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Station Cone Algorithm for Linear Programming

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Abstract: Recently we have proposed anew method combininginterior and exterior approaches to solve linear programming problems. This method uses an interior point, and from there connected to the vertex of the so called station cone which is also a solution of the dual problem. This allows us to determine the entering vector and the new station cone. Here in this paper, we present a new modified algorithm for the case, when at each iteration we determine a new interior point. The new building interior point moves toward the optimal vertex. Thanks to the shortened from both inside and outside, the new version allows to find quicker the optimal solution. The computational experiments show that the number of iterations of the new modified algorithm is significantly smaller than that of the second phase of the dual simplex method.

Keywords: Linear programming, simplex method, station cone

1. Introduction

Inventing linear programming by Danzig [4] in 1947 is recognized as one of the greatest mathematical discoveries of the 20th century. Since then, thousands of papers and monographs have appeared and dedication to this important mathematical field [see 1, 5, 12, 13]. Great sense of Danzig's simplex method is probably not in the mathematical difficulty level, which is at the level of broad application in all areas of human life. Therefore any extension or modification of the simplex algorithm toward better are welcome.

In 1979 Khachian has opened a new horizon for the linear programming as prove that the linear programming problem be solved in polynomial time [7]. Khachian's ellipsoid method gives a bound of $O(n^5L)$ arithmetic operations on number with $O(nL)$ digits. Despite its major theoretical advance, the ellipsoid method had little practical impact as the simplex method is more efficient for many classes of linear programming problems [1, 8, 12, 13].

Other important invention of the linear programming was in 1984, when Kamarkar [6] proposed a new projective method for linear programming which requires $O(n^{3.5}L)$ operations. Kamarkar's algorithm not only improved Khachian's theoretical worst-case polynomial bound but in fact provides practical test results better than the simplex method. There are several important open problems in the theory of linear programming. One of them is the question: To be or not a variations of simplex algorithm that run in polynomial time?

In [3] Chu N.N, Duong P.C and Hue L.T have proposed a new algorithm combining interior and exterior approaches to solve linear programming problems. This method can be viewed as a variation of simplex method in combination with interior approach. Here in this paper, we present a new modified algorithm for the case, when at each iteration we determine a new interior point. The new building interior point moves toward the optimal vertex. Thanks to the shortened from both inside and outside, the new version allows to find quicker the optimal solution.

The paper is organized as follows. In section 2 we introduce the concept of station cone which is fundametal important for the construction of the algorithm. In section 3, we describe the criterion of

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selecting the leaving variables. The section 4 proposes the selecting rule for entering vectors. The section 5 describes the ogirinal algorithm in [3]. The new modified algorithm is presenting in section 6. The section 7 presents the computational experiments. Finally, some discussions have been made in section 8.

2. Station Cone

For the convenience of the reader, we would like to briefly present here the concept station cone[3]. Consider a linear programming problem in the matrix form

$$\begin{aligned} &\max\ \langle c,x\rangle \\ &x\in P:=\{x|Ax\le b,\ x\ge 0\}, \end{aligned} \tag{2.1}$$

where $c\in\mathbb{R}^n, A\in A^{mxn}, b\in\in\mathbb{R}^m, \forall x\in\mathbb{R}^n$. Let $A_1, A_2,...,A_m$ denote the row vectors.Through this paper we suppose that (2.1) and its dual problem are nondegenerated.We also suggest the feasible region P of (2.1) has strict interior points. For simplicity of argument, we assume that the matrix A has full column rank n and $n < m$.

Let $I_n=\{i_1,i_2,...,i_n\}\subset\{1,2,...,m\}$ such that the vectors $A_i,\ i\in I_n$ are linear independent. This means the vector $A_i,\ i\in I_n$ establish a basis of R^n. Therefore any vector $A_l\in\mathbb{R}^n$ can be expressed as a linear combination of the vectors $A_i, i\in I_n$. Let λ_{li_k} be the linear coefficient of the vector A_l in the basis $A_{i_k},\ i_k\in I_n$, then

$$a_{lj}=\sum_{k=1}^{n}\lambda_{li_k}a_{i_kj},\ j=1,2,...,n,\ l=1,2,...,m.$$

Consider the system of homogeneous linear inequalities

$$A_{i_k}x\le 0,\ \ i_k\in I_n. \tag{2.2}$$

We indeed need to introduce the following definition.

Definition 1. *Thelinear inequality*

$$A_l x\le 0 \tag{2.3}$$

is called the consequent linear inequality of the system (2.2) if and only if all the solutions of the system (2.2) satisfy the linear inequality (2.3).

We need the following well known result in theory of linear inequalities.

Theorem 2.1 [2]. *The linear inequality (2.3) is a consequent linear inequality of the system (2.2) if and only if*

$$A_l=\sum_{k=1}^{n}\lambda_{li_k}A_{i_k},\ \lambda_{li_k}\ge 0,\ i_k\in I_n$$

Definition 2. *Let polyhedral cone M be defined by system*

$$A_{i_1}x\le b_{i_1}, A_{i_2}x\le b_{i_2}\ ,\ ....,A_{i_n}x\le b_{i_n},$$

where $A_{i_1}, A_{i_2},...,A_{i_n}$ *are linear independent. Then M is called a station cone if the vector c is a nonnegative linear combination of the vectors* $A_{i_1}, A_{i_2},...,A_{i_n}$. *The vertex x* is called a station solution and the vectors* $A_{i_1}, A_{i_2},...,A_{i_n}$ *is called a basis of a station cone.*

In other words, the solutions of the system of linear inequalities that create the station cones satisfy the inequality $\langle c,x\rangle\le\langle c,x^*\rangle$, whereas x^* is the vertex of the station cones. This is equal to the fact that the inequality $\langle c,x\rangle\le\langle c,x^*\rangle$ is the consequent inequality of the system of the linear inequalities, which formulate the station cone. This also means that the vector c is the nonnegative linear combination of the basic vectors of the station cone.

We have the following result

Theorem 2.2 [3]. *If the station solution x* satisfies all the constraints of the problem (2.1) then x* is an optimal solution.*

3. Selecting the Leaving Vector

In this section, for convenience, we will repeat the rule for selecting the leaving vector [3]. Let $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$ be the basis of the station cone and

$$c = \sum_{k=1}^{n} \lambda_{k0} A_{i_k},\ A_j = \sum_{k=1}^{n} \lambda_{kj} A_{i_k},\quad j = 1,2,\ldots m$$

Then from definion 2.1 follows that

$$\lambda_{ko} \geq 0,\quad \forall k = 1,2,\ldots n.$$

We assume that all λ_{ko} are strictly positive, i.e.

$$\lambda_{k0} > 0,\ k = 1,2,\ldots,n.$$

It is obvious that $\lambda_{k0} > 0,\ k = 1,2,\ldots,n;$ $\lambda_{k0} = 0,\ k = n+1,\ldots,m$ is a basis solution of the dual problem of (2.1):

$$\min\ \left\{ \langle b, \lambda \rangle \setminus A^T \lambda \geq c^T, \lambda \geq 0 \right\} \qquad (3.1)$$

where $\lambda \in R^m$. The assumption $\lambda_{k0} > 0,$ $k = 1,2,\ldots,n$ means that the dual problem (3.1) is nondegenerated.

So we have proved the following

Theorem 2.3 [3]. *Let $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$ be the basis of the station cone. Suppose we replaced A_{i_r} by A_s. Then $A_{i_1}, \ldots, A_{i_{r-1}}, A_s, A_{i_{r+1}}, \ldots, A_{i_n}$ is the basis of the station cone if the leaving vector A_{i_r} was chosen by condition*

$$\frac{\lambda_{r0}}{\lambda_{rs}} = \min_k \frac{\lambda_{k0}}{\lambda_{ks}},\ \lambda_{ks} > 0,\ \lambda_{rs} > 0. \qquad (3.4)$$

Theorem 2.4 [3]. *Among the coefficients λ_{ks}, k = 1, 2,..., n at least one λ_{rs} exists such that $\lambda_{rs} > 0$.*

4. Selecting the Entering Vector

The idea of algorithm in [3] is moving from one vertex x^k of a station cone M^k to another vertex $x^{[k+1]}$ of another station cone M^{k+1} with a better value of the objective function. The movement depends on the cutting hyperplane $A_s x = b_s$ which will be defined by the intersection of the feasible polytope *P* and the segment connecting the vertex x^k of the station cone M^k and the given interior point $O \in P$. The movement stops when the vertex x^k of the station cone M^k becomes a feasible point.

Let *O* be a strict interior point of *P*. Denoted by $0^i,\ i = 1,2,\ldots,n$ the projections of O onto n facets of the station cone M^k. Let $H_i,\ i = 1,2,\ldots,n$ be the intersection points of the boundary of *P* and the segments $0,\ 0^i,\ i = 1,2,\ldots,n$. Then the new point O^* will be calculated by the following formula

$$O^* = \frac{1}{n+1}\left(\sum_{i=1}^{n} H_i + O \right) \qquad (4.5)$$

It is obvious that O^* in (4.5) is the barycenter of the polytope $H_1, H_2, \ldots, H_n, 0.$ Let us connect the point O^* with vertex x^k of the station cone M^k. Let z^k denote the intersection point of P and $[O^*, x^k]$, such that $z^k \in P, (z^k, x^k] \notin P$. Then the inequality $A_s x \leq b_s$ with $A_s z^k = b^s$ will bechosen as entering variable. This means the inequality $A_s x \leq b_s$ will enter the next station cone $M^{\{k+1\}}$(if $A_i z^k = b_i$ for some $H_1, H_2, \ldots, H_n, 0.$ then we can choose any $i \in \{i_1, i_2, \ldots, i_k\}$). The point z^k will be calculated as follows.

Denote $\overline{I} \subset \{1,2,\ldots,m\}$ such that $A_i\ x^k > bi,$ i = 1, 2, …, *m*, we have to find $\lambda_i,\ i \in \overline{I}$ such that $A_i z_i = b_i,\ i \in \overline{I},$ i.e.

$$A_i(\lambda_i O + (1-\lambda_i)x^k) = b_i,\ 0 < \lambda_i < 1,\ i \in \overline{I}.$$

Therefore

$\lambda_s =$
$\max\left\{ \lambda_i \,\middle|\, z_i = \lambda_i O + (1-\lambda_i)x^k,\ A_i z_i = b_i,\ 0 < \lambda_i < 1,\ \mathrm{i} \in \overline{\mathrm{I}} \right\}$

will define the cutting hyperplane $A_s x = b_s$ and A_s is the entering vector into the next station cone

M^{k+1}. If $A_i z^k = b_i$ for $i \in \{i_1, i_2, ..., i_k\}$ then we can choose any $i \in \{i_1, i_2, ..., i_k\}$.

Theorem 2.5 [3]. *Let* x^k *be a vertex of* M^k *at step k. Suppose* x^k *is a unique optimal solution of* $\langle c, x \rangle$, $\forall x \in M^k$. *Then*

$$\langle c, x^{k+1} \rangle < \langle c, x^k \rangle .$$

Remark 2.1. The assumption x^k is a unique optimal solution of $\langle c, x \rangle$ on M^k which is equivalent to the assumption that the vector c is a strict positive linear combination of the basis vectors of M^k i.e. $\lambda_{k0} > 0,\ \forall k = 1, 2, ..., n$. This means the dual problem (3.1) of (2.1) is nondegenerated.

5. Interior Exterior Algorithm

For convernience, we will describe here the algorithm was proposed in [3].

1. Initialization

Determine the starting station cone M. Calculate the point O^* by formula (4.5).

Let: $M^k = M$; $O = O^*$.

2. Step $(\boldsymbol{k = 1, 2, ...})$

If the vertex x^k of the station cone M^k is a feasible point of P, then x^k is an optimal solution. In the contrary case, select the inequality $A_s x \leq b_s$ for entering the station cone and define the inequality $A_{i_r} x \leq b_{i_r}$ for leaving the station cone. Determine the new station cone $M^{\{k+1\}}$ with the vertex $x^{\{k+1\}}$.

Go to next step $k = k + 1$.

Remark.Except for the calculation for finding the entering variable, each step of the above algorithm is a simplex pivot.

With the assumption that the dual problem (3.1) of (2.1) is nondegenerated, then

Theorem2.6 [3]. *The above algorithm produces an optimal solution after a finite number of iterations.*

6. New Modified Algorithm

Unlike the algorithm in Section 5, in the section below,we will develop algorithms that at each iteration k will have to find new points O_k. The sequence of interior points O_k moves toward optimal vertex. And so we conduct parallel two asymptoticall, from outside to inside and from the inside out. The interior piont O_k will be defined as follows:

$$O_{k+1} = \lambda_k O_k + (1 - \lambda_k) z^k, \lambda_k = \frac{1}{2^n}$$

Clearly that O_{k+1} is an interior point of P. Let

$$y^k = \frac{2^n - 2}{2^n} O_k + \frac{2}{2^n} z^k$$

Wenoticed O_{k+1} is also an interior point of the following problem

max<c,x>

$$x \epsilon P^{k+1} \{x | Ax \leq b. x \geq 0, (c, x) \geq (c, y^k)\}.(6.1)$$

Obviously, the constraint $(c, x) \geq (c, y^k)$ has eliminated part of the feasible region P^k. So the problem (6.1) has smaller feasible region after each iteration.

In [9,10] K. G. Murty has shown that from the interior point $O_{k+1} \epsilon P^{k+1}$, can build the biggest sphere in P^{k+1} with center O^*_{k+1} on the hyperplane $(c, x) = (c, y^k)$, and the construction sphere requires polynomial computational complexity. Here for simplicity, instead of finding the center O^*_{k+1} of the biggest sphere, we find the point O^*_{k+1} as in (4.5). Let $O_{k+1} = O^*_{k+1}$. For convenience, we will call the algorithm which is described below as station cone algorithm.

1. Initialization

Determine the starting station cone M. Calculate the point O^* by formula (4.5).Let:$M^k = M$, $P^k = P$, $O_k = O^*$.

2. Step $(\boldsymbol{k = 1, 2, ...})$

If the vertex x^k of the station cone M^k is a feasible point of P^k, then x^k is an optimal solution. In the contrary case, select the inequality $A_s x \leq b_s$ for entering the station cone and define the inequality $A_{i_r} x \leq b_{i_r}$ for leaving the station cone. Determine the new station cone $M^{\{k+1\}}$ with the vertex $x^{\{k+1\}}$.

Calculate the points : z^k ,y^k, O_{k+1}, O_{k+1}^* . Let $O_{k+1} = O_{k+1}^*$.

Go to next step $k = k + 1$.

7. Computational Experiences

The statione conealgorithm have been tested, using MatLab, on a set of randomly generated linear problems [11] of the form

$$\begin{cases} \max \ \langle c, x \rangle \\ \quad Ax \le b, \end{cases} \quad (7.1)$$

Where $c = (1, 1, \ldots, 1) \in R^n$, A is the full matrix of ($n \times m$) with a_{ij} is randomly generated from the interval [0,1), the vector b has been chosen such that the hyperplanes $\langle A_i, x \rangle = b_i$, $i = 1, \ldots, m$ are tangent to the sphere(0, 1) with center at origin and radius r = 1.To ensure that (7.1) has a finite optimal solution we add the constraints

$$x_i \le 1, \ i = 1, 2, \ldots, n. \quad (7.2)$$

The optimal solution and objective function value of ((7.1)-(7.2)) have been retested by simplex and dual algorithm from MatLab. We tested several hundreds of examples. Due to the limited framework of the article, here we print out 2 table results.

Table 1 $150 \le n \le 300$, $200 \le m \le 700$.

n	m	Problem	Iterations	
			STATION CONE	SIMPLEX
		1	1014	12367
150	200	2	1251	14973
		3	957	11586
		1	1245	14868
150	250	2	1123	13834
		3	976	11676
		1	2238	25476
200	300	2	2153	24650
		3	2630	28314
		1	3183	36429
250	300	2	3242	37878
		3	3465	41568
		1	5357	66946
250	500	2	5403	63309
		3	5162	68936
300	600	1	7746	108541
350	700	1	10065	158096

Table 2. n = 40, 100, 300, 400, 500; m=200,1000

n	m	Problem	Iterations	
			STATION CONE	DUAL SIMPLEX
		1	184	1250
40	200	2	196	1633
		3	212	1485
		1	621	7654
100	200	2	719	8547
		3	708	8288
300	1000	1	7843	238321
400	1000	2	11456	393562
500	1000	3	18305	587656

8. Discussion

1. The main purpose of the station cone algorithm is simultaneously approaching from inside out and outside in. From the outside is moving in from one to otherstation cone. From the inside out is the building sequence of interior points, such that each subsequent point near the optimal solution than the previous point.

2. Each iteration of the statione conealgorithm is a simplex pivot.If we can figure out how to build point O_k so that their numbers are limited on by a polynimial, thenwe can construct a polynomial algorithm for linear programming.

3. Test data is generated randomly. All input matrix are full density. This has made the tests take quite time, especially when *m* and *n* are large enough.

4. Test results show that, with the increase of *m* and *n*, the number of iterations of the dual simplex rose much faster than the number of iterations of the station cone algorithm. In other words, the station cone algorithm has more advantages when *m* and *n* are large numbers.

5.We believe that there is a class of linear programming which allow to construct a polynomial sequence of interior points converging to the optimal solution. And we will try to find a such class of linear programming in future research work.

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