta x^i} + Y^i\frac{\partial}{\partial y^i} + P^i\frac{\partial}{\partial p_i}\right)$$

$$= -X^i\delta p_i + P^i dx^i \quad (25)$$

and

$$-dK^2 = -\frac{\delta K^2}{\delta x^i}dx^i - \frac{\partial K^2}{\partial y^i}\delta y^i - \frac{\partial K^2}{\partial p_i}\delta p^i \quad (26)$$

By means of (20); using (25) and (26) , it is obtained

$$X^i = \frac{\partial K^2}{\partial p_i}, Y^i = 0, P^i = -\frac{\delta K^2}{\delta x^i} \quad (27)$$

and so the Hamiltonian vector field is found as follows:

$$X_{K^2} = \frac{\partial K^2}{\partial p_i}\frac{\delta}{\delta x^i} - \frac{\delta K^2}{\delta x^i}\frac{\partial}{\partial p_i} \quad (28)$$

Assume that a curve $\alpha: I \subset \mathbb{R} \rightarrow \widetilde{T^{*2}M}$

$$\alpha(t) = \left(x^i(t), y^i(t), p_i(t)\right) \quad (29)$$

be an integral curve of the Hamiltonian vector field of the Hamiltonian vector field X_{K^2}, i.e,

$$X_{K^2}(\alpha(t)) = \dot{\alpha}(t), t \in I \quad (30)$$

where

$$\dot{\alpha}(t) = \frac{dx^i}{dt}\frac{\delta}{\delta x^i} + \frac{dy^i}{dt}\frac{\partial}{\partial y^i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i} \quad (31)$$

Considering (30), if we equal (28) and (31) we find

$$\frac{dx^i}{dt} = \frac{\partial K^2}{\partial p_i}, \frac{dy^i}{dt} = 0, \frac{dp_i}{dt} = -\frac{\delta K^2}{\delta x^i} \quad (32)$$

Hence, the equations introduced in (32) are called *Hamilton-Jacobi equations* on the Riemann almost contact model of space $C^{(2)n}$ and the triple (H^{*2m}, ϕ, X_{K^2}) is said to be a *Hamiltonian mechanical system* on the Riemann almost contact model H^{*2m} of the space $C^{(2)n}$.

5. Corollary

Considering together with Euler-Lagrange (19) and Hamilton-Jacobi (32) equations, we find

$$g_{ij}\frac{\partial K^2}{\partial p_i} = p_i \text{ or } g_{ij} = \frac{p_i}{\frac{\partial K^2}{\partial p_i}} \quad (33)$$

if the functions L and K^2 are accepted to be equal then using equations (19), (32) and (33) we get

$$\frac{\partial}{\partial t}\left(\frac{\delta K^2}{\delta x^i}\right) = p_i \quad (34)$$

As shown that it can be written to be independent p_i of g_{ij}

6 Discussion

From sections 3 and 4 we conclude that Lagrangian and Hamiltonian dynamics in classical mechanics and field theory can be intrinsically characterized on the Riemann Almost Contact Model of a Cartan Space of Order Two. The paths of semispray ξ on the almost Riemann model $H^{2m} = \left(\widetilde{T^{*2}M}, G, F\right)$ of manifold $C^{(2)n}$ are the solutions of the Euler-Lagrange equations obtained in (19) on H^{2m} of manifold $C^{(2)n}$. Also the solutions of Hamilton-Jakobi equations obtained in (30) on H^{*2m} of manifold $C^{(2)n}$ are paths of vector field X_{K^2} on the almost Riemann model H^{*2m} of manifold $C^{(2)n}$. Consequently it is shown that these equations are very important for researchers working the analytics problems about Cartan spaces.

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