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An Analytıc Method for Interval Bimatrix Games

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Abstract: This paper deals with mxn two-person non-zero sum games with interval pay-offs. An analytic method for solving such games is given. A pair of Nash Equilibrium is found by using the method. The analytic method is effective to find at least one Nash Equilibrium (N.E) for two-person bimatrix games. Therefore, the analytic method for two-person bimatrix games is adapted to interval bimatrix games.

Keywords: Bimatrix game, Nash equilibrium, interval payoff, interval matrix

1. Introduction

Interval game theory concerns about making decisions under conflict caused by opposing interests. Interval analysis allows us to compute with sets on the real line. In applications, interval analysis provides rigorous enclosure of solutions to model equations. For details, one can refer to [4]. Interval valued matrix, whose entries are closed intervals is proposed by many researchers to model some kind of uncertainty.

We present, in this paper an analytic method for interval bimatrix games.The method for bimatrix games provides both an elementary proof of the existence of equilibrium points and an efficient computational method for finding at least one equilibrium point. This method is an algorithm to find N.E. The algorithm was first introduced in [1].

2. Interval Numbers

An interval number $\tilde{a}$ is closed subset of real numbers.Moreover its represented as follows,

$$\tilde{a} = [a_L, a_R] = \{x \in \mathbb{R}: a_L \leq x \leq a_R\}$$

in which a_L and a_R are respectively referred to as the lower and upper bound of the interval $\tilde{a}$ and $a_L \leq a_R$. If $a_L = a_R$, then $\tilde{a} = [a, a]$ is a real number. Midpoint and half-width of an interval number $\tilde{a}$ is defined as follows,

$$m(\tilde{a}) = \frac{a_L + a_R}{2}, w(\tilde{a}) = \frac{a_R - a_L}{2}$$

The set of all interval numbers is represented by $\tilde{\mathbb{R}}$.

2.1 Basic Interval Arithmetic

Let $\tilde{a} = [a_L, a_R]$ and $\tilde{b} = [b_L, b_R]$ be two interval numbers. The arithmetic operations are defined as follows,

$$\tilde{a} + \tilde{b} = [a_L + b_L, a_R + b_R]$$

$$\tilde{a} - \tilde{b} = [a_L - b_R, a_R - b_L]$$

$$\tilde{a}\tilde{b} = [minS, maxS], S = \{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}$$

$$\frac{\tilde{a}}{\tilde{b}} = \tilde{a}.(1/\tilde{b}), (0 \notin \tilde{b}$$

and

$$\frac{1}{\tilde{b}} = \{\tilde{b}: (\tfrac{1}{\tilde{b}}) \in \tilde{b}\} = [\tfrac{1}{b_R}, \tfrac{1}{b_L}]\ \frac{\tilde{a}}{\tilde{b}} = \tilde{a}.\left(\tfrac{1}{\tilde{b}}\right) =$$

$$[a_L, a_R]\left[\tfrac{1}{b_R}, \tfrac{1}{b_L}\right] =$$

$$\left[\min\left\{\tfrac{a_L}{b_R}, \tfrac{a_L}{b_L}, \tfrac{a_R}{b_R}, \tfrac{a_R}{b_L}\right\}, \max\left\{\tfrac{a_L}{b_R}, \tfrac{a_L}{b_L}, \tfrac{a_R}{b_R}, \tfrac{a_R}{b_L}\right\}\right]$$

$$\alpha \in \mathbb{R} \text{ için } \alpha\tilde{a} = \alpha[a_L, a_R] = \begin{cases} \alpha[a_L, a_R], & \alpha \geq 0 \\ \alpha[a_R, a_L], & \alpha < 0 \end{cases}$$

$$\tilde{a}.\frac{1}{\tilde{a}} \neq 1$$

$$\tilde{a} + (-\tilde{a}) \neq 0$$

2.2 Comparasion Between Interval Numbers

An extensive research and wide coverage on interval

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arithmetic and its applications can be found in [5]. A brief comparasion on different interval orders is given in [5] on the basis of decision makers opinion.

Let $\tilde{a} = [a_L, a_R]$ and $\tilde{b} = [b_L, b_R]$ be two disjoint interval numbers. $\tilde{a}$ is less than $\tilde{b}$ if and only if $a_R < a_L$. This is denoted by $\tilde{a} < \tilde{b}$ and the relation is an extension of " $<$ " on the real line. If the closed interval numbers are overlapping, then we use the acceptability index idea suggested by [6].

Let $\widetilde{\mathbb{R}}$ be the set of all closed intervals on the real line $\mathbb{R}$. The function

$$\mathcal{A}: \widetilde{\mathbb{R}} x \widetilde{\mathbb{R}} \longrightarrow [\infty, 0)$$

such that

$$\mathcal{A}(\tilde{a} \prec \tilde{b}) = \frac{m(\tilde{b}) - m(\tilde{a})}{w(\tilde{b}) + w(\tilde{a})}$$

$(w(\tilde{b}) + w(\tilde{a}) \neq 0)$ is called *acceptability function.* Thus, the number $\mathcal{A}(\tilde{a} \prec \tilde{b})$ is called grade of acceptability of the $\tilde{a}$ to be inferior to $\tilde{b}$.

$i)$ $\mathcal{A}(\tilde{a} \prec \tilde{b}) \geq 1$ when $m(\tilde{b}) > m(\tilde{a})$ and $a_R \leq b_L$

$ii)$ $0 < \mathcal{A}(\tilde{a} \prec \tilde{b}) < 1$ when $m(\tilde{b}) > m(\tilde{a})$ and $b_L \leq a_R$

$iii)$ $\mathcal{A}(\tilde{a} \prec \tilde{b}) = 0$ when $m(\tilde{b}) = m(\tilde{a})$. In this case, an extensive research about comparasion of $\tilde{a}$ and $\tilde{b}$ can be found in [6].

Example 1. Let $\tilde{a} = [10,20]$ and $\tilde{b} = [24,28]$ be two interval numbers.Then,

$$\mathcal{A}(\tilde{a} \prec \tilde{b}) = \frac{26 - 15}{2 + 5} = \frac{11}{7} > 1.$$

Hence, $\tilde{a}$ is less than $\tilde{b}$ with full satisfaction.

Example 2. Let $\tilde{a} = [1,5]$ and $\tilde{b} = [3,17]$ be two interval numbers.Then,

$$\mathcal{A}(\tilde{a} \prec \tilde{b}) = \frac{10 - 3}{7 + 2} = \frac{7}{9} \epsilon (0,1).$$

Hence, $\tilde{a}$ is less than $\tilde{b}$ with grade of satisfaction $\frac{7}{9}$.

3. Interval Bimatrix Games

A bimatrix game is a two player game, player I and player II, player I has m pure strategies $\{I_1, I_2, \ldots, I_m\}$ while player II has n pure strategies $\{II_1, II_2, \ldots, II_n\}$. There is no longer a value c ,such that

$$H_I(I_i, II_j) + H_{II}(I_i, II_j) = c$$

where $H_I(I_i, II_j)$, $H_{II}(I_i, II_j)$ are expected payoffs of player I and player II, respectively. That is, the pay-offs of player I or player II do not give information about pay-off of the other one. Thus, if player I plays I_i and player II plays II_j , then the payoff is as follows

$$H(I_i, II_j) = \left(H_I(I_i, II_j), H_{II}(I_i, II_j)\right).$$

If player I selects the strategy i and player II selects the strategy j , then $[a_{ijL}, a_{ijR}]$ and $[b_{ijL}, b_{ijR}]$ are payoffs of the player I and player II, respectively. Hence, a bimatrix game is determined by a pair of matrix $(\tilde{A}, \tilde{B})$

When payoff matrix whose entries are interval numbers, is given, the payyoff matris of two-person non-zero sum game $(\tilde{A}, \tilde{B})$ is represented as follows:

$$(\tilde{A}, \tilde{B}) = \begin{bmatrix} ([a_{11L}, a_{11R}], [b_{11L}, b_{11R}]) & [a_{12L}, a_{12R}], [b_{12L}, b_{12R}] & \ldots & [a_{1mL}, a_{1mR}], [b_{1mL}, b_{1mR}] \\ [a_{12L}, a_{12R}], [b_{12L}, b_{12R}] & [a_{22L}, a_{22R}], [b_{22L}, b_{22R}] & \ldots & [a_{2mL}, a_{2mR}], [b_{2mL}, b_{2mR}] \\ \vdots & \vdots & \vdots & \vdots \\ [a_{1mL}, a_{1mR}], [b_{1mL}, b_{1mR}] & [a_{2mL}, a_{2mR}], [b_{2mL}, b_{2mR}] & \ldots & [a_{nmL}, a_{nmR}], [b_{nmL}, b_{nmR}] \end{bmatrix}$$

A mixed strategy set of player I is

$$S_I = \left\{ x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \ \forall i = 1, \ldots, m \right\}.$$

Similarly, a mixed strategy set of player II is

$$S_{II} = \left\{ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : y_i \geq 0, \sum_{j=1}^{n} y_j = 1 \ \forall i = 1, \ldots, n \right\}.$$

When player I plays mixed strategy of $x \in S_I$ and player II plays mixed strategy of $y \in S_{II}$, player I receives payoff

$$H_I(x,y) = \sum_{j=1}^{n}\sum_{i=1}^{m} x_i\big[a_{ijL}, a_{ijR}\big]y_j$$

and player II receives payoff

$$H_{II}(x,y) = \sum_{j=1}^{n}\sum_{i=1}^{m} x_i\big[b_{ijL}, b_{ijR}\big]y_j.$$

Let $x^* \in S_I$, $y^* \in S_{II}$ be mixed strategies for player I and player II, respectively. If the following inequalities

$$H_I(x,y^*) \leq H_I(x^*,y^*)$$
$$H_{II}(x^*,y) \leq H_{II}(x^*,y^*)$$

are satisfied for arbitrary $x \in S_I$, $y \in S_{II}$ then pair of (x^*,y^*) is called strategies of equilibrium (or pair of equilibrium) of the game.

4. An Analytic Method

Let $\tilde{A}$, $\tilde{B}$ be mxn pay-off matrixs for player I and player II, respectively. We assume that player I has m pure-strategy and player II has n pure-strategy

$$\tilde{A} = \begin{bmatrix} [a_{11L}, a_{11R}] & \cdots & [a_{1nL}, a_{1nR}] \\ \vdots & \vdots & \vdots \\ [a_{m1L}, a_{m1R}] & \cdots & [a_{mnL}, a_{mnR}] \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} [b_{11L}, b_{11R}] & \cdots & [b_{1nL}, b_{1nR}] \\ \vdots & \vdots & \vdots \\ [b_{m1L}, b_{m1R}] & \cdots & [b_{mnL}, b_{mnR}] \end{bmatrix}$$

and so let x^*, y^* be respectively $mx1$ mixed strategy vector for player I and $nx1$ mixed strategy vector for player II.

Let each components of interval matrix $\tilde{A}$ and $\tilde{B}$ be positive definite. We assume that all entries of $\tilde{A} = [a_{ijL}, a_{ijR}]$ and $\tilde{B} = \big[b_{ijL}, b_{ijR}\big]$, $1 \leq i \leq m, 1 \leq j \leq n$ for all i,j are different length and are not nested intervals.

$$\textstyle\sum_{i=1}^{m} x_i = 1 \; x_i \geq 0 \; i = 1,\dots,m \qquad (4.1)$$

$$\textstyle\sum_{j=1}^{n} y_j = 1 \; y_j \geq 0 \; j = 1,\dots,n \qquad (4.2)$$

$$\tilde{A}y^* \leq \big(x^{*T}\tilde{A}y^*\big)([1,1])_{mx1} = H_{\tilde{A}}([1,1])_{mx1} \quad (4.3)$$

$$\tilde{B}^T x^* \leq \big(x^{*T}\tilde{B}y^*\big)([1,1])_{nx1} = H_{\tilde{B}}([1,1])_{nx1} \quad (4.4)$$

If we take into account (4.1), (4.2), (4.3) conditions, then we can work out Nash Equilibrium.

$$d = \max_{i,j}(t_{ij}, s_{ij}) + [1,1], i = 1,\dots,m \; j = 1,\dots,n, t_{ij} = \big[a_{ijL}, a_{ijR}\big], s_{ij} = \big[b_{ijL}, b_{ijR}\big]$$

$$\tilde{A}_1 = d\tilde{E} - \tilde{A} = \begin{pmatrix} [d,d] & [d,d] \\ [d,d] & [d,d] \end{pmatrix} - \begin{pmatrix} [a_{11L}, a_{11R}] & [a_{1nL}, a_{1nR}] \\ [a_{m1L}, a_{m1R}] & [a_{mnL}, a_{mnR}] \end{pmatrix} = \begin{pmatrix} t_{11} & t_{1n} \\ t_{m1} & t_{mn} \end{pmatrix} > 0$$

where $t_{ij} = [d,d] - \big[a_{ijL}, a_{ijR}\big], i = 1,\dots,m \; j = 1,\dots,n$

$$\tilde{B}_1 = d\tilde{E} - \tilde{B} = \begin{pmatrix} [d,d] & [d,d] \\ [d,d] & [d,d] \end{pmatrix} - \begin{pmatrix} [b_{11L}, b_{11R}] & [b_{1nL}, b_{1nR}] \\ [b_{m1L}, b_{m1R}] & [b_{mnL}, b_{mnR}] \end{pmatrix} = \begin{pmatrix} s_{11} & s_{1n} \\ s_{m1} & s_{mn} \end{pmatrix} > 0$$

where $s_{ij} = [d,d] - \big[b_{ijL}, b_{ijR}\big], i = 1,\dots,m \; j = 1,\dots,n$

$$\tilde{E} = \big[e_{ijL}, e_{ijR}\big] = [1,1], i = 1,\dots,m \; j = 1,\dots,n$$

the following conditions are equivalent to (4.1), (4.2), (4.3)

$$\tilde{B}_1^T \tilde{x} \geq ([1,1])_{nx1} \qquad (4.5)$$

$$\tilde{x} \geq 0 \qquad (4.6)$$

$$\Big(\tilde{y}, \big(\tilde{B}_1^T \tilde{x} - ([1,1])_{nx1}\big)\Big) = 0 \qquad (4.7)$$

$$\tilde{A}_1^T \tilde{y} \geq ([1,1])_{mx1} \qquad (4.8)$$

$$\tilde{y} \geq 0 \qquad (4.9)$$

$$\Big(\tilde{x}, \big(\tilde{A}_1^T \tilde{y} - ([1,1])_{mx1}\big)\Big) = 0 \qquad (4.10)$$

(3.4.1) can be expressed as follows:

$$\tilde{B}_1^T \tilde{x} \leq (d \textstyle\sum_{i=1}^{m} \tilde{x}_i - 1)([1,1])_{nx1} \qquad (4.11)$$

$x^* = \frac{\tilde{x}}{\sum_{i=1}^{m} \tilde{x}_i} \Longrightarrow \tilde{x} = x^* \sum_{i=1}^{m} \tilde{x}_i$ then, if we substitute $\tilde{x}$ into (4.11), we obtain the following inequalities;

$$\tilde{B}_1^T x^* \sum_{i=1}^{m} \tilde{x}_i \leq (d \sum_{i=1}^{m} \tilde{x}_i - 1)([1,1])_{nx1}$$

$$\tilde{B}_1^T x^* \leq \left(d - \frac{1}{\sum_{i=1}^{m} \tilde{x}_i}\right)([1,1])_{nx1} \dots (*)$$

Therefore, (4.3) is equivalent (*). The following statement

$$x^{*T} \tilde{B} y^* = d - \frac{1}{\sum_{i=1}^{m} \tilde{x}_i}$$

is added to (4.5) then, (4.3) is verified when conditions of (4.5), (4.6), (4.7) are satisfied and similarly (4.2) is verified when conditions of (4.8), (4.9), (4.10) are satisfied.

Then, optimal strategies x^*, y^* and values of games H_I, H_{II} are obtained as follows,

$$x^* = \frac{\tilde{x}}{\sum_{i=1}^{m} \tilde{x}_i}, y^* = \frac{\tilde{y}}{\sum_{j=1}^{n} \tilde{y}_j}$$

$$H_I = d - \frac{1}{\sum_{j=1}^{n} \tilde{y}_j}, H_{II} = d - \frac{1}{\sum_{i=1}^{m} \tilde{x}_i}.$$

4.1 Lemke-Howson Algorithm

I) Calculating matrixes $\tilde{A}_1$, $\tilde{B}_1$

1) $d = \max_{i,j}(t_{ij}, s_{ij}) + [1,1], i = 1, \dots, m\ j = 1, \dots, n$

2)

$$\tilde{A}_1 = d\tilde{E} - \tilde{A} > 0$$
$$\tilde{B}_1 = d\tilde{E} - \tilde{B} > 0,$$

$\tilde{E} = [e_{ijL}, e_{ijR}] = [1,1], i = 1, \dots, m\ j = 1, \dots, n$

II) Determining initial strategy vectors x^0, y^0;

3) Forming tableau $\tilde{A}_0^*$

$$A_0^* = \begin{array}{c} \begin{array}{cccccccc} t_1^1 & t_2^1 & \cdots & t_m^1 & e_1 & e_2 & \cdots & e_n \end{array} \\ \begin{bmatrix} t_{11}^1 & t_{21}^1 & \cdots & t_{m1}^1 & \vdots [1,1] & [0,0] & \cdots & [0,0] \\ t_{12}^1 & t_{22}^1 & \cdots & t_{m2}^1 & \vdots [0,0] & [1,1] & \cdots & [0,0] \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ t_{1n}^1 & t_{2n}^1 & \cdots & t_{mn}^1 & \vdots [0,0] & [0,0] & \cdots & [1,1] \end{bmatrix} \end{array}$$
$$= [\tilde{A}_1^T : \tilde{I}]$$

4) Let's choose initial strategy vector y^0 ;

$$y^{0^T} = \left(\frac{1}{a}\ 0 \dots 0\right)$$

where $a = \min_i t_{i1}^1, i = 1,2, \dots, m$ and $0 = [0,0]$ is an interval number. a is a minimal element of first column of matrix $\tilde{A}_1$ and i^* is a row giving minimal element of first column of matrix $\tilde{A}_1$.

5) Similiarly, tableau $\tilde{B}_0^* = [\tilde{B}_1 : \tilde{I}]$ is formed as in third step. Moreover, base $(f_1, f_2, \dots f_m)$ replace with base $(e_1, e_2, \dots e_n)$

6) Let's choose initial strategy vector x^0 :

$$x^{0^T} = \left(0\ 0\ \dots \frac{1}{b}\ 0 \dots 0\right)$$

where $b = \min_j s_{i^*j}^1, j = 1,2, \dots, n$ and $(0 = [0,0]$ is an interval number.) b is a minimal element of i^*. row of matrix $\tilde{B}_1$ and j^* is a column giving minimal element of i^* row of matrix $\tilde{B}_1$.

(III) Controlling Equilibrium Conditions

7) We can control whether equilibrium conditions are provided, using the following method.

Determining sets of $p(x)$ and (x) :

$$p(x) = \{f_i, s_j^{1T} \mid f_i^T x = [0,0], s_j^{1T} x - [1,1] = [0,0]\}$$

Similarly,

$$q(y) = \{e_j, t_i^{1T} \mid e_j^T y = [0,0], t_i^{1T} y - [1,1] = [0,0]\}.$$

Then, we can express sets of $p(x^0)$ and (y^0) :

$$p(x^0) = \{f_1, f_2, \dots, f_{i^*-1}, s_{j^*}, f_{i^*+1}, \dots, f_m\}$$
$$q(y^0) = \{t_{i^*}, e_2, \dots, e_n\}.$$

After then, we control whether equilibrium conditions are satisfied with $M(x, y)$.

$$M(x, y) = \left\{e_r, f_s \middle| \begin{matrix} e_r \in M(x,y)\ if\ e_r \in q(y)\ or\ s_r^1 \in p(x) \\ f_s \in M(x,y)\ if\ f_s \in p(x)\ or\ t_s^1 \in q(y) \end{matrix}\right\}$$

If $M(x_i, y_j) = \{e_1, \dots, e_n, f_1, f_2, \dots, f_m\}$, then pair of (x_i, y_j) is equilibrium state. If equilibrium condition is satisfied, then we pass on 15^{th} step otherwise, pass on 8^{th} step.

(IV) Changing The Bases

8) Base $(e_1, e_2, \dots, e_n)$ replaces with $q(y^0)$ and base $(f_1, \dots, f_m)$ replaces with $p(x^0)$. Tableaus of $\tilde{A}_1^*, \tilde{B}_1^*$ are formed by means of one method which is similar to simplex.

Tableau $\tilde{A}_1^*$ is as folows

$$\tilde{A}_1^* = \begin{bmatrix} \tilde{\alpha}_{11} & \tilde{\alpha}_{21} & \dots & \tilde{\alpha}_{i^*-1,1} & 1 & \tilde{\alpha}_{i^*+1,1} & \dots & \tilde{\alpha}_{m1} & q^{11} & \dots & q^{1n} \\ \tilde{\alpha}_{12} & \tilde{\alpha}_{22} & \dots & \tilde{\alpha}_{i^*-1,2} & 0 & \tilde{\alpha}_{i^*+1,2} & \dots & \tilde{\alpha}_{m2} & q^{21} & \dots & q^{2n} \\ \dots & & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \tilde{\alpha}_{1n} & \tilde{\alpha}_{2n} & \dots & \tilde{\alpha}_{i^*-1,n} & 0 & \tilde{\alpha}_{i^*+1,n} & \dots & \tilde{\alpha}_{mn} & q^{n1} & \dots & q^{nn} \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_n \end{matrix}$$

$$\xi_1 - 1 \quad \xi_2 - 1 \quad \dots \quad \xi_{i^*-1} - 1 \quad \xi_{i^*} - 1 \quad \xi_{i^*+1} - 1 \quad \dots \quad \xi_m - 1 \quad y_1^0 \quad \dots \quad y_n^0$$

where $1 = [1,1], 0 = [0,0]$ are interval numbers and $\tilde{\alpha}_{i1} = \frac{t'_{i1}}{t'_{i^*1}}$, $\tilde{\alpha}_{ij} = t'_{ij} - \tilde{\alpha}_{i1} t'_{i^*j}, j \neq 1$.

9) In tableau $\tilde{A}_1^*$, instead of bottom row, corresponding to the base $(e_1, e_2, \dots, e_n)$

$$y^{0^T} = (y_1^0 \, y_2^0 \dots y_n^0)$$

the vector is written.

$$\xi_i = t_i^{1T} y^0, i = 1,2, \dots, m$$

where t_i^1 $(i = 1,2, \dots, m)$ are columns of tableau $\tilde{A}_0^*$.

10) Let's work out values of λ^*, λ^{**}

$$\lambda_j^* = \min_{\substack{\alpha_{kj}<0 \\ q^{jr}<0 \\ 1\le k\le m \\ 1\le r\le n}} \left\{ -\frac{\xi_k - 1}{\alpha_{kj}}, -\frac{y_r^0}{q^{jr}} \right\}, \lambda_j^{**}$$

$$= \max_{\substack{\alpha_{sj}>0 \\ q^{jt}>0 \\ 1\le s\le m \\ 1\le t\le n}} \left\{ -\frac{\xi_s - 1}{\alpha_{sj}}, -\frac{y_t^0}{q^{jt}} \right\}$$

where either value of λ_j^* or value of λ_j^{**} is zero. Elements of column λ in matrix A_1^* consist of λ_j^* or λ_j^{**} which are non-zero.

11) By similar arguments, tableau $\tilde{B}_1^*$ is formed. Hence, μ_i is obtained instead of λ_j.

(V) Determining Optimal Strategies and Game-Value

12) Let's determine all of strategies x and y as follows:

$$\begin{cases} x_i^T = x^{0^T} + \mu_i p^i, i = 1,2, \dots, m \\ y_i^T = y^{0^T} + \lambda_j q^j, j = 1,2, \dots, n \end{cases}$$

where p^i, q^j are rows of matrixs P and Q which are in tableaus of $\tilde{B}_1^*$ and $\tilde{A}_1^*$,respectively.

13) Let's now determine sets of $q(y_j)$ and $p(x_i)$ for value λ_j minimal value $(\lambda_j = \lambda_j^*)$ or maximum value $(\lambda_j = \lambda_j^{**})$ is obtained from the following statement

$$-\frac{\xi_k - [1,1]}{\alpha_{kj}} \text{ or } -\frac{y_r^0}{q^{jr}}$$

We assume that it is $-\frac{\xi_k - [1,1]}{\alpha_{kj}}$.

$$q(y_j) = \{q(y^0) \cup t_m\} \setminus \{e_j\}$$

Similarly,

$$p(x_i) = \{p(x^0) \cup s_m\} \setminus \{f_i\}$$

14) We can control the equilibrium condition as in 7^{th} step. We specify $M(x_i, y_j)$ for every pair of (x_i, y_j). If the equilibirum condition is satisfied, then we pass on 15^{th} step. Otherwise, we pass on 4^{th} step and minimal element of second column of $\tilde{A}_1^T$ is found. Thus, initial strategy vector y^0 is as follows

$$y^{0^T} = \left(0 \ \frac{1}{a} \ 0 \dots 0\right)$$

where $a = \min_i t_{i1}^1, i = 1,2, \dots, m$. then, we supply all steps of the algorithm.

15) Let $(\tilde{x}, \tilde{y})$ be a pair of point N.E. So, optimal strategies x^*, y^* and expected pay-offs H_I, H_{II} for player I and player II, respectively are as follows:

$$x^* = \frac{\tilde{\boldsymbol{x}}}{\sum_{i=1}^m \tilde{\boldsymbol{x}}_i}, y^* = \frac{\tilde{\boldsymbol{y}}}{\sum_{j=1}^n \tilde{\boldsymbol{y}}_j}$$

$$H_I = d - \frac{1}{\sum_{j=1}^n \tilde{\boldsymbol{y}}_j}, \ H_{II} = d - \frac{1}{\sum_{i=1}^m \tilde{\boldsymbol{x}}_i}$$

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