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More about Bernoulli Numbers

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Abstracts: Bernoulli Numbers are coded with Deterministic Redundancy of Arithmetic Operations, adding and multiplying or exponent, in Natural Number System. And based on the redundancy, a process for obtaining the Bernoulli Numbers is elaborated.

Key Words: Arithmetic Operation, Number Process, Repeating, Matrices, and Determinant.

Giving $s_0=0$ and build the below number sequence.

$$s_i=s_{i-1}+d(i),\ i\in N$$

Then d(i) is also a number sequence and the sum of the first n terms of d(i) can be described as below, which is independent from the s sequence by which it is carried though.

$$\sum d(i)=s_n-s_0=s_n \tag{a}$$

Consider d(i)=i^p, we have s_n as below.

$$s_n=\sum i\hat{}p$$

By observation, the above sum of n-items can be expressed with a (p+1)'s polynomial of n as below.

$$s_n=\sum a_{ij}*n\hat{}j,\ i,\ j=0 \text{ to } p+1$$

with p+2 items, a_{ij}'s, j=0 to p+1, are the Bernoulli Numbers<1>, and the below set of linear equations of a_{ij} are formed.

$$\sum a_{ij}*n\hat{}j=\sum i\hat{}p,\ i=0 \text{ to } n \text{ and } j=0 \text{ to } p+1$$

Which will be rational Diophantine (*Non Deterministic*) if n<p+1 or there are solutions for a_{ij} for which at least p+2 equations can be formed, i.e. j=0 to p+1. And giving p, n∈0 & N and p<n, without loss of generality, it is obvious to have the below results.

$$a_{0j}=0 \text{ for any } j=0 \text{ to } p+1$$

$$\sum a_{ij}*n\hat{}j=\sum i\hat{}p \text{ for } j=1 \text{ to } p+1 \text{ and } i=1 \text{ to } n \tag{b}$$

i.e. a p+1-degree polynomial of n is enough to define ∑i^p, i=1 to n, the sum of p-power of the consecutive natural numbers equal to and less than n, *The Deterministic Redundancy appears when n>p+1*.

By analogy, n^(p+1) can be a summation of p-degree polynomial of "i" for i=1 to n. i.e. the below set of linear equations for b_{pj}.

$$n^{p+1}=\sum b_{pi}*j\hat{}i,\ i=0 \text{ to } p,\ j=1 \text{ to } n \tag{c}$$

It is easy to observe that the solutions for b_{pi}'s correspond to entries of the Pascal's Triangle, and the below equations.

$$b_{pi}=(-1)^{p+i}*\binom{i}{p+1},\ i=0 \text{ to } p \tag{d}$$

And by expanding equations (c), we have:

$$n^{p+1}=\sum(b_{pp}*j^p+b_{pp-1}*j^{p-1}+b_{pp-2}*j^{p-2}+\ldots$$
$$+b_{p2}*j^2+b_{p1}*j+b_{p0}),\ j=1 \text{ to } n.$$

That is,

$$\sum j^p=[n^{p+1}-\sum(b_{pp-1}*j^{p-1}+b_{pp-2}*j^{p-2}+\ldots$$
$$+b_{p2}*j^2+b_{p1}*j+b_{p0})]/b_{pp} \tag{e}$$

In which, $\sum j^p$ can be expressed by a summation of $\sum j^k$ combination and k=0 to p-1 and is less then p, which can be ordered down to p=0, i.e. ∑1=n.

And we have:

$$a_{p+1p+1}=1/b_{pp}=1/(p+1),$$

$$a_{p+1p}=-(-1)*b_{pp-1}/(b_{p-1p-1}*b_{pp})=1/2.$$

Actually, the Bernoulli Numbers a_{ij} can be obtained by the below iteration process.

$$p=0,\ b_{00}=1, \text{ then } a_{11}=1, \text{ i.e. } \sum 1=n$$

$$p=1,\ b_{10}=-1,\ b_{11}=2, \text{ then } a_{22}=1/2,\ a_{21}=1/2, \text{ i.e.}$$
$$\sum j=n^2/2+n/2$$

$$p=2,\ b_{20}=1,\ b_{21}=-3,\ b_{22}=3,$$

and we have the equations (e) as below.

$$\sum j^2=\{n^3-\sum[(-3)*j+1]\}/3=n^3/3+$$
$$\sum j-\sum 1/3=n^3/3+n^2/2+n/2-n/3=n^3/3+n^2/2+n/6, \text{ i.e.}$$
$$a_{31}=1/6,\ a_{32}=1/2,a_{33}=1/3$$

$p=3$, $b_{30}=-1$, $b_{31}=4$, $b_{32}=-6$, $b_{33}=4$,

and then we have the equations (e) as below.

$\sum j^3=\{n^4-\sum[(-6)*j^2+4*j-1]\}/4=n^4/4+\sum j^2*6/4-\sum j+\sum 1/4=n^4/4+n^3/2+n^2/4$, i.e.

$a_{41}=0$, $a_{42}=1/4$, $a_{43}=1/2$, $a_{44}=1/2$.

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Mathematics is there to be discovered and to be recognized.

References

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