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Bernoulli Numbers & Vandermonde Matrices

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Abstracts: Matrix structuring is a very beautiful way to place Bernoulli numbers, by which a new view to the numbers is opened. Natural Numbers are mathematics seeds and Natural Number System (NNS) breeds the whole world mathematically.

Key Words: Algebra, Arithmetic Operation, Number Process, Repeating, Matrices, and Determinant.

Giving $s_0=0$ and build the below number sequence.

$$s_i=s_{i-1}+d(i),\ i\in N$$

Then d(i) is also s number sequence and the sum of the first n terms of d(i) can be described as below.

$$\sum d(i)=s_n-s_0=s_n \tag{a}$$

This Equation (a) is the seed for differentials and integrals.

Consider d(i)=i^p, we have s_n as below.

$$s_n=\sum i\text{^}p$$

By observation, the above sum of n-items can be expressed with a (p+1)'s polynomial as below.

$$s_n=\sum a_i*n\text{^}i,\ i=0 \text{ to } p+1$$

with p+2 items, and the below set of linear equations of a_j are formed.

$$\sum a_i*n\text{^}i=\sum j\text{^}p,\ i=0 \text{ to } p+1 \text{ and } j=0 \text{ to } n$$

Which will be rational Diophantine if n<p+1 or there are solutions for a_i for which at least p+2 equations can be formed, i.e. j=0 to p+1. And giving p, n∈0 & N and p<n, without loss of generality, it is obvious to have the below results.

$$a_0=0 \text{ for any } p$$

$$\sum a_i*n\text{^}i=\sum j\text{^}p \text{ for } i=1 \text{ to } p+1 \text{ and } j=1 \text{ to } n \tag{b}$$

i.e. p+1 degree polynomial of n is enough to define ∑j^p, j=1 to n, the sum of p powers of the consecutive natural numbers equal to and less than n, which, the Equation (b), is *the seed for Fourier Transformations*.

The above linear equations can also be rewritten as below in general.

$[r_{ij}]x[a_{ij}]=[s_{ij}]$

While i, j=1 to p+1 and $r_{ij}=i\text{^}j$, $s_{ij}=\sum k\text{^}(j-1)$ and k=1 to i.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots\cdots & 1 & 1 \\ 2 & 2^2 & 2^3 & 2^4 & \cdots\cdots & 2^p & 2^{p+1} \\ 3 & 3^2 & 3^3 & 3^4 & \cdots\cdots & 3^p & 3^{p+1} \\ 4 & 4^2 & 4^3 & 4^4 & \cdots\cdots & 4^p & 4^{p+1} \\ \cdots\cdots & & & & & & \\ p & p^2 & p^3 & p^4 & \cdots\cdots & p^p & p^{p+1} \\ (p+1) & (p+1)^2 & \ldots & & & (p+1)^p & (p+1)^{p+1} \end{pmatrix} \times \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1p} & a_{1p+1} \\ 0 & a_{22} & a_{23} & \cdots & a_{2p} & a_{2p+1} \\ 0 & 0 & a_{33} & \cdots & a_{2p} & a_{2p+1} \\ \cdots\cdots & & & & & \\ \cdots\cdots & & & & & \\ 0 & 0 & 0 & \cdots & a_{pp} & a_{pp+1} \\ 0 & 0 & 0 & \cdots & 0 & a_{p+1p+1} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1p} & s_{1p+1} \\ s_{21} & s_{22} & s_{23} & \cdots & s_{2p} & s_{2p+1} \\ s_{31} & s_{32} & s_{33} & \cdots & s_{2p} & s_{2p+1} \\ \cdots & & \boxed{s_{ij}=\sum k^{j-1},\ k=1 \text{ to } i} & & & \\ \cdots & & & & & \\ s_{p1} & s_{p2} & s_{p3} & \cdots & s_{pp} & s_{pp+1} \\ s_{p+11} & s_{p+12} & \cdots & & s_{p+1p} & s_{p+1p+1} \end{pmatrix}_{[1]} \tag{c}$$

e.g. (Matrices are written in a line, one row after another.)

p=0, r_{11}=1, s_{11}=1 and a_{11}=1;

p=1, $r_{11}/r_{12}/r_{21}/r_{22}$=1/1/2/4, $s_{11}/s_{12}/s_{21}/s_{22}$=1/1/2/3 and $a_{11}/a_{12}/a_{21}/a_{22}$=1/"1/2"/0/"1/2";

p=2, $r_{11}/r_{12}/r_{13}/r_{21}/r_{22}/r_{23}/r_{31}/r_{32}/r_{33}$=1/1/1/2/4/8/3/9/27,

$s_{11}/s_{12}/s_{13}/s_{21}/s_{22}/s_{23}/s_{31}/s_{32}/s_{33}$=1/1/1/2/3/5/3/6/14,

$a_{11}/a_{12}/a_{13}/a_{21}/ a_{22}/a_{23}/a_{31}/ a_{32}/a_{33}$=1/"1/2"/"1/6"/0/"1/2"/"1/2"/0/0/"1/3";

…………

Det|[r_{ij}]|=∏k!, k=1 to p+1.

[r_{ij}] is a Vandermonde matrix and [s_{ij}] is a matrix with sums of powers of all related natural numbers, and [a_{ij}] can be called *matrix with Bernoulli numbers*, which is upper triangle. So,

Det|[s_{ij}]|=∏k!, k=1 to p.

Because [sij] can obviously be transformed to a Vandermonde by replacing k-row with each item subtracting corresponding item of k-1-row, k=2 to p+1. And so,

Det|[a_{ij}]|=1/(p+1)!.

Giving a natural number p, the corresponding Bernoulli numbers a_{ip} (i=1 to p) can be obtained by solving the below linear p-equation set, i.e. j=1 to p.

[r_{ji}][a_{ip}]=[s_{ip}]

While r_{ji}=j^i, s_{ip}=∑i^(p-1)

Triangulating [r_{ij}]

Giving [r_{ij}]$_p$ of p-order square matrix, after triangulation, elements of the matrix will be transformed into below forms.

r_{1j} unchanged, j=1 to p,

r_{21}=0, r_{2j}=R2_{2j}/R1_{11}[2], while R1_{11}=Det|r_{11}|=r_{11}, R2_{2j}=Det|[$r_{11}/r_{1j}/r_{21}/r_{2j}$]| and j=2 to p,

$r_{31}=r_{32}$=0, r_{3j}=R3_{3j}/R2_{22}, while R3_{3j}=Det|[$a_{11}/r_{12}/r_{1j}/r_{21}/r_{22}/r_{2j}/r_{31}/r_{32}/r_{3j}$]| and j=3 to p,

…………

$r_{p-11}=r_{p-12}=\ldots=r_{p-1p-2}$=0, r_{p-1j}=R(p-1)$_{p-1j}$/R(p-2)$_{p-2p-2}$,

While R(p-1)$_{p-1j}$=Det|[$r_{11}/\ldots/r_{1p-2}/r_{1j}/\ldots/r_{p-11}/\ldots/r_{p-1p-2}/r_{p-1j}$]| and j=p-1 and p, and

$r_{p1}=r_{p2}=\ldots=r_{pp-1}$=0, r_{pp}=Rp$_{pp}$/R(p-1)$_{p-1p-1}$.

While Rp$_{pp}$=Det|[r_{ij}]|, and

Until here the [r_{ij}] has been transformed into an upper-triangle-matrix. And r_{kk} (k=1 to p+1) for the above Vandermonde matrix [r_{ij}] can be obtained as below after triangulation process.

r_{kk}=k!.

And for [s_{ij}], s_{kk}=(k-1)!, after triangulation. So,

$a_{kk}=s_{kk}/r_{kk}$=1/k, k=1 to p+1.

And it is also not very difficult to get a_{kk+1}=1/2 (k=1 to p).

[s_{ij}] & [r_{ij}] and Vandermonde Matrix

After a transformation of $s_{kj}=s_{kj}-s_{k-1j}$ (k=2 to p+1) and s_{1j}=1 (j=1 to p+1), [s_{ij}] becomes a unit standardized Vandermonde [v_{ij}][3] while the same for [r_{ij}] after $r_{ij}=r_{ij}/r_{i1}$ transformation, r_{i1}=i.

v_{ij}=i^(j-1) and i, j=1 to p+1

The determinant of |s_{ij}| keeps the same after the above said transformation, i.e. ∏k! (k=1 to p), but the determinant of |r_{ij}| should be multiplied by (p+1)!, which is exactly the Det|a_{ij}|.

By continuing triangulating $[r_{ij}]$ and $[s_{ij}]$, it is obvious to conclude the below result, in which the left one is $[r_{ij}]$ and the right one $[s_{ij}]$.

$$
\begin{array}{c|c}
r\backslash c & \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots j \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \ldots \\ i \end{array} &
\left(\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots 1 \\
0 & 2 & 6 & 14 & 30 & 62 & 126 & 254 & \ldots 2*s_{2j} \\
0 & 0 & 6 & 36 & 150 & 540 & 1806 & 5796 & \ldots 3*s_{3j} \\
0 & 0 & 0 & 24 & 240 & 1560 & 8400 & 40824 & \ldots 4*s_{4j} \\
0 & 0 & 0 & 0 & 120 & 1800 & 16800 & 126000 & \ldots 5*s_{5j} \\
0 & 0 & 0 & 0 & 0 & 720 & 15120 & 191520 & \ldots 6*s_{6j} \\
0 & 0 & 0 & 0 & 0 & 0 & 5040 & 141120 & \ldots 7*s_{7j} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 40320 & \ldots 8*s_{8j} \\
\ldots & \ldots & \ldots & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots i*s_{ij}
\end{array}\right)
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots j \end{array} \\
\left(\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots 1 \\
0 & 1 & 3 & 7 & 15 & 31 & 63 & 127 & \ldots 1+r_{2j-1} \\
0 & 0 & 2 & 12 & 50 & 180 & 602 & 1932 & \ldots r_{2j-1}+r_{3j-1} \\
0 & 0 & 0 & 6 & 60 & 390 & 2100 & 10206 & \ldots r_{3j-1}+r_{4j-1} \\
0 & 0 & 0 & 0 & 24 & 360 & 3360 & 25200 & \ldots r_{4j-1}+r_{5j-1} \\
0 & 0 & 0 & 0 & 0 & 120 & 2520 & 31920 & \ldots r_{5j-1}+r_{6j-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 720 & 20160 & \ldots r_{6j-1}+r_{7j-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5040 & \ldots r_{7j-1}+r_{8j-1} \\
\ldots & \ldots & \ldots & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots r_{i-1j-1}+r_{ij-1}
\end{array}\right)
\end{array}
$$

These are very interesting and beautiful upper triangle matrices. We can generate the both by giving their 1st rows, which could be seen *as the seeds and the soil,* simply with arithmetic adding and multiplying.

$s_{ii-k}=r_{ii-k}=0$, k=1 to i-1 and i=1 to p+1 (upper-triangulated)

$s_{22}=r_{11}+r_{21}=1$ (sum by adding numbers' power)

$r_{22}=2*s_{22}=2$ (raise the power by multiplying)

$s_{2j}=r_{1j-1}+r_{2j-1}=2$^(j-1)-1 (sum by adding numbers' power)

$r_{2j}=2*s_{2j}$ (j=3 to p+1) (raise the power by multiplying)

…………

$s_{pp}=r_{p-1p-1}+r_{p-1p-2}$ (sum by adding numbers' power)

$r_{pp}=p*s_{pp}$ (raise the power by multiplying)

$s_{pj}=r_{p-1j-1}+r_{pj-1}$ (sum by adding numbers' power)

$r_{pj}=p*s_{pj}$ (j=p & p+1) (raise the power by multiplying)

$s_{p+1p+1}=r_{pp}+r_{pp-1}$ (sum by adding numbers' power)

$r_{p+1p+1}=(p+1)*s_{p+1p+1}$ (raise the power by multiplying)

then the two (p+1)×(p+1) matrices are completely generated.

And the below equations about the two matrices are observed deductively.

$r_{kk}=k*s_{kk}=k*r_{k-1k-1}=k*(k-1)*s_{k-1k-1}$

Considering the Equation (c), we have the below equation.

$s_{ij}=\sum r_{ik}*a_{kj}$ (i=1 to p+1 and k=i to j, j≥i) (d)

i.e.

$a_{ii}=s_{ii}/r_{ii}=1/i$

$a_{ii+1}=(r_{ii+1}/r_{ii})/[i*(i+1)]=1/2$, because

$r_{ii+1}/r_{ii}=i*s_{ii+1}/r_{ii}=i*(r_{i-1i}/r_{ii}+1)=r_{i-1i}/r_{i-1i-1}+i=i*(i+1)/2$ and i=1 to p, by iteration process in equation (a), which also holds for s_{ij}.

It is not difficult to verify the below observed equations.

$s_{k+1k+1}=2*s_{k-1k}$, k=2 to p

$r_{ij}/r_{ik}=s_{ij}/s_{ik}$ (k=i to j)

$s_{ii+1}/s_{i+1i+1}-1/2=r_{ii+1}/r_{i+1i+1}=(i+1)/2$ (i=1 to p+1)

$s_{ii+1}=s_{i+2i+2}/2$

…………

Giving a natural number p, by iteration process, a complete matrix of Bernoulli numbers can be generated with the below equation.

$[a_{ij}]_{p+1p+1}=[r_{ij}]^{-1}{}_{p+1p+1}[s_{ij}]_{p+1p+1}$.

Mathematics is there to be discovered and to be recognized.

Notes:

[1] $[s_{ij}]$ is as below.

$$[s_{ij}]=\begin{pmatrix} 1 & 1 & 1 & \cdots\cdots & 1 & 1 \\ 2 & 3 & 5 & \cdots\cdots & s_{2p} & s_{2p+1} \\ 3 & 6 & 14 & \cdots\cdots & s_{2p} & s_{2p+1} \\ \cdots\cdots\cdots\cdots & & & & & \\ \cdots\cdots\cdots\cdots & & & & & \\ p & s_{p2} & & \cdots\cdots & s_{pp} & s_{pp+1} \\ p+1 & s_{p+12} & & \cdots\cdots & s_{p+1p} & s_{p+1p+1} \end{pmatrix}$$

$$\boxed{\begin{array}{l} s_{ip}=\sum k^{p-1},\ k=1 \text{ to } i \\ s_{pj}=\sum k^{j-1},\ k=1 \text{ to } p \\ s_{ij}=\sum k^{j-1},\ k=1 \text{ to } i \end{array}}$$

[2] $R(k)_{ij}$ indicates the determinant of a k-order square matrix of the below pattern (i and j=k+1 to p), and the parentheses for k can be omitted.

$$R(k)_{ij}= \operatorname{Det}\begin{pmatrix} r_{11} & r_{12} & \cdots\cdots\cdots\cdots & r_{1k-1} & r_{1j} \\ r_{21} & r_{22} & \cdots\cdots\cdots\cdots & r_{2k-1} & r_{2j} \\ r_{31} & r_{32} & \cdots\cdots\cdots\cdots & r_{3k-1} & r_{3j} \\ \cdots\cdots\cdots\cdots & & & & \\ r_{k-11} & r_{k-12} & \cdots\cdots & r_{k-1k-1} & r_{k-1j} \\ r_{i1} & r_{i2} & r_{i3}\cdots\cdots\cdots\cdots & r_{k-1i} & r_{ij} \end{pmatrix}$$

Then $R0_{ij}=1$ (i and j=1 to p), $R1_{ij}=r_{11}$ (i and j=2 to p), and $Rp_{00}=[r_{ij}]$ (i and j=1 to p).

[3] Giving an n-order standardized Vandermonde $v_{ij}=a_i$^(j-1) and i, j=1 to n, $[v_{ij}]$ is called a Unit Standardized Vandermonde when $a_i=i$ (i=1 to n).

References

[1] Teubner Taschenbuch der Mathematik, E. Zeidler.