More about Bernoulli Numbers

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Abstracts: Bernoulli Numbers are coded with Deterministic Redundancy of Arithmetic Operations, adding and multiplying or exponent, in Natural Number System. And based on the redundancy, a process for obtaining the Bernoulli Numbers is elaborated.

Key Words: Arithmetic Operation, Number Process, Repeating, Matrices, and Determinant.

Given \( s_0 = 0 \) and build the below number sequence.

\[
s_i = s_{i-1} + d(i), \quad i \in \mathbb{N}
\]

Then \( d(i) \) is also a number sequence and the sum of the first \( n \) terms of \( d(i) \) can be described as below, which is independent from the \( s \) sequence by which it is carried through.

\[
\sum d(i) = s_n, \quad s_0 = s_n
\] (a)

Consider \( d(i) = i \cdot p \), we have \( s_n \) as below.

\[
s_n = \sum i \cdot p
\]

By observation, the above sum of \( n \)-items can be expressed with a \((p+1)\)'s polynomial of \( n \) as below.

\[
s_n = \sum a_{ij} \cdot n^j, \quad i, j = 0 \text{ to } p+1
\]

with \( p+2 \) items, \( a_{ij} \)'s, \( j = 0 \text{ to } p+1 \), are the Bernoulli Numbers in the below set of linear equations of \( a_{ij} \) are formed.

\[
\sum a_{ij} \cdot n^j = \sum i \cdot p, \quad i=0 \text{ to } n \text{ and } j=0 \text{ to } p+1
\]

Which will be rational Diophantine (Non Deterministic) if \( n < p+1 \) or there are solutions for \( a_{ij} \) for which at least \( p+2 \) equations can be formed, i.e. \( j=0 \text{ to } p+1 \). And giving \( p, n \in \mathbb{N} \) and \( p < n \), without loss of generality, it is obvious to have the below results.

\[
a_{ij} = 0 \text{ for any } j = 0 \text{ to } p+1
\]

\[
\sum a_{ij} \cdot n^j = \sum i \cdot p \text{ for } j=1 \to p+1 \text{ and } i=1 \to n
\] (b)

i.e. a \((p+1)\)-degree polynomial of \( n \) is enough to define \( \sum i \cdot p \), \( i=1 \text{ to } n \), the sum of \( p \)-power of the consecutive natural numbers equal to and less than \( n \), The Deterministic Redundancy appears when \( n > p+1 \).

By analogy, \( n'\cdot(p+1) \) can be a summation of \( p \)-degree polynomial of “\( i \)” for \( i=1 \to n \). i.e. the below set of linear equations for \( b_{pj} \).

\[
n^{p+1} = \sum b_{pj} \cdot i^j, \quad i=0 \to p
\]

It is easy to observe that the solutions for \( b_{pj} \)'s correspond to entries of the Pascal’s Triangle, and the below equations.

\[
b_{pj} = (-1)^{p+1} \binom{i}{p+1}, \quad i=0 \to p
\] (c)

And by expanding equations (c), we have:

\[
n^{p+1} = \sum (b_{pp} \cdot i^p + b_{pp-1} \cdot j^p + b_{pp-2} \cdot j^p + \ldots + b_{p2} \cdot j^2 + b_{p1} \cdot j + b_{p0})
\]

That is,

\[
\sum j^p = \sum (b_{pp} \cdot i^p + b_{pp-1} \cdot j^p + b_{pp-2} \cdot j^p + \ldots + b_{p2} \cdot j^2 + b_{p1} \cdot j + b_{p0})
\]

By analogy, \( n'j^p \) can be expressed by a summation of \( \sum \) combination and \( k=0 \text{ to } p-1 \) and is less then \( p \), which can be ordered down to \( p=0 \), i.e. \( \sum 1=n \).

And we have:

\[
a_{p+1p+1} = 1/b_{pp} = 1/(p+1),
\]

\[
a_{p+1p} = (-1) \cdot b_{pp-1}/(b_{p+1p-1} \cdot b_{pp}) = 1/2.
\]

Actually, the Bernoulli Numbers \( a_{ij} \) can be obtained by the below iteration process.

\[
p=0, \quad b_{00}=1, \text{ then } a_{11}=1, \text{ i.e. } \sum 1=n
\]

\[
p=1, \quad b_{10}=1, \quad b_{11}=2, \text{ then } a_{22}=1/2, \quad a_{21}=1/2, \text{ i.e. } \sum j=n^2/2+n/2
\]

\[
p=2, \quad b_{20}=1, \quad b_{21}=-3, \quad b_{22}=3,
\]

and we have the equations (e) as below.

\[
\sum j^3 = \{3 \cdot \sum (i-3)i^3 \}/3 = n^3/3+
\]

\[
\sum j \cdot \sum 1/3 = n^3/3+n^2/2+n/2-n/3 = n^3/3+n^2/2+n/6, \text{ i.e. }
\]

\[
a_{31}=1/6, \quad a_{32}=1/2, a_{33}=1/3
\]
p=3, b_{30}=-1, b_{31}=4, b_{32}=-6, b_{33}=4, and then we have the equations (e) as below.

\[ \sum_{j}^{3} = \{ n^4 - \sum ((-6)j^2 + 4j - 1) \}/4 = n^4/4 + \sum j^2/6 + \sum j + 1/4 = n^4/4 + n^3/2 + n^2/4, \text{ i.e.} \]

\[ a_{41} = 0, a_{42} = 1/4, a_{43} = 1/2, a_{44} = 1/2. \]

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Mathematics is there to be discovered and to be recognized.

References