Journal of Mathematics and System Science 6 (2016) 147-158
doi: 10.17265/2159-5291/2016.04.003

Dynamic IS-LM model with Philips Curve and International Trade

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Received: January 27, 2016 / Accepted: February 25, 2016 / Published: April 25, 2016.

Abstract: In this article, we will show the Kaldor-Philips business cycle model with Hopf bifurcation theorem of the three dimensions, and the International trade model along with the dynamic IS-LM model by utilizing Hopf bifurcation theorem of the four dimensions. In section 2, we will present the Kaldor-Philips type dynamic IS-LM model. In section 3, we will also the dynamic IS-LM-international trade model of Hopf bifurcation theoremof four dimensions. In the last, but not least, I will present the policy implications of the Kaldor-Philips business cycle and International trade models along with the dynamic IS-LM model.

1. Introduction

Some Post Keynesians question whether or not inflation targeting is compatible with the Post Keynesian Economy (Davidson, 2006; Lima & Setterfield, 2008; Palley, 2006; Setterfield, 2006; Dos Santos, 2014).

Setterfield (2006) tried to show that inflation targeting is compatible with Post Keynesian economics, especially, when policy used to achieve the inflation target explicitly acknowledge (1) there is the demand-driven nature of the real income generating process; and (2) the importance of conflicting process of income distribution for determining the rate of inflation.

In this article, we will show the Kaldor-Philips business cycle model with Hopf bifurcation theorem of the three dimensions, and the International trade model along with the dynamic IS-LM model by utilizing Hopf bifurcation theorem of the four dimensions.

In section 2, we will present the Kaldor-Philips type dynamic IS-LM model. In section 3, we will also the dynamic IS-LM-international trade model of Hopf bifurcation theoremof four dimensions. In the last, but not least, I will present the policy implications of the Kaldor-Philips business cycle and International trade models along with the dynamic IS-LM model.

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2. Dynamic IS-LM Model with Philips Curve

As the institutional characters of modern capitalist economy, we can point out the instability of the financial market and the increasingly important of the decision-making processes of the business firms. Besides, we can consider the correspondences of the government and the central bank in the cases of the booms and recessions.

We can see the literature the stabilization policy as follows Philips defined three types of stabilization policy. Those are following; (i) proportional stabilization policy: government expenditure is proportional to the size of error; (ii) derivative stabilization policy: government expenditure is increased in response to the direction of the change of the real income; and (iii) integral stabilization policy: government expenditure corresponds to the calculative or integralerror in the past (Nagatani 1981, pp.164-165). Asada (1997 ch.2) analyzed the implication of the cycle caused by policy in the macro-dynamics of business cycles, and distinguish effective cases from ineffectiveness as the stabilization policy.

Torre (1977) applied to the analysis of the business cycle Hopf Bifurcation theoremwith the dynamic IS-LM model.

2.1 Dynamic IS-LM Model with Philips Curve: Analytical Approach

Notation

y = real national income, y^T = target national income along with fullemployment (exogenous variable), I = real investment, S = real saving, k = real capital stock, δ = depreciation rate of capital (exogenous variable), G = government expenditure (exogenous variable), T = income tax, M = nominal money supply (exogenous variable), p = price level, L = money demand, r = nominal interest rate, π^e = expected rate of inflation (exogenous variable).

Let us consider the dynamic IS-LM-PC model with Kaldorian investment function, Keynesian liquidity preference function and dynamic Philips curve by using Hopf bifurcation theorem of three dimensions. We don't consider the open economy.

$$\frac{dy}{dt} = \alpha(I + G - S - T), \alpha > 0 \quad (1)$$

$$I = I(y, k, r - \pi^e),\ I_y > 0,\ I_k < 0,\ I_r < 0 \quad (2)$$

$$G = G_0, G_0 > 0 \quad (3)$$

$$S = S(y), S_y > 0 \quad (4)$$

$$T = T(y),\ T_y > 0 \quad (5)$$

$$\frac{dk}{dt} = I(y, k, r) - \delta k, \delta > 0 \quad (6)$$

$$\frac{M}{p} = L(y, r),\ L_y > 0,\ L_r < 0 \quad (7)$$

$$\frac{dp}{dt} = \gamma(y - y^T), \gamma > 0 \quad (8)$$

Eq.(3-2-1) is the dynamic equation with the excess demand of goods and services market. Eq. (3-2-2) is the investment function with the increasing function of income, and the decreasing function of real interest rate. Eq. (3-2-3) is government expenditure function.Eq. (3-2-4) is saving function. Eq. (3-2-5) is the income tax. Eq. (3-2-6) is the dynamic function of capital accumulation. Eq.(3-2-7) is equilibrium condition of the money market, which means the real money supply equals the real money demand, which is the increasing function of the real income, and decreasing function of nominal interest rate. Eq. (3-2-8) is the dynamic equation of the price level with the reflective function between the real output and target output along with the full employment. Reduced (3-2-7) to the nominal interest rate, equation is as follows:

$$r = r(y, P),\ r_y > 0,\ r_p > 0 \quad (9)$$

From the above economic system, we can get the following system of dynamic equations:

$$\text{(i)}\ \frac{dy}{dt} = \alpha\left(I(y, k, r(y, p)) + G_0 - S(y) - T(y)\right),\ \alpha > 0$$

$$\text{(ii)}\frac{dk}{dt} = I(y, k, r(y, p)) - \delta k, \delta > 0 \quad (10)$$

$$\text{(iii)}\frac{dp}{dt} = \gamma(y - y^T), \gamma > 0$$

(Assumption 1) The signs of derivatives are as follows:

$$I_y > 0,\ I_k < 0,\ I_r < 0, S_y > 0,\ T_y > 0,\ r_y > 0,\ r_p > 0.$$

(Assumption 2) All functions in the system are differential, and all functions without the investment function are defined linearly. The investment function is defined as the function which has the Kaldorian non-linearity.

$$I_{yy} < 0\ for\ y > y^*, \text{ and } I_{yy} > 0\ for\ y < y^*$$

From (Assumption 1) and (Assumption 2), the above dynamic system has the only equilibrium (y*, k*, p*) independent of the parameters α and γ. To analyze the dynamic system of three dimensions, we assume as follows:

(Assumption 3) There is an equilibrium (y*, k*,p*) in the space {(y, k, p)|y>0, k>0, p>0}. Linearizing the system around the equilibrium, the Jacobian matrix J* is as follows:

$$J^* = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & 0 & 0 \end{pmatrix} \quad (11)$$

Where

$F_{11} = \alpha(I_y + I_r r_y - S_y - T_y), F_{12} = \alpha I_k < 0, F_{13} = \alpha I_r r_p < 0,\ F_{21} = I_y > 0, F_{22} = I_k - \delta < 0, F_{23} = I_r r_p < 0, F_{31} = \gamma > 0.$

Suppose that all derivatives are estimated at the equilibrium point. We fix the parameter, γ. The characteristic equation of the Jacobian matrix (3-2-11) is as follows:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \quad (12)$$

where $a_1 = -trace\, J^* = -F_{11} - F_{22} = -\alpha(I_y + I_r r_y - S_y - T_y) - (I_k - \delta)$

$a_2 = F_{22}F_{33} - F_{23}F_{32} + F_{11}F_{33} - F_{13}F_{31} + F_{11}F_{22} - F_{12}F_{21}$

$= -\alpha\gamma I_r r_p + \alpha(I_y + I_r r_y - S_y - T_y)(I_k - \delta) - \alpha I_k I_y$

$a_3 = \det J^* = -\alpha\gamma\delta I_r r_p > 0,$

and a_1, a_2 *and* a_3 are functions of α each other.

When the equilibrium loses the stability as a parameter changes, Hopf bifurcation occurs.

Then, we assume as follows:

(Assumption 4) On the equilibrium,

$$(I_y + I_r r_y - S_y - T_y)$$

must be positive, but enough to be small for all $a_1, a_2 > 0$.

By the way, we will prove the existence of closed orbits in the dynamic system (3-2-10) by using Hopf bifurcation theorem.

Proposition 1. Under (Assumption 1) ～ (Assumption 4), if the bifurcation parameter α is near to the bifurcation value, then $\alpha > \alpha_0$ or $\alpha < \alpha_0$, there is a closed orbit at least around an equilibrium of the dynamic system (3-2-10).

(proof)

In proving the local stability of the dynamic system, the very useful theorem is Routh-Hurwitz theorem. In the case of three dimensions, if $a_1, a_2, a_3 > 0$ and $a_1a_2 - a_3 > 0$, it is shown that the real parts of eigenvalues are negative. If $a_1, a_2, a_3 > 0$ and $a_1a_2 - a_3 > 0$, it is shown that the real parts of conjugated complex roots are zero, there are no real roots which are equivalent to zero.

Then, suppose α as the parameter of the bifurcation and assume an initial value which Routh-Hurwitz conditions are filled. By (Assumption 4), a marginal increase in α implies a marginal decrease in a_1, i.e.,

$$\frac{\partial a_1}{\partial \alpha} = -(I_y + I_r r_y - S_y - T_y) < 0,$$

at last a_1 will become to zero. Besides, a marginal increase in α implies a marginal increase in a_3, i.e.,

$$\frac{\partial a_3}{\partial \alpha} = -\gamma\delta I_r r_p > 0$$

The sign of $\partial a_2/\partial \alpha$ is ambiguous, but the existence of a value α_0 with the consequence $a_1a_2 - a_3 = 0$ can nevertheless be demonstrated. If $\partial a_2/\partial \alpha > 0$, the product a_1a_2 will be equal to infinite on the upper line of $\alpha = \hat{\alpha}$ at last, and

$a_1a_2 - a_3 = \{\alpha(I_y + I_r r_y - S_y - T_y) + (I_k - \delta)\}\{\alpha\gamma I_r r_p - \alpha(I_y + I_r r_y - S_y - T_y)(I_k - \delta) + \alpha I_k I_y\} + \alpha\gamma\delta I_r r_p = 0$

$$a_1a_2 - a_3 = A\alpha_0^2 - B\alpha_0 = 0$$

$A = \gamma I_r r_p(I_y + I_r r_y - S_y - T_y) - (I_y + I_r r_y - S_y - T_y)^2(I_k - \delta) + \gamma\delta(I_y + I_r r_y - S_y - T_y)I_r r_p$

$B = -\gamma I_r r_p(I_k - \delta) + (I_y + I_r r_y - S_y - T_y)(I_k - \delta)^2 - \gamma\delta I_r r_p(I_k - \delta)$

Then, $\alpha > 0$, $\alpha_0 = B/A$

The Jacobian of (3-2-9) has a pair of pure imaginary eigenvalues and no other eigenvalues with zero real part on the bifurcation value α_0.

For $\alpha < \alpha_0$, $a_1a_2 - a_3$ is negative. Then, a pair of conjugated complex root $\lambda(\alpha_0)$, that ensures

$a_1a_2 - a_3 = 0$ cannot become $\lambda(\alpha_0) + \lambda(\overline{\alpha_0})$ =0.

Then, for $\alpha < \alpha_0$, the real part of a pair of conjugated complex root is non zero. The case, in which closed orbits arise at $\alpha < \alpha_0$ is called the subcritical case: closed orbits enclose stable fixed points $(y, k, p)|_{\alpha<\alpha_0}$, implying that the orbits repelling. The equilibrium becomes unstable at $\alpha > \alpha_0$, there is no orbit. Then, (Assumption 1)～(Assumption 4), if the bifurcation parameter α is near to the bifurcation value, then for $\alpha > \alpha_0$ or $\alpha < \alpha_0$ there is a closed orbit at least around an equilibrium of the dynamic system (3-2-10). (Q.E.D.)

2.2 Numerical Simulation of Dynamic IS-LM Model with Philips Curve

Along with the above dynamic IS-LM model with Philips curve, we will show the numerical simulation as follows:

$$\frac{dy}{dt} = \alpha[0.1(Y-23)^3 + 0.05K + 0.05p_0 + G_0 - (0.1(Y-23) + 0.13Y - 0.65)]$$

$$\frac{dk}{dt} = 0.1(Y-23)^3 + 0.05K + 0.05p_0 - 0.05K$$

$$\frac{dp}{dt} = 0.3(Y_0 - Y)$$

$$G_0 = 0.05,\ p_0 = 0,\ Y_0 = 23,\ K_0 = 45$$

if α=2, G_0=0.05, p_0=0, Y_0=23, and K_0=45, we will show the result as follows:

Besides, if α=8, G_0=0.05, p_0=0, Y_0=23, and K_0=45, we will show the result as follows:

We will consider the implication of this model. Let us consider the case which the real national income

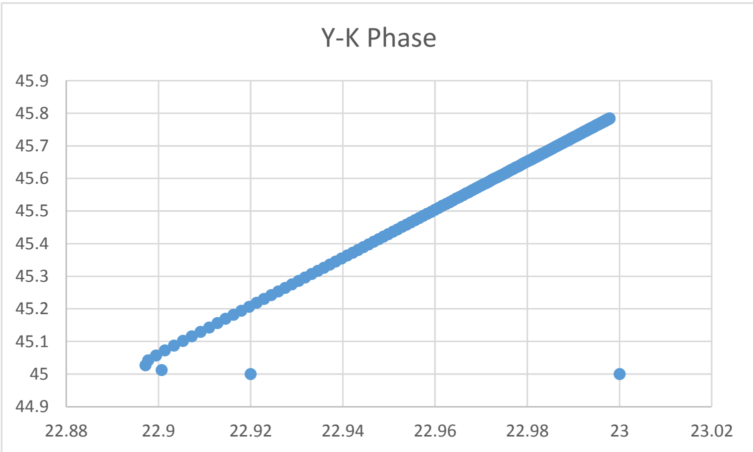

Fig. 1 Y-K Phase Diagram(α=2, G_0=0.05,p_0=0,Y_0=23, and K_0=45).

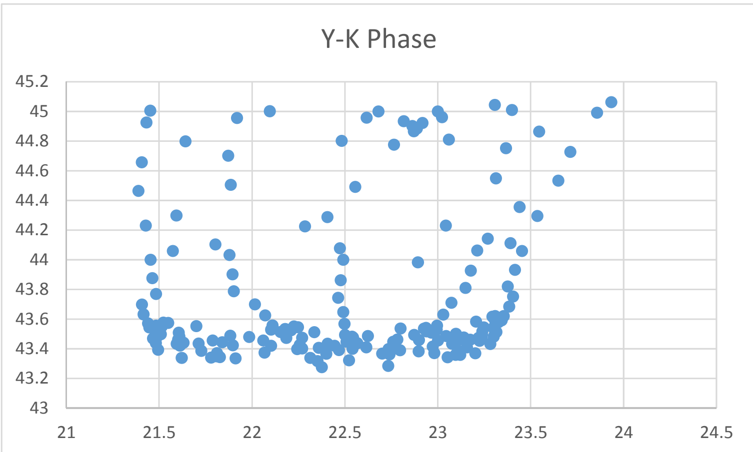

Fig. 2 Y-K Phase Diagram(α=8, G_0=0.05,p_0=0,Y_0=23, and K_0=45).

becomes smaller than the equilibrium value by some exogenous causes for simplicity. The decrease in the real income will lead to the decrease of consumption and decrease of investment. And also the decrease in the real income will lead to the decrease of the money demand and income tax. And if the real national income become smaller than the equilibrium value, the actual rate of inflation will be adapted by the gap between the actual and target level of real income.

3. International Trade withthe dynamic IS-LM model

Lorentz (1987) has developed a very simple IS-LM model which examines the effect of economic linkages on the economic dynamical system of a macro-open economy along with the dynamic IS-LM model.

We will extend the following dynamic macroeconomic model with two-country, international trade. We assume the price levelsin each country are exogenous variables, and the change rate of money supply in each country will regulate dynamically by the net export of each country.

3.1 Dynamic IS-LM Model with International Trade: Analytical Approach

Notation

y_i = real national income in country *i* (*i* = 1, 2), I_i=real investment in country i (i=1,2), S_i=real saving in country *i* (*i* = 1, 2), M_i= nominal money supply in country *i* (*i* = 1, 2), p_i = price level in country *i* (*i* = 1, 2) (exogenous variables), L_i =money demand in country *i* (*i* = 1, 2), $M_i^T =$ nominal money supply in country *i* (*i* = 1, 2) (exogenous variable), r_i= nominal interest rate in country *i* (*i* = 1, 2), $\pi_i^e =$ expected rate of inflation in country *i* (*i* = 1, 2) (exogenous variable).

$$\frac{dy_1}{dt} = \alpha_1\big(I_1(y_1, r_1 - \pi_1^e) - S_1(y_1) + EX_1(y_2) - IM_1(y_1)\big), \alpha_1 > 0 \quad (13)$$

$$I_{1_{y_1}} > 0, I_{1_{r_1}} < 0, 0 < S_{1_{y_1}} \leq 0, EX_{1_{y_2}} > 0, IM_{y_1} > 0$$

$$\frac{dy_2}{dt} = \alpha_2\big(I_2(y_2, r_2 - \pi_2^e) - S_2(y_2) + EX_2(y_1) - IM_2(y_2)\big), \alpha_2 > 0 \quad (14)$$

$$I_{2_{y_2}} > 0, I_{2_{r_2}} < 0, 0 < S_{2_{y_2}} \leq 0, EX_{2_{y_1}} > 0, IM_{2_{y_2}} > 0$$

$$M_1/\overline{p_1} = L_1(y_1, r_1), \frac{\partial L_1}{\partial y_1} > 0, \frac{\partial L_1}{\partial r_1} < 0 \quad (15)$$

$$M_2/\overline{p_2} = L_2(y_2, r_2), \frac{\partial L_2}{\partial y_2} > 0, \frac{\partial L_2}{\partial r_2} < 0 \quad (16)$$

$$\frac{dM_1}{dt} = \gamma_1\big(EX_1(y_2) - IM_1(y_1)\big) + \beta_1(M_1^T - M_1), \gamma_1 > 0, \beta_1 > 0 \quad (17)$$

$$\frac{dM_2}{dt} = \gamma_2\big(EX_2(y_1) - IM_2(y_2)\big) + \beta_2(M_2^T - M_2), \gamma_2 > 0, \beta_2 > 0 \quad (18)$$

Eq.(4-1) is the dynamic equation with the excess demand of goods and services market of country 1 in the open economy. Eq. (4-2) is the dynamic equation with the excess demand of goods and services market of country 2 in the open economy. Eq. (4-3) is equilibrium condition of the money market of country 1, which means the real money supply equals the real money demand, which is the increasing function of the real income, and decreasing function of nominal interest rate. Eq. (4-4) is equilibrium condition of the money market of country 2, which means the real money supply equals the real money demand, which is the increasing function of the real income, and decreasing function of nominal interest rate. Eq. (4-5) is the dynamic equation of the nominal money supply of country 1, and Eq. (4-6) is the dynamic equation ofnominal money supply of country 2.

Eq. (4-3) and (4-4) are reduced to the form as follows:

$$r_1 = r_1(y_1, M_1),\ r_{1y1} > 0, r_{1_{M_1}} < 0 \quad (19)$$

$$r_2 = r_2(y_2, M_2),\ r_{2y2} > 0, r_{2_{M_2}} < 0 \quad (20$$

From the above economic system, we can get the following system of dynamic equations:

$$(i)\frac{dy_1}{dt} = \alpha_1\big(I_1(y_1, r_1(y_1, M_1)) - S_1(y_1) + EX_1(y_2) - IM_1(y_1)\big), \alpha_1 > 0$$

$$(ii)\frac{dy_2}{dt} = \alpha_2\big(I_2(y_2, r_2(y_2, M_2)) - S_2(y_2) + EX_2(y_1) - IM_2(y_2)\big), \alpha_2 > 0$$

$$\text{(iii)}\frac{dM_1}{dt} = \gamma_1\big(EX_1(y_2) - IM_1(y_1)\big) + \beta_1(M_1^T - M_1), \gamma_1 > 0, \beta_1 > 0 \quad (4\text{-}9)$$

$$(iv)\frac{dM_2}{dt} = \gamma_2\big(EX_2(y_1) - IM_2(y_2)\big) + \beta_2(M_2^T - M_2), \gamma_2 > 0, \beta_2 > 0$$

(Assumption 5) The signs of derivatives are as follows:

$$I_{1_{y_1}} > 0, I_{1_{r_1}} < 0, 0 < S_{1_{y_1}} \leq 0, EX_{1_{y_2}} > 0, IM_{1_{y_1}} > 0$$

$$I_{2_{y_2}} > 0, I_{2_{r_2}} < 0, 0 < S_{2_{y_2}} \leq 0, EX_{2_{y_1}} > 0, IM_{2_{y_2}} > 0$$

$$r_{1y1} > 0, r_{1_{M_1}} < 0,\ r_{2y2} > 0, r_{2_{M_2}} < 0$$

(Assumption 6) All functions in the system are differential, and all functions without the investment function are defined linearly. The investment function is defined as the function which has the Kaldorian non-linearity.

$$I_{1_{y_1y_1}} < 0\ for\ y_1 > y_1{}^*,$$

and

$$I_{1_{y_1y_1}} > 0\ for\ y_1 < y_1{}^*$$

$$I_{2_{y_2y_2}} < 0\ for\ y_2 > y_2{}^*,$$

and

$$I_{2_{y_2y_2}} > 0\ for\ y_2 < y_2{}^*$$

(Assumption 7) There is an equilibrium (y_1*, y_2*,M_1*,M_2*) in the space $\{(y_1, y_2, M_1, M_2)\ |y_1>0, y_2>0,\ M_1>0, M_2 > 0\}$. Linearizing the system around the equilibrium, the Jacobian matrix J* is as follows:

$$J^* = \begin{pmatrix} J_{11} & J_{12} & J_{13} & 0 \\ J_{21} & J_{22} & 0 & J_{24} \\ J_{31} & J_{32} & J_{33} & 0 \\ J_{41} & J_{42} & 0 & J_{44} \end{pmatrix} \quad (21)$$

$$J_{11} = \alpha_1(I_{1_{y_1}} + I_{1_{r_1}} r_{1y1} - S_{1y1} - IM_{1_{y_1}}),$$
$$J_{12} = \alpha_1 EX_{1_{y_2}} > 0,\ J_{13} = \alpha_1 I_{1_{r_1}} r_{1_{M_1}} > 0,$$
$$J_{21} = \alpha_2 EX_{2_{y_1}} > 0,$$
$$J_{22} = \alpha_2(I_{2_{y_2}} + I_{2_{r_2}} r_{2y2} - S_{2y2} - IM_{2_{y_2}}),$$
$$J_{24} = \alpha_2 I_{2_{r_2}} r_{2_{M_2}} > 0,\ J_{31} = -\gamma_1 IM_{1_{y_1}} < 0,$$
$$J_{32} = \gamma_1 EX_{1_{y_2}} > 0,\ J_{33} = -\beta_1 < 0,$$
$$J_{41} = \gamma_2 EX_{2_{y_1}} > 0,$$
$$J_{42} = -\gamma_2 IM_{2_{y_2}} < 0,\ J_{44} = -\beta_2 < 0.$$

Hopf bifurcation theorem(existence part)

Consider the following general system

$$\frac{dy}{dt} = \varphi(y, \alpha) \quad \text{(H)}$$

and suppose that for each α in the relevant interval it has an isolated equilibrium point $y_e = y_e(\alpha)$. Assume that the Jacobian matrix of φ with respect to y, evaluated at $(y_e(\alpha), \alpha)$ has the following properties:

(H1) it possesses a pair of simple complex conjugate eigenvalues $\theta(\alpha) \pm i\omega(\alpha)$ that become pure imaginary at the critical value α_0 of the parameter, i.e., $\theta(\alpha_0)$=0, while $\omega(\alpha) \neq 0$, and no other eigenvalues with zero real part exist at $(y_e(\alpha_0), \alpha_0)$;

$$\text{(H2)}\frac{d\theta(\alpha)}{d\alpha}|_{\alpha=\alpha_0} \neq 0$$

THEN system (H) has a family of periodic solutions.

Since the existence part of the theorem leaves us as regards the nature of the cycle, there are in principle various possibilities. One possibility is that orbits spiral from $y_e(\alpha)$ when $\alpha > \alpha_0$ toward a stable limit cycle.This is called a *supercritical Hopf bifurcation*. Another is that an unstable cycle exists for $\alpha < \alpha_0$, inside of which all orbits spiral in toward $y_e(\alpha)$. This is called a subcritical Hopf bifurcation. Other possibilities may also exist[1].

Suppose that all derivatives are estimated at the equilibrium point. We fix the parameter, γ. The characteristic equation of the Jacobian matrix (4-10) is a four-dimensional system. In the case of a four-dimensional system we have the following useful theorem due to Asada and Yoshida (2003).[2]

[1] G. Gandolfo(2009), p.481.
[2] Ibid., pp.481-482.

Asada-Yoshida Theorem

(i) the polynomial equation

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \qquad (22)$$

has a pair of pure imaginary roots and two roots with non-zero real parts if and only if either of the following set of conditions (A) or (B) is satisfied:

(A) $a_1 a_3 > 0$, $a_4 \neq 0$, and $\Phi \equiv a_1 a_2 a_3 - {a_1}^2 a_4 - {a_3}^2 = 0$

(B) $a_1 = 0, a_3 = 0,$ and $a_4 < 0$.

(ii) The polynomial equation(4-11) has a pair of pure imaginary roots and two roots with negative real parts if and only if the following set of conditions (C) is satisfied:

(C) $a_1 > 0, a_3 > 0, a_4 > 0$ and $\Phi \equiv a_1 a_2 a_3 - {a_1}^2 a_4 - {a_3}^2 = 0$

where

$$a_1 = -\text{trace J}* = -J_{11} - J_{22} - J_{33} - J_{44} = -\alpha_1(I_{1_{y_1}} + I_{1_{r_1}} r_{1y1} - S_{1y1} - IM_{1_{y_1}}) - \alpha_2(I_{2_{y_2}} + I_{2_{r_2}} r_{2y2} - S_{2y2} - IM_{2_{y_2}}) + \beta_1 + \beta_2$$

$$a_2 = \boldsymbol{F_{12}}^* + \boldsymbol{F_{13}}^* + \boldsymbol{F_{14}}^* + \boldsymbol{F_{23}}^* + \boldsymbol{F_{24}}^* + \boldsymbol{F_{34}}^*$$
$$= J_{11}J_{22} - J_{12}J_{21} - J_{13}J_{21} + J_{11}J_{24} + J_{11}J_{32} - J_{12}J_{31} + J_{11}J_{33} - J_{13}J_{31} + J_{11}J_{42} - J_{12}J_{41} - J_{13}J_{41} + J_{21}J_{32} - J_{22}J_{31} - J_{24}J_{31} + J_{21}J_{42} - J_{22}J_{41} + J_{21}J_{44} - J_{24}J_{41} + J_{31}J_{42} - J_{32}J_{41} - J_{33}J_{41} + J_{31}J_{44}$$
$$= \alpha_1\alpha_2(I_{1_{y_1}} + I_{1_{r_1}} r_{1y1} - S_{1y1} - IM_{1_{y_1}})\left(I_{2_{y_2}} + I_{2_{r_2}} r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) - \alpha_1\alpha_2 EX_{1_{y_2}} EX_{2_{y_1}}$$
$$- \alpha_1\alpha_2 I_{1_{r_1}} r_{1_{M_1}} EX_{2_{y_1}} + \alpha_1\alpha_2 I_{2_{r_2}} r_{2_{M_2}} \left(I_{1_{y_1}} + I_{1_{r_1}} r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right) + \alpha_1\gamma_1 EX_{1_{y_2}} IM_{1_{y_1}}$$
$$- \alpha_1\beta_1(I_{1_{y_1}} + I_{1_{r_1}} r_{1y1} - S_{1y1} - IM_{1_{y_1}}) + \alpha_1\gamma_1 I_{1_{r_1}} r_{1_{M_1}} IM_{1_{y_1}} + \alpha_1\gamma_2(I_{1_{y_1}} + I_{1_{r_1}} r_{1y1} - S_{1y1} - IM_{1_{y_1}}) IM_{2_{y_2}} - \alpha_1\gamma_2 EX_{1_{y_2}} EX_{2_{y_1}} + \alpha_2\gamma_1 EX_{2_{y_1}} EX_{1_{y_2}}$$
$$+ \alpha_2\gamma_1\left(I_{2_{y_2}} + I_{2_{r_2}} r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) IM_{1_{y_1}} + \alpha_2\gamma_1 I_{2_{r_2}} r_{2_{M_2}} IM_{1_{y_1}} - \alpha_2\gamma_2 EX_{2_{y_1}} IM_{2_{y_2}}$$
$$- \alpha_2\gamma_2\left(I_{2_{y_2}} + I_{2_{r_2}} r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) EX_{2_{y_1}} - \alpha_2\beta_2 EX_{2_{y_1}} - \alpha_2\gamma_2 I_{2_{r_2}} r_{2_{M_2}} EX_{2_{y_1}}$$
$$+ \gamma_1\gamma_2 IM_{1_{y_1}} IM_{2_{y_2}} - \gamma_1\gamma_2 EX_{1_{y_2}} EX_{2_{y_1}} + \gamma_1\beta_2 IM_{1_{y_1}} + \beta_1\gamma_2 EX_{2_{y_1}}$$

$$a_3 = -\alpha_2\beta_1\beta_2\left(I_{2_{y_2}} + I_{2_{r_2}} r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) + \alpha_2\gamma_2\beta_1 IM_{2_{y_2}} I_{2_{r_2}} r_{2_{M_2}}$$
$$- \alpha_1\beta_1\left(I_{1_{y_1}} + I_{1_{r_1}} r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right) + \alpha_1\beta_2\gamma_1 IM_{1_{y_1}} I_{1_{r_1}} r_{1_{M_1}}$$
$$+ \alpha_1\alpha_2\beta_2\left(I_{1_{y_1}} + I_{1_{r_1}} r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right)\left(I_{2_{y_2}} + I_{2_{r_2}} r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right)$$
$$- \alpha_1\alpha_2\gamma_2 IM_{2_{y_2}}\left(I_{1_{y_1}} + I_{1_{r_1}} r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right) I_{2_{r_2}} r_{2_{M_2}} + \alpha_2\alpha_1\beta_2 EX_{2_{y_1}} EX_{1_{y_2}}$$
$$- \gamma_2\alpha_1\beta_2 EX_{1_{y_2}} EX_{2_{y_1}}$$
$$+ \alpha_1\alpha_2\beta_1\left(I_{2_{y_2}} + I_{2_{r_2}} r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right)\left(I_{1_{y_1}} + I_{1_{r_1}} r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right)$$
$$+ \alpha_2\alpha_1\beta_1 EX_{1_{y_2}} EX_{2_{y_1}} - \alpha_2\alpha_1\gamma_1 EX_{1_{y_2}} I_{1_{r_1}} r_{1_{M_1}} EX_{2_{y_1}}$$
$$- \gamma_1\alpha_1\alpha_2\left(I_{2_{y_2}} + I_{2_{r_2}} r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) I_{1_{r_1}} r_{1_{M_1}} IM_{1_{y_1}}$$

$$a_4 = \det J^* = J_{11}J_{22}J_{33}J_{44} - J_{11}J_{24}J_{33}J_{42} - J_{12}J_{21}J_{33}J_{44} - J_{13}J_{21}J_{32}J_{44} - J_{13}J_{22}J_{31}J_{44} + J_{13}J_{24}J_{31}J_{42} + J_{12}J_{24}J_{33}J_{41} - J_{13}J_{24}J_{32}J_{41}$$

$$= \alpha_1\alpha_2\beta_1\beta_2\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right)\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right)$$
$$- \alpha_1\alpha_2\beta_1\gamma_2 IM_{2_{y_2}} I_{2_{r_2}} r_{2_{M_2}}\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right) - \alpha_1\alpha_2\beta_1\beta_2 EX_{2_{y_1}} EX_{1_{y_2}}$$
$$+ \alpha_1\alpha_2\gamma_1\beta_2 EX_{1_{y_2}} EX_{2_{y_1}} I_{1_{r_1}} r_{1_{M_1}} - \alpha_1\alpha_2\gamma_1\beta_2 IM_{1_{y_1}}\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) I_{1_{r_1}} r_{1_{M_1}}$$
$$+ \alpha_1\alpha_2\gamma_1\gamma_2 IM_{2_{y_2}} IM_{1_{y_1}} I_{2_{r_2}} r_{2_{M_2}} I_{1_{r_1}} r_{1_{M_1}} - \alpha_1\alpha_2\beta_1\gamma_2 EX_{2_{y_1}} I_{2_{r_2}} r_{2_{M_2}} EX_{1_{y_2}}$$
$$- \alpha_1\alpha_2\gamma_1\gamma_2 EX_{2_{y_1}} EX_{1_{y_2}} I_{2_{r_2}} r_{2_{M_2}} I_{1_{r_1}} r_{1_{M_1}}$$

and a_1, a_2, a_3 and a_4 are functions of $\alpha_1, \alpha_2, \beta_1$, and β_2 each other.

When the equilibrium loses the stability as a parameter changes, Hopf bifurcation occurs.

Then, we assume as follows:

(Assumption 8) On the equilibrium,

$$\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right)$$

and

$$\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right)$$

must be positive, but enough to be small for all $a_1, a_3 > 0$.

Proposition 2. Under (Assumption 5) ～ (Assumption 8), if the bifurcation parameter α is near to the bifurcation value, then $\alpha_1 > \alpha_{10}$, $\alpha_2 > \alpha_{20}$ or $\alpha_1 < \alpha_{10}$, $\alpha_2 < \alpha_{20}$, there is a closed orbit at least around an equilibrium of the dynamic system (4-9).

(proof)

In proving the local stability of the dynamic system, the very useful theorem is Routh-Hurwitz theorem. In the case of four dimensions, if $a_1, a_2, a_3, a_4 > 0$ and $a_1a_2a_3 - a_1^2a_4 - a_3^2 > 0$, it is shown that the real parts of eigenvalues are negative. If $a_1, a_2, a_3, a_4 > 0$ and $a_1a_2a_3 - a_1^2a_4 - a_3^2 > 0$, it is shown that the real parts of conjugated complex roots are zero, there are no real roots which are equivalent to zero.

Then, suppose α as the parameter of the bifurcation and assume an initial value which Routh-Hurwitz conditions are filled. By (Assumption 8), a marginal increase in α implies a marginal decrease in a_1, i.e.,

$$\frac{\partial a_1}{\partial \alpha_1} = -\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right) < 0,$$

$$\frac{\partial a_1}{\partial \alpha_2} = -\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) < 0$$

at last a_1 will become to zero. Besides, a marginal increase in α implies a marginal increase in a_3, i.e.,

$$\frac{\partial a_3}{\partial \alpha_1} = \beta_1\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right) + \beta_2\gamma_1 IM_{1_{y_1}} I_{1_{r_1}} r_{1_{M_1}}$$
$$+ \alpha_1\alpha_2\beta_2\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right)\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right)$$
$$- \alpha_2\gamma_2 IM_{2_{y_2}}\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right) I_{2_{r_2}} r_{2_{M_2}} + \alpha_2\beta_2 EX_{2_{y_1}} EX_{1_{y_2}} - \gamma_2\beta_2 EX_{1_{y_2}} EX_{2_{y_1}}$$
$$+ \alpha_2\beta_1\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right)\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right)$$
$$+ \alpha_2\beta_1 EX_{1_{y_2}} EX_{2_{y_1}} - \alpha_2\gamma_1 EX_{1_{y_2}} I_{1_{r_1}} r_{1_{M_1}} EX_{2_{y_1}}$$
$$- \gamma_1\alpha_2\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) I_{1_{r_1}} r_{1_{M_1}} IM_{1_{y_1}}$$

$$\frac{\partial a_3}{\partial \alpha_2} = -\beta_1\beta_2\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) + \gamma_2\beta_1 IM_{2_{y_2}} I_{2_{r_2}} r_{2_{M_2}}$$
$$+\alpha_1\beta_2\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right)\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right)$$
$$-\alpha_1\gamma_2 IM_{2_{y_2}}\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right) I_{2_{r_2}} r_{2_{M_2}} + \alpha_1\beta_2 EX_{2_{y_1}} EX_{1_{y_2}} + \alpha_1\beta_1 (I_{2_{y_2}}$$
$$+ I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}})(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}) + \alpha_1\beta_1 EX_{1_{y_2}} EX_{2_{y_1}}$$
$$-\alpha_1\gamma_1 EX_{1_{y_2}} I_{1_{r_1}} r_{1_{M_1}} EX_{2_{y_1}} - \gamma_1\alpha_1\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) I_{1_{r_1}} r_{1_{M_1}} IM_{1_{y_1}}$$

$$\frac{\partial a_4}{\partial \alpha_1} = \alpha_2\beta_1\beta_2\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right)\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right)$$
$$-\alpha_2\beta_1\gamma_2 IM_{2_{y_2}} I_{2_{r_2}} r_{2_{M_2}}\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right) - \alpha_2\beta_1\beta_2 EX_{2_{y_1}} EX_{1_{y_2}}$$
$$+\alpha_2\gamma_1\beta_2 EX_{1_{y_2}} EX_{2_{y_1}} I_{1_{r_1}} r_{1_{M_1}} - \alpha_2\gamma_1\beta_2 IM_{1_{y_1}}\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) I_{1_{r_1}} r_{1_{M_1}}$$
$$+\alpha_2\gamma_1\gamma_2 IM_{2_{y_2}} IM_{1_{y_1}} I_{2_{r_2}} r_{2_{M_2}} I_{1_{r_1}} r_{1_{M_1}} - \alpha_2\beta_1\gamma_2 EX_{2_{y_1}} I_{2_{r_2}} r_{2_{M_2}} EX_{1_{y_2}}$$
$$-\alpha_2\gamma_1\gamma_2 EX_{2_{y_1}} EX_{1_{y_2}} I_{2_{r_2}} r_{2_{M_2}} I_{1_{r_1}} r_{1_{M_1}}$$

$$\frac{\partial a_4}{\partial \alpha_2} = \alpha_1\beta_1\beta_2\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right)\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right)$$
$$-\alpha_1\beta_1\gamma_2 IM_{2_{y_2}} I_{2_{r_2}} r_{2_{M_2}}\left(I_{1_{y_1}} + I_{1_{r_1}}r_{1y1} - S_{1y1} - IM_{1_{y_1}}\right) - \alpha_1\beta_1\beta_2 EX_{2_{y_1}} EX_{1_{y_2}}$$
$$+\alpha_1\gamma_1\beta_2 EX_{1_{y_2}} EX_{2_{y_1}} I_{1_{r_1}} r_{1_{M_1}} - \alpha_1\gamma_1\beta_2 IM_{1_{y_1}}\left(I_{2_{y_2}} + I_{2_{r_2}}r_{2y2} - S_{2y2} - IM_{2_{y_2}}\right) I_{1_{r_1}} r_{1_{M_1}}$$
$$+\alpha_1\gamma_1\gamma_2 IM_{2_{y_2}} IM_{1_{y_1}} I_{2_{r_2}} r_{2_{M_2}} I_{1_{r_1}} r_{1_{M_1}} - \alpha_1\beta_1\gamma_2 EX_{2_{y_1}} I_{2_{r_2}} r_{2_{M_2}} EX_{1_{y_2}}$$
$$-\alpha_1\gamma_1\gamma_2 EX_{2_{y_1}} EX_{1_{y_2}} I_{2_{r_2}} r_{2_{M_2}} I_{1_{r_1}} r_{1_{M_1}}$$

The sign of ${\partial a_2}/{\partial \alpha}$ is ambiguous, but the existence of a value α_0 with the consequence $a_1a_2a_3 - a_1^2a_4 - a_3^2 = 0$ can nevertheless be demonstrated.

The Jacobian of (4-10) has a pair of pure imaginary eigenvalues and no other eigenvalues with zero real part on the bifurcation value α_{10}, α_{20}.

For $\alpha_1 > \alpha_{10}$, $\alpha_2 > \alpha_{20}$, $a_1a_2a_3 - a_1^2a_4 - a_3^2$ is negative. Then, a pair of conjugated complex root $\lambda(\alpha_0)$, that ensures $a_1a_2a_3 - a_1^2a_4 - a_3^2$ cannot become $\lambda(\alpha_0) + \lambda(\overline{\alpha_0}) = 0$.

Then, for $\alpha_1 < \alpha_0$, the real part of a pair of conjugated complex root is non zero. The case, in which closed orbits arise at $\alpha_1 < \alpha_{10}$, $\alpha_2 < \alpha_{20}$ is called the subcritical case: closed orbits enclose stable fixed points, implying that the orbits repelling. The equilibrium becomes unstable at $\alpha > \alpha_0$, there is no orbit. Then, (Assumption 5)～(Assumption 8), if the bifurcation parameter α is near to the bifurcation value, then for $\alpha_1 > \alpha_{10}, \alpha_2 > \alpha_{20}$ or $\alpha_1 < \alpha_{10}, \alpha_2 < \alpha_{20}$ there is a closed orbit at least around an equilibrium of

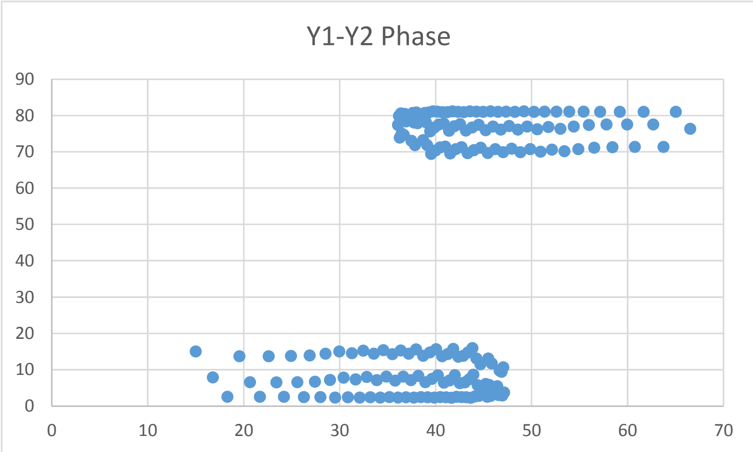

Fig. 3 Y1-Y2 Phase Diagram (α_1=6.8, α_2=7.8, ξ12=4, ξ21=4, γ_1=0.2, γ_2 =0.2, β_1 =0.2, β_2=0.2).

the dynamic system (4-9). (Q.E.D.)

3.2 Numerical Analysis of Dynamic IS-LM model with International Trade

Along with the above dynamic IS-LM model with International Trade, we will show the numerical simulation as follows:

$$(i)\frac{dy_1}{dt} = \alpha_1((8arctany_1 - 0.000001y_1M_1 - 0.15y_1 + 0.0001 * \xi 12 * Y2 - 0.0001 * \xi 21 * Y1)), \alpha_1 > 0$$

$$(ii)\frac{dy_2}{dt} = \alpha_2(((8arctany_2 - 0.000001y_2M_2 - 0.15y_2 + 0.0001 * \xi 21 * Y1 - 0.0001 * \xi 12 * Y2)), \alpha_2 > 0$$

$$(iii)\frac{dM_1}{dt} = \gamma_1(0.0001 * \xi 12 * Y2 - 0.0001 * \xi 21 * Y1) + \beta_1(4 - M_1), \gamma_1 > 0, \beta_1 > 0$$

$$(iv)\frac{dM_2}{dt} = \gamma_2(0.0001 * \xi 21 * Y1 - 0.0001 * \xi 12 * Y2) + \beta_2(4 - M_2), \gamma_2 > 0, \beta_2 > 0$$

If α_1 =6.8, α_2 =7.8, ξ12=4, ξ21=4, γ_1 =0.2, γ_2 =0.2, β_1 =0.2, β_2=0.2, we will show the result as follows:

We will consider the implication of the interregional trade macroeconomic model with two country, international trade. Let us consider the real national income of country 1 becomes smaller than the equilibrium value by some exogenous reason. The decrease of real income of country 1 will lead to the decrease of consumption, investment, and import in country 1 and export of country 2. The increase of net export of country 2 will lead to the increase of the real income of country 2. Besides, the decrease of real income of country 1 will also lead to the decrease of the rate of change of money supply in country 2 and the increase of the rate of change of money supply in country 1. The increase of the rate of change of money supply in country 1 will lead to decrease of interest rate of country 1, and that will show the upward shift of investment of country 1. It will lead to the increase of the real income of country 1.

The decrease of the rate of change of money supply in country 2 will lead to the increase of the interest rate, and also show the decrease of the investment of country 2. And it will lead to the decrease of the real income of country 2.

4. Concluding Remarks

We will consider the implication of this model. Let us consider the case which the real national income becomes smaller than the equilibrium value by some exogenous causes for simplicity. The decrease in the real income will lead to the decrease of consumption and decrease of investment. And also the decrease in the real income will lead to the decrease of the money demand and income tax. And if the real national income become smaller than the equilibrium value, the actual rate of inflation will be adapted by the gap between the actual and target level of real income.

We will consider the regional macroeconomic model with two-country, international trade. We assume the price levels in each country are exogenous variables, and the change rate of money supply in each country will regulate dynamically by the net export of each country.

We will consider the implication of the interregional trade macroeconomic model with two country, international trade. Let us consider the real national income of country 1 becomes smaller than the equilibrium value by some exogenous reason. The decrease of real income of country 1 will lead to the decrease of consumption, investment, and import in country 1 and export of country 2. The increase of net export of country 2 will lead to the increase of the real income of country 2. Besides, the decrease of real income of country 1 will also lead to the decrease of the rate of change of money supply in country 2 and the increase of the rate of change of money supply in country 1. The increase of the rate of change of money supply in country 1 will lead to decrease of interest rate of country 1, and that will show the upward shift of investment of country 1. It will lead to the increase of the real income of country 1.

The decrease of the rate of change of money supply in country 2 will lead to the increase of the interest rate, and also show the decrease of the investment of country 2. And it will lead to the decrease of the real income of country 2.

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