Integral Representation of the Manifold Solution for New Class of the Volterra Type Integral Equation with a Boundary Singularity in the Case, When Kernel Contain Logarithmic Singularity and its Power

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Abstract: In this work we suggestion new methods investigation the model Volterra type integral equation with logarithmic singularity, kernel which consisting from composition polynomial function with logarithmic singularity and function with singular point. The problem investigation this type integral equation at $n = 2m$ reduce to problem investigate the Volterra type integral equation (1) for $n = 2$ the theory for which was constructed in [2]. In this work, we investigation integral equation (1) at $n = 2m + 1$. In this case, we investigate integral equation (1) reduction it's to m integral equation type [2]

$$\varphi(x) + \int_a^x \left[p_1 + p_2 \ln \left(\frac{x-a}{t-a} \right) \right] \frac{\varphi(t)}{t-a} dt = f(x) \quad (I)$$

and one the following integral equation [1]

$$\omega(x) + p_3 \int_a^x \frac{\omega(t)}{t-a} dt = g(x). \quad (II)$$

Key words: singular kernel, volterra type integral equation, boundary singularity, logarithmic singularity.

1. Introduction

Let $\Gamma = \{x: a < x < b\}$ be a set of point on the real axis and consider an integral equation

$$\varphi(x) + \int_a^x \left[\sum_{j=1}^n A_j \ln^{j-1} \left(\frac{x-a}{t-a} \right) \right] \frac{\varphi(t)}{t-a} dt = f(x), \quad (1)$$

where $A_j (1 \leq j \leq n)$ are given constants, $f(x)$ are given function in Γ and $\varphi(x)$ to be found. The solution to this equation is sought in the class of function $\varphi(x) \in C(\overline{\Gamma})$ vanishing at the singular point $x = a$ i.e. ,

$$\varphi(x) = o[(x-a)^\varepsilon] \quad \varepsilon > 0, \text{ and } x \rightarrow a.$$

Support that the solution of the integral equation (1) is the function $\varphi(x) \in C^{(n)}(\Gamma)$. Besides, let in equation (1) function $f(x) \in C^{(n)}(\Gamma)$ too. Then differentiating integral equation (1) n times, we obtained the following $n - th$ order degenerate

differential equation

$$(D_x)^n \varphi(x) + A_1 (D_x)^{n-1} \varphi(x) + A_2 (D_x)^{n-2} \varphi(x) + 2! A_3 (D_x)^{n-3} \varphi(x) + \dots (n-1)! A_n \varphi(x) = (D_x)^n f(x), \quad (2)$$

where $D_x = (x-a) \frac{d}{dx}$.

The homogeneous differential equation corresponding to (2) has the following characteristic equation

$$\lambda^n + A_1 \lambda^{n-1} + A_2 \lambda^{n-2} + 2! A_3 \lambda^{n-3} + 3! A_4 \lambda^{n-4} + \dots + (n-1)! A_n = 0. \quad (3)$$

To the problem investigation different cases ($n = 1, n = 2, n = 3$) for the integral equation (1) the papers [1]-[3] are devoted. The case of equation (1) when the parameters $A_j (1 \leq j \leq n)$ are such that the roots of the characteristic equation (3) are real and different is investigated in [3]. But in the investigation of the other cases for the roots of the characteristic equation (3), for representation of the manifold solution of the equation (1) other difficulties of analytical character arise.

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In this connection in this work we offer a representation of the manifold solution of the integral equation (1) by the solution of the integral equation (1) for $n = 1, n = 2$, the theory for which was constructed in [1] ($n = 1$) and [2] ($n = 2$). For $n = 2$ in [2], other possibility cases are investigated. This gives the possibility depending on whether $n = 2m$ or $n = 2m + 1$, to write out the solution of equation (1) in an explicit form in connection with m second order algebraic equations.

In the case, when in the integral equation (1) the parameters A_j ($1 \leq j \leq n$) are such that the roots of the characteristic equation (3) are real, different and positive we have the following confirmation.

Theorem 1. *Let in the integral equation (1) the parameters A_j ($1 \leq j \leq n$) be such that the roots of the characteristic equation (3) λ_j ($1 \leq j \leq n$) are real, different and positive, the function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior*

$$f(x) = o[(x-a)^{\delta_1}], \quad \delta_1 > \lambda, \\ \lambda = \max(\lambda_1, \lambda_2, \dots, \lambda_n) \text{ at } x \rightarrow a.$$

Then integral equation (I) in the class of function $\varphi(x) \in C(\overline{\Gamma})$ vanishing in point $x = a$ is always solvability and its solution is given by formula

$$\varphi(x) = \sum_{k=1}^n (x-a)^{\lambda_k} C_k + f(x) + \\ \frac{1}{\Delta_0} \int_a^x \left[\sum_{k=1}^n (-1)^k \lambda_k^3 \left(\frac{x-a}{t-a} \right)^{\lambda_k} \right] \frac{f(t)}{t-a} dt \equiv \\ K_1^{-1} [C_1, C_2, C_3, \dots, C_n, f(x)],$$

where C_k ($1 \leq k \leq n$) -arbitrary constants, Δ_0 is n -th order Vander monde determinant from λ_j ($1 \leq j \leq n$).

Right now we investigate the integral equation (1) for $n = 3$ and $n = 4$ in a different way.

2. The Case, When $n = 3$

In this case, equation (1) has the following form

$$\varphi(x) + \int_a^x \left[A_1 + A_2 \ln \left(\frac{x-a}{t-a} \right) + A_3 \ln^2 \left(\frac{x-a}{t-a} \right) \right] \frac{\varphi(t)}{t-a} dt = f(x) \quad (4)$$

Introduce the following integral operators

$$T_{p_1, p_2}(\varphi) \equiv \varphi(x) + \int_a^x \left[p_1 + p_2 \ln \left(\frac{x-a}{t-a} \right) \right] \frac{\varphi(t)}{t-a} dt \quad (5)$$

and

$$\Pi_{p_3}(\varphi) = \varphi(x) + p_3 \int_a^x \frac{\varphi(t)}{t-a} dt. \quad (5_1)$$

If, in integral equation (4) the parameters A_j ($1 \leq j \leq 3$) with constants p_1, p_2, p_3 connect by means of following equality

$$A_1 = p_1 + p_3, \quad A_2 = p_2 + p_1 p_3, \quad 2A_3 = p_2 p_3, \quad (6)$$

then the equation (4) we represented in form

$$T_{p_1, p_2}[\Pi_{p_3}(\varphi)] = f(x). \quad (7)$$

Designate

$$\Pi_{p_3}(\varphi) = \omega(x). \quad (8)$$

Then problem found the solution of the integral equation (4) reduce to problem solution integral equation

$$T_{p_1, p_2}(\omega) = f(x). \quad (9)$$

The integral equation (9) corresponding the following characteristic equation

$$\lambda^2 + p_1 \lambda + p_2 = 0. \quad (10)$$

In depend from the roots of equation (10), signs constants p_1, p_2 and p_3 , in according [2] and [1] first time we found solution integral equation (9). Obtained solution equation (9) function $\omega(x)$ by substituting to right part equation (8), after solving integral equation (8), we found solution integral equation (7).

In the case, when in integral equation (9) $p_1 < 0, p_2 > 0, D = p_1^2 - 4p_2 > 0$, according to [2] the solution integral equation (9) is given by formula

$$\omega(x) = (x-a)^{\lambda_1} C_1 + (x-a)^{\lambda_2} C_2 + f(x)$$

$$- \frac{1}{\sqrt{D}} \int_a^x \left[\lambda_2^2 \left(\frac{x-a}{t-a} \right)^{\lambda_2} - \lambda_1^2 \left(\frac{x-a}{t-a} \right)^{\lambda_1} \right] \frac{f(t)}{t-a} dt \equiv \\ \left(T_{p_1, p_2}^{C_1, C_2} \right)^{-1} (f), \quad (11)$$

where $\lambda_1 = \frac{|p_1| + \sqrt{D}}{2}, \lambda_2 = \frac{|p_1| - \sqrt{D}}{2}, C_1, C_2$ - arbitrary

constants.

At $p_3 < 0$ if solution integral equation (8) exist, then according to [1] it's possible represented in following form

$$\begin{aligned} \varphi(x) = \\ (x-a)^{|p_3|} C_3 + \omega(x) - p_3 \int_a^x \left(\frac{x-a}{t-a} \right)^{|p_3|} \frac{\omega(t)}{t-a} dt \equiv \\ (\Pi_{p_3}^{C_3})^{-1}(\omega), \end{aligned} \quad (12)$$

where C_3 –arbitrary constant.

The solution of the type (11) and (12) exist, if $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = 0[(x-a)^{\delta_2}], \quad \delta_2 > \lambda_1 \text{ at } x \rightarrow a \quad (13)$$

and $\omega(x) \in C(\overline{\Gamma})$, $\omega(a) = 0$ with asymptotic behavior

$$\omega(x) = 0[(x-a)^{\delta_3}], \quad \delta_3 > |p_3| \text{ at } x \rightarrow a.$$

If $\lambda_2 > |p_3|$ and function $f(x)$ satisfied the condition of the (13), then solution the integral equation (4) or (7) exist and give by formula

$$\varphi(x) = (\Pi_{p_3}^{C_3})^{-1} \left[(T_{p_1, p_2}^{1, C_1, C_2})^{-1}(f) \right]. \quad (14)$$

So, we proofed the following confirmation

Theorem 2. Let the parameters A_j ($1 \leq j \leq 3$) in equation (4) with constants p_1, p_2, p_3 in integral operators T_{p_1, p_2}, Π_{p_3} connected by means of formula (6). Besides let $p_1 < 0, p_2 > 0, p_3 < 0, p_1^2 - 4p_2 > 0$ ($A_1 < 0, A_2 > 0, A_3 < 0$), $\lambda_2 > |p_3|$ and let the function $f(x) \in C(\overline{\Gamma})$ $f(a) = 0$ with asymptotic behavior (13). Then the integral equation (4) or (7) in class of function $\varphi(x) \in C(\overline{\Gamma})$ vanishing in point $x = a$ is always solvability: its general solution contains three arbitrary constants and is given by the formula (14).

In the case, when in integral equation (7) $p_1 > 0, p_2 > 0, p_3 > 0, p_1^2 - 4p_2 > 0$ ($A_1 > 0, A_2 > 0, A_3 > 0$) the following result is true.

Theorem 3. Let the parameters A_j ($1 \leq j \leq 3$) in equation (4) with constants p_1, p_2, p_3 in integral operators T_{p_1, p_2}, Π_{p_3} connected by means of formula (6). Besides let $p_1 > 0, p_2 > 0, p_3 > 0, p_1^2 -$

$4p_2 > 0$ ($A_1 > 0, A_2 > 0, A_3 > 0$), and let the function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = 0[(x-a)^\varepsilon], \quad \varepsilon > 0 \text{ as } x \rightarrow a. \quad (15)$$

Then the integral equation (4) or (7) in class of function $\varphi(x) \in C(\overline{\Gamma})$ vanishing in point $x = a$ has a unique solution, which is given by the formula

$$\varphi(x) = (\Pi_{p_3})^{-1} \left[(T_{p_1, p_2})^{-1}(f) \right],$$

where

$$(\Pi_{p_3})^{-1}(f) = f(x) - p_3 \int_a^x \left(\frac{t-a}{x-a} \right)^{p_3} \frac{f(t)}{t-a} dt$$

$$(T_{p_1, p_2})^{-1}(f) = f(x) -$$

$$\frac{1}{\sqrt{D}} \int_a^x \left[\lambda_2^2 \left(\frac{t-a}{x-a} \right)^{|\lambda_2|} - \lambda_1^2 \left(\frac{t-a}{x-a} \right)^{|\lambda_1|} \right] \frac{f(t)}{t-a} dt.$$

Remark 1. Assertion similar in Theorem 1 also hold in the following cases: 1) $p_1 < 0, p_2 < 0, p_3 > 0, D > 0$; 2) $p_1 < 0, p_2 < 0, p_3 < 0, D > 0$; 3) $p_1 > 0, p_2 < 0, p_3 > 0, D > 0$; 4) $p_1 > 0, p_2 < 0, p_3 < 0, D > 0$;

If the roots of the characteristic equation (10) real, equal and positive, that is $\lambda_1 = \lambda_2 = \lambda = -\frac{p_1}{2}$, $p_1 < 0$, then solution integral equation (9) is give formula

$$\begin{aligned} \omega(x) = (x-a)^{\frac{|p_1|}{2}} [C_4 + \ln(x-a)C_5] + f(x) - \\ \int_a^x \left(\frac{x-a}{t-a} \right)^{\frac{|p_1|}{2}} \left[p_1 - p_2 \ln \left(\frac{x-a}{t-a} \right) \right] \frac{f(t)}{t-a} dt \equiv \\ (T_{p_1, p_2}^{2, C_4, C_5})^{-1}(f). \end{aligned} \quad (16)$$

At $p_3 < 0$ solution equation (8) is given by formula

$$\varphi(x) = (\Pi_{p_3}^{C_6})^{-1}(\omega). \quad (17)$$

The solution of the type (16) and (17) exist, if $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = 0[(x-a)^{\delta_4}], \delta_4 > \frac{|p_1|}{2} \text{ at } x \rightarrow a \quad (18)$$

and $\omega(x) \in C(\overline{\Gamma})$, $\omega(a) = 0$ with asymptotic behavior

$$\omega(x) = 0[(x-a)^{\delta_5}], \delta_5 > |p_3| \text{ at } x \rightarrow a.$$

If $\lambda_2 > |p_3|$ and function $f(x)$ satisfied the condition of the (18), then solution of the integral equation (4) or (7) exist and give by formula

$$\varphi(x) = (\Pi_{p_3}^{C_6})^{-1} \left[(T_{p_1, p_2}^{2, C_4, C_5})^{-1} (f) \right]. \quad (19)$$

So, we proofed the following confirmation

Theorem 4. Let the parameters A_j ($1 \leq j \leq 3$) in equation (4) with constants p_1, p_2, p_3 in integral operators T_{p_1, p_2}, Π_{p_3} connected by means of formula (7). Besides let the roots of the characteristic equation (10) real, equal and positive, that is $\lambda_1 = \lambda_2 = \lambda = -\frac{p_1}{2}$, $p_1 < 0$, $p_1^2 = 4p_2$, $p_3 < 0$ ($A_1 > 0, A_2 > 0, A_3 > 0$), $p_1 < 2p_3$ and let the function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior (18). Then the integral equation (4) or (7) in class of function $\varphi(x) \in C(\overline{\Gamma})$ vanishing in point $x = a$ is always solvability: its general solution contains three arbitrary constants and is given by the formula (19).

From theorem 4 and integral representation (19) follows:

Theorem 5. Let the parameters A_j ($1 \leq j \leq 3$) in equation (4) with constants p_1, p_2, p_3 in integral operators T_{p_1, p_2}, Π_{p_3} connected by means of formula (6). Besides let the roots of the characteristic equation (10) real, equal and negative, that is $\lambda_1 = \lambda_2 = \lambda = -\frac{p_1}{2}$, $p_1 > 0$, $p_1^2 = 4p_2$, $p_3 > 0$ ($A_1 < 0, A_2 > 0, A_3 < 0$), $|p_1| > 2|p_3|$ and let the function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior (15). Then the integral equation (4) or (7) in class of function $\varphi(x) \in C(\overline{\Gamma})$ vanishing in point $x = a$ has a unique solution, is given by the formula

$$\varphi(x) = (\Pi_{p_3})^{-1} (T_{p_1, p_2}^{2, 0, 0})^{-1} (f),$$

where

$$(T_{p_1, p_2}^{2, 0, 0})^{-1} (f) = f(x) -$$

$$\int_a^x \left(\frac{t-a}{x-a} \right)^{\frac{p_1}{2}} [p_1 - p_2 \ln \left(\frac{x-a}{t-a} \right)] \frac{f(t)}{t-a} dt.$$

In this case, when the roots of the characteristic equation (10) is complex and conjugate, that is $\lambda_1 = \lambda = A + iB$, $\lambda_2 = \bar{\lambda} = A - iB$ where $A = \frac{-p_1}{2}$,

$B = \frac{\sqrt{D_1}}{2}$, $D_1 = 4p_2 - (p_1)^2$, $p_2 < 0$, then according to [2] solution integral equation (10) is give by formula

$$\begin{aligned} \omega(x) = (x-a)^{\frac{|p_1|}{2}} \{ \cos[B \ln(x-a)] C_7 \\ + \sin[B \ln(x-a)] C_8 \} + \\ + f(x) - \\ 2 \int_a^x \left(\frac{x-a}{t-a} \right)^{\frac{|p_1|}{2}} \left[-A \cos \left[B \ln \left(\frac{x-a}{t-a} \right) \right] \right. \\ \left. + B \sin \left[B \ln \left(\frac{x-a}{t-a} \right) \right] \right] \frac{f(t)}{t-a} dt \equiv \\ \equiv (T_{p_1, p_2}^{3, C_7, C_8})^{-1} (f), \end{aligned} \quad (20)$$

where C_7, C_8 – arbitrary constants, if $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = 0[(x-a)^{\delta_6}], \delta_6 > \frac{|p_1|}{2} \text{ at } x \rightarrow a. \quad (21)$$

Substitute obtained valued $\omega(x)$ in formula

$$\varphi(x) = (\Pi_{p_3}^{C_9})^{-1} (\omega),$$

we found the solution of the integral equation (4) or (5) in this case

$$\varphi(x) = (\Pi_{p_3}^{C_9})^{-1} \left[(T_{p_1, p_2}^{3, C_7, C_8})^{-1} (f) \right], \quad (22)$$

when $p_1 < 0$, $p_3 < 0$, $4p_2 - (p_1)^2 > 0$, $|p_1| > 2|p_3|$.

So, in this case, we proof the following confirmation

Theorem 6. Let the parameters A_j ($1 \leq j \leq 3$) in equation (4) with constants p_1, p_2, p_3 in integral operators T_{p_1, p_2}, Π_{p_3} connected by means of formula

(7). Besides let the roots of the characteristic equation (11) is complex and conjugate, that is $\lambda_1 = \lambda = A + iB$, $\lambda_2 = \bar{\lambda} = A - iB$ where $A = \frac{-p_1}{2}$, $B = \frac{\sqrt{D_1}}{2}$, $D_1 = 4p_2 - (p_1)^2$, $p_1 < 0$ and $p_3 < 0$, $|p_1| > 2|p_3|$, the function $f(x) \in C(\bar{\Gamma})$, $f(a) = 0$ with asymptotic behavior (21). Then the integral equation (4) or (7) in class of function $\varphi(x) \in C(\bar{\Gamma})$ vanishing in point $x = a$ is always solvability: its general solution contains three arbitrary constants and is given by the formula (22).

In the case, when $p_1 > 0$ and $p_3 > 0$, we have the following confirmation

Theorem 7. Let the parameters A_j ($1 \leq j \leq 3$) in equation (4) with constants p_1 , p_2 , p_3 in integral operators T_{p_1, p_2} , Π_{p_3} connected by means of formula (7). Besides let the roots of the characteristic equation (11) satisfy condition of the theorem 7, besides $p_1 < 0$ and $p_3 < 0$. Let $p_1 > 0$, $p_3 > 0$, $p_1 > 2p_3$, the function $f(x) \in C(\bar{\Gamma})$, $f(a) = 0$ with asymptotic behavior (15). Then the integral equation (4) or (7) in class of function $\varphi(x) \in C(\bar{\Gamma})$ vanishing in point $x = a$, has a unique solution, which is given by the formula

$$\varphi(x) = (\Pi_{p_3})^{-1} (T_{p_1, p_2}^{3,0,0})^{-1} (f),$$

where

$$(T_{p_1, p_2}^{3,0,0})^{-1} (f) = f(x) - 2 \int_a^x \left(\frac{t-a}{x-a} \right)^{\frac{p_1}{2}} \left[-A \cos \left[B \ln \left(\frac{x-a}{t-a} \right) \right] + B \sin \left[B \ln \left(\frac{x-a}{t-a} \right) \right] \right] \frac{f(t)}{t-a} dt.$$

In particular if the roots of the algebraic equation (10) imaginary that is, if $\lambda_1 = iB$, $\lambda_2 = -iB$, $B = \frac{\sqrt{D_1}}{2}$, $D_1 = 4p_2 - (p_1)^2$, $p_2 > 0$ and if solution integral equation (9), exist then its may be represented in form

$$\omega(x) == f(x) - 2B \int_a^x \left[\sin \left[B \ln \left(\frac{x-a}{t-a} \right) \right] \right] \frac{f(t)}{t-a} dt.$$

Besides if $p_3 < 0$ then substitute valued $\omega(x)$ in formula (12), we obtained solution integral equation (1) in this case in form

$$\begin{aligned} \varphi(x) &= (x-a)^{|p_3|} C_3 + f(x) \\ &- \int_a^x \left[2p_3 + \frac{2B^2 p_3}{p_3^2 + B^2} \cos \left[B \ln \left(\frac{x-a}{t-a} \right) \right] + \frac{2B^3}{p_3^2 + B^2} \sin \left[B \ln \left(\frac{x-a}{t-a} \right) \right] + p_3 \left(\frac{p_3^2 - B^2}{p_3^2 + B^2} \right) \left(\frac{x-a}{t-a} \right)^{|p_3|} \right] \frac{f(t)}{t-a} dt. \end{aligned} \quad (23)$$

The solution of the type (23) exist, if $f(x) \in C(\bar{\Gamma})$ $f(a) = 0$ with asymptotic behavior

$$f(x) = 0[(x-a)^{\delta_7}], \quad \delta_7 > |p_3| \text{ at } x \rightarrow a. \quad (24)$$

So, in this case, we have the following confirmation:

Theorem 8. Let the parameters A_j ($1 \leq j \leq 3$) in equation (4) with constants p_1 , p_2 , p_3 in integral operators T_{p_1, p_2} , Π_{p_3} connected by means of formula (7). Besides let the roots of the characteristic equation (10) imaginary and $f(x) \in C(\bar{\Gamma})$, $f(a) = 0$ with asymptotic behavior (24), $p_3 < 0$. Then the integral equation (4) or (7) in class of function $\varphi(x) \in C(\bar{\Gamma})$ vanishing in point $x = a$ is always solvability: its general solution contains one arbitrary constant and is given by the formula (23), where C_3 - arbitrary constant.

3. The Case, When $n = 4$

In the case when $n = 4$, we represent equation (1) in the following form

$$\prod_{j=1}^2 T_{p_j, q_j}(\varphi) = f(x) \text{ or } T_{p_1, q_1} T_{p_2, q_2}(\varphi) = f(x) \text{ where}$$

$$T_{p_j, q_j}(\varphi) \equiv \varphi(x) + \int_a^x \left[p_j + q_j \ln \left(\frac{x-a}{t-a} \right) \right] \frac{\varphi(t)}{t-a} dt \quad (1 \leq j \leq 2),$$

where the constants $p_j, q_j, j = 1, 2$ are connected with the parameters A_j ($1 \leq j \leq 4$) in the following way

$$\begin{aligned} A_1 &= p_1 + p_2, \quad A_2 = q_1 + q_2 + p_1 p_2, \\ 2! A_3 &= p_1 q_2 + p_2 q_1, \quad 3! A_4 = q_1 q_2. \end{aligned} \quad (25)$$

Introduce the new unknown function

$$\omega(x) = T_{p_2, q_2}(\varphi), \quad (26)$$

and we arrive at the solution of the following integral equation

$$T_{p_1, q_1}(\omega) = f(x), \quad (27)$$

whose theory which developed in [2].

In dependence on the roots of the algebraic equation [2]

$$(\lambda)^2 + p_1\lambda + q_1 = 0 \quad (28)$$

we write out the solution integral equation (27), and we find the unknown function (x) . After substituting the found value (x) into (26), in dependence on the roots of the algebraic equation [2]

$$(\mu)^2 + p_2\mu + q_2 = 0 \quad (29)$$

we find the solution of the integral equation (1) in the case $n = 4$.

For example, in the case when the roots of the equation (28) real, equal and negative the solution of the integral equation (27) is given by formula [2]

$$\begin{aligned} \omega(x) &= (x-a)^{\frac{|p_1|}{2}} [C_1 + C_2 \ln(x-a)] + f(x) \\ &\quad - \int_a^x \left(\frac{x-a}{t-a} \right)^{\frac{|p_1|}{2}} \left[p_1 \right. \\ &\quad \left. - q_1 \ln \left(\frac{x-a}{t-a} \right) \right] \frac{f(t)}{t-a} dt \\ &\equiv \left(T_{p_1, q_1}^{1, C_1, C_2} \right)^{-1} (f). \end{aligned}$$

Analogously, when the roots of the equation (29) are real, equal and negative, the solution of the integral equation (27) given by the formula

$$\varphi(x) = \left(T_{p_2, q_2}^{1, C_3, C_4} \right)^{-1} (\omega). \quad (30)$$

Substituting the value of (x) Into (30), we see, that, if the solution of the integral equation (1) in this case exist, then it can be represented in the following form

$$\varphi(x) = \left(T_{p_2, q_2}^{1, C_3, C_4} \right)^{-1} \left(T_{p_1, q_1}^{1, C_1, C_2} \right)^{-1} (f). \quad (31)$$

It is easily seen, that the solution of the type (31) exist, if $|p_2| > |p_1|$, $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_2}], \quad \delta_2 > \frac{|p_2|}{2}, \text{ at } x \rightarrow a \quad (32)$$

and $C_1 = 0, C_2 = 0$.

So, we have proved the following assertion.

Theorem 9. *Let in integral equation (1) $n = 4$, the parameters $A_j (1 \leq j \leq 4)$ are connected with the coefficients the algebraic equation (28), (29) by formula (25), $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior (32). Besides the roots of the algebraic equation (28), (29) are real, equal, negative and $|p_2| > |p_1|$. Then the integral equation (1) is always solvable in the class of function $\varphi(x) \in C(\overline{\Gamma})$ vanishing of the point $x = a$ and its solution is given by formula*

$$\varphi(x) = \left(T_{p_2, q_2}^{1, C_3, C_4} \right)^{-1} \left(T_{p_1, q_1}^{1, 0, 0} \right)^{-1} (f), \quad (33)$$

where $C_k (k = 3, 4)$ – arbitrary constants.

Remark 2. The solution of the type (33) we can write in the following form

$$\begin{aligned} \varphi(x) &= (x-a)^{\frac{|p_1|}{2}} [C_3 + C_4 \ln(x-a)] + f(x) \\ &\quad - \int_a^x \left\{ \left(\frac{x-a}{t-a} \right)^{\frac{|p_1|}{2}} \left[D_1 + D_2 \ln \left(\frac{x-a}{t-a} \right) \right] \right. \\ &\quad \left. + \left(\frac{x-a}{t-a} \right)^{\frac{|p_2|}{2}} \left[D_3 \right. \right. \\ &\quad \left. \left. + D_4 \ln \left(\frac{x-a}{t-a} \right) \right] \right\} \frac{f(t)}{t-a} dt, \end{aligned}$$

where

$$\begin{aligned} D_1 &= p_1 + \frac{2p_1p_2}{|p_1| - |p_2|} + \frac{4q_1p_2}{(|p_1| - |p_2|)^2} \\ &\quad - \frac{4p_1q_2}{(|p_1| - |p_2|)^2}, \\ D_2 &= -q_1 - \frac{2q_1p_2}{|p_1| - |p_2|}, \\ D_3 &= 1 - \frac{2p_1p_2}{|p_1| - |p_2|} + \frac{4q_1p_2}{(|p_1| - |p_2|)^2} \\ &\quad + \frac{4p_1q_2}{(|p_1| - |p_2|)^2}, \\ D_4 &= \frac{8q_2p_1}{(|p_1| - |p_2|)^2}. \end{aligned}$$

In the case, when $|p_2| < |p_1|$ we have the

following confirmation

Theorem 10. Let fulfillment any condition the theorem 8, besides $|p_2| > |p_1|$. Let $|p_2| < |p_1|$ and $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_3}], \delta_3 > \frac{|p_1|}{2} \text{ at } x \rightarrow a.$$

Then integral equation (1) in class of function $\varphi(x) \in C(\overline{\Gamma})$ vanishing in point $x = a$ is always solvability and its solution is given by formula (31), where C_k ($k = 1, 2, 3, 4$) –arbitrary constants.

Remark 3. The solution of the type (31) we can write in the following form

$$\begin{aligned} \varphi(x) &= (x-a)^{\frac{|p_2|}{2}} [C_3 + C_4 \ln(x-a)] \\ &+ (x-a)^{\frac{|p_1|}{2}} [C_5 + C_6 \ln(x-a)] + \\ f(x) &- \int_a^x \left\{ \left(\frac{x-a}{t-a} \right)^{\frac{|p_1|}{2}} [D_1 + D_2 \ln \left(\frac{x-a}{t-a} \right)] \right. \\ &+ \left(\frac{x-a}{t-a} \right)^{\frac{|p_2|}{2}} [D_3 \\ &+ D_4 \ln \left(\frac{x-a}{t-a} \right)] \Bigg\} \frac{f(t)}{t-a} dt \end{aligned}$$

where C_k ($k = 3, 4, 5, 6$) –arbitrary constants.

When the roots of the equation (28) complex and conjugate, that is $\lambda_1 = A_1 + iB_1$, $\lambda_2 = A_1 - iB_1$,

where $A_1 = -\frac{p_1}{2}$, $B_1 = \frac{\sqrt{D_1}}{2}$, $D_1 = 4q_1 - p_1^2$ then

the solution of the integral equation (27) given by formula [2]

$$\begin{aligned} \omega(x) &= (x-a)^{A_1} \{ \cos[B_1 \ln(x-a)]C_1 \\ &+ \sin[B_1 \ln(x-a)]C_2 \} + \\ f(x) &- 2 \int_a^x \left(\frac{x-a}{t-a} \right)^{A_1} \left\{ B_1 \sin \left[B_1 \ln \left(\frac{x-a}{t-a} \right) \right] \right. \\ &- A_1 \cos \left[B_1 \ln \left(\frac{x-a}{t-a} \right) \right] \Bigg\} \frac{f(t)}{t-a} dt \\ &\equiv \left(T_{p_1, q_1}^{2, C_1, C_2} \right)^{-1} (f). \end{aligned} \quad (34)$$

The solution of the type (34) exist, if $p_1 < 0$, $D_1 > 0$, $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_3}], \delta_3 > A_1, \text{ at } x \rightarrow a,$$

Analogously if the solution of the integral equation (26) at $p_2 < 0$,

$D_2 = 4q_2 - p_2^2 > 0$, $\omega(x) \in C(\overline{\Gamma})$, $\omega(a) = 0$ with asymptotic behavior

$$\omega(x) = o[(x-a)^{\delta_3}], \delta_3 > A_2, \text{ at } x \rightarrow a,$$

exist, then its solution is given by formula

$$\begin{aligned} \varphi(x) &= (x-a)^{A_2} \{ \cos[B_2 \ln(x-a)]C_3 \\ &+ \sin[B_2 \ln(x-a)]C_4 \} + \end{aligned}$$

$$\begin{aligned} \omega(x) &- 2 \int_a^x \left(\frac{x-a}{t-a} \right)^{A_2} \left\{ B_2 \sin \left[B_2 \ln \left(\frac{x-a}{t-a} \right) \right] \right. \\ &- A_2 \cos \left[B_2 \ln \left(\frac{x-a}{t-a} \right) \right] \Bigg\} \frac{\omega(t)}{t-a} dt \\ &\equiv \left(T_{p_2, q_2}^{2, C_3, C_4} \right)^{-1} (\omega). \end{aligned} \quad (35)$$

In formula (35) in place of (x) substitute its valued from formula (34), we obtained general solution of the integral equation (1) at the case $n = 4$, when the roots of the algebraic equation (28), (29) complex and conjugate.

$$\varphi(x) = \left(T_{p_2, q_2}^{2, C_3, C_4} \right)^{-1} \left(T_{p_1, q_1}^{2, C_1, C_2} \right)^{-1} (f) \quad (36)$$

So, we proof the following confirmation

Theorem 11. Let in integral equation (1) $n = 4$, parameters A_j ($1 \leq j \leq 4$) with coefficients algebraic equation (28), (29) connected by formula (25). $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior (28). Besides the roots of algebraic equation (28), (29) complex and conjugate, that is $\lambda_1 = A_1 + iB_1$, $\lambda_2 = A_1 - iB_1$, where $A_1 = -\frac{p_1}{2}$, $B_1 = \frac{\sqrt{D_1}}{2}$, $D_1 = 4q_1 - p_1^2$ and $\mu_1 = A_2 + iB_2$, $\mu_2 = A_2 - iB_2$, where $A_2 = -\frac{p_2}{2}$, $B_2 = \frac{\sqrt{D_2}}{2}$, $D_2 = 4q_2 - p_2^2$, p_1, p_2

negative and $|p_2| > |p_1|$. Let fulfillment any condition the theorem 8 besides $|p_2| > |p_1|$. Let $|p_2| < |p_1|$ and $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_3}], \delta_3 > \frac{|p_1|}{2}, \text{ at } x \rightarrow a,$$

Then integral equation (1) in class of function $\varphi(x) \in C(\overline{\Gamma})$ vanishing in point $x = a$ is always solvability and its solution is given by formula (36), where C_k ($k = 1, 2, 3, 4$) – arbitrary constants.

4. General Case

In general, the problem of finding the solution of the integral equation (1) for any integer n is reduced to the problem of the study of the integral equation (1) for even and odd n .

4.1 The Case, When $n = 2m$

The case, when $n = 2m$ equation (1), we represented in following form

$$\prod_{j=1}^m T_{p_j, q_j}(\varphi) = f(x), \quad (37)$$

where constants p_j, q_j ($1 \leq j \leq m$) the coefficients of the following characteristic equations

$$(\lambda^{(j)})^2 + p_j \lambda^{(j)} + q_j = 0 \quad (1 \leq j \leq m). \quad (38)$$

Later on by $\lambda_k^{(j)}$ ($k = 1, 2, 1 \leq j \leq m$) we denote the roots of the characteristic equations (38).

Integral equation (1) we can represented in the form (37), if in (1) parameters A_j ($1 \leq j \leq n$) with the roots of characteristic equations (38) connected with following formula

$$\begin{aligned} A_1 &= -\sum_{j=1}^m (\lambda_1^{(j)} + \lambda_2^{(j)}), \quad A_2 = \sum_{\substack{j,k=1 \\ j \neq k}}^m (\lambda_1^{(k)} \lambda_2^{(j)}) + \\ &\sum_{\substack{j,k=1 \\ j \neq k}}^m (\lambda_1^{(k)} \lambda_1^{(j)}) + \sum_{\substack{j,k=1 \\ j \neq k}}^m (\lambda_2^{(k)} \lambda_2^{(j)}), \quad 2! A_3 = \\ &\sum_{\substack{j,k,s=1 \\ j \neq k \neq s}}^m (\lambda_1^{(k)} \lambda_2^{(j)} \lambda_1^{(s)}) + \sum_{\substack{j,k,s=1 \\ j \neq k \neq s}}^m (\lambda_2^{(k)} \lambda_1^{(j)} \lambda_2^{(s)}) \\ (n-1)! A_n &= \prod_{j=1}^m \lambda_1^{(j)} \lambda_2^{(j)} \end{aligned} \quad (39)$$

In (37) introduce the new unknown functions

$$\begin{aligned} m(x) &= T_{p_m, q_m}(\varphi), \quad \psi_{m-1}(x) = T_{p_{m-1}, q_{m-1}}(m), \\ \psi_{m-2}(x) &= T_{p_{m-2}, q_{m-2}}(m-1), \dots, \quad \psi_2(x) = \\ &T_{p_1, q_1}(1), \quad T_{p_1, q_1}(1) = f(x). \end{aligned}$$

So, in this case the problem found the solution of the integral equation (1), reduce to problem found solution m splitting system integral equation Volterra type (9).

In particular if all the roots of characteristic equation (38) real, equal, negative and

$$|p_m| > |p_{m-1}| > |p_{m-2}| \dots \dots > |p_1|, \quad (40)$$

function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_m}], \delta_m > |p_m|, \text{ at } x \rightarrow a, \quad (41)$$

then the solution of the integral equation (1) is given by formula

$$\varphi(x) = \prod_{j=1}^m \left(T_{p_{m-j+1}, q_{m-j+1}}^{1, C_1^{m-j+1}, C_2^{m-j+1}} \right)^{-1} (f), \quad (42)$$

where C_1^{m-j+1}, C_2^{m-j+1} ($1 \leq j \leq m$) – arbitrary constants.

So, we proof the following confirmation

Theorem 12. Let in integral equation (1) $n = 2m$, parameters A_j ($1 \leq j \leq n$) with coefficients algebraic equations (38) connected by formula (39). Besides let all the roots of characteristic equation (38) real, equal and negative, function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior (41) and in (38) parameters p_j ($1 \leq j \leq m$) be satisfied condition (40). Then integral equation (I) in class of function $\varphi(x) \in C(\overline{\Gamma})$, vanishing in point $x = a$ is always solvable and its general solution contains $2m$ arbitrary constants and is given by the formula (42), where C_k^{m-j+1} ($k = 1, 2, 1 \leq j \leq m$) – arbitrary constants.

Remark 4. The representation integral equation (1) in the form (37) give possibility, in depend of the roots of the algebraic equations (38), when $n = 2m$, write out the general solution integral equation (1) in this case.

4.2 The Case, When $n = 2m + 1$

Let $n = 2m + 1$. In this case equation (1), we

represented in following form

$$\prod_{j=1}^m T_{p_j, q_j}(\varphi) = f(x), \quad (43)$$

where

$$\omega(x) = \varphi(x) + p_{m+1} \int_a^x \frac{\varphi(t)}{t-a} dt \equiv \Pi_{m+1}(\varphi), \quad (44)$$

$p_j, q_j (1 \leq j \leq m)$ the coefficients the algebraic equation

$$(\mu^{(j)})^2 + p_j \mu^{(j)} + q_j = 0 \quad (1 \leq j \leq m). \quad (45)$$

In this case, equation (43) we represented in following form

$$\varphi(x) + \int_a^x \left[\sum_{j=1}^{2m} B_j \ln^{j-1} \left(\frac{x-a}{t-a} \right) \right] \frac{\varphi(t)}{t-a} dt = f(x), \quad (46)$$

where parameters $B_j (1 \leq j \leq 2m)$ with roots of the algebraic equations (45) connected by formula

$$B_1 = -\sum_{j=1}^m (\mu_1^{(j)} + \mu_2^{(j)}), \quad B_2 = \sum_{j,k=1}^m (\mu_1^{(k)} \mu_2^{(j)}) +$$

$$\sum_{j \neq k}^m (\mu_1^{(k)} \mu_1^{(j)}) + \sum_{j,k=1}^m (\mu_2^{(k)} \mu_2^{(j)}), \quad 2! B_3 =$$

$$\sum_{j,k,s=1}^m (\mu_1^{(k)} \mu_2^{(j)} \mu_1^{(s)}) + \sum_{j \neq k \neq s}^m (\mu_2^{(k)} \mu_1^{(j)} \mu_2^{(s)})$$

$$(n-1)! B_n = \prod_{j=1}^m \mu_1^{(j)} \mu_2^{(j)}, \quad (47)$$

where $\mu_1^{(j)}, \mu_2^{(j)} (1 \leq j \leq m)$ the roots of the algebraic equations (45).

In formula (46) in place of (x) , substitute its valued by formula (44) and we observe that

$$\int_a^x \ln^{j-1} \left(\frac{x-a}{t-a} \right) \left[\int_a^t \frac{\varphi(\tau)}{\tau-a} d\tau \right] \frac{dt}{t-a} =$$

$$\frac{1}{j} \int_a^x \ln^j \left(\frac{x-a}{t-a} \right) \frac{\varphi(t)}{t-a} dt,$$

we will be have

$$\varphi(x) + \int_a^x \left[\sum_{j=1}^{2m+1} A_j \ln^{j-1} \left(\frac{x-a}{t-a} \right) \right] \frac{\varphi(t)}{t-a} dt = f(x), \quad (48)$$

where $A_1 = p_{m+1} + B_1, \quad A_2 = B_2 + B_1 p_{m+1},$

$$A_3 = B_3 + \frac{B_2 p_{m+1}}{2},$$

$$A_4 = B_4 + \frac{B_3 p_{m+1}}{3} \dots \dots \dots, \quad A_{2m} = B_{2m} +$$

$$\frac{B_{2m-1} p_{m+1}}{2m-1}, \quad A_{2m+1} = \frac{B_{2m} p_{m+1}}{2m}.$$

In this equality together $B_j (1 \leq j \leq 2m)$ substituting its significance from formulas (47), we have

$$A_1 = p_{m+1} - \sum_{j=1}^m (\mu_1^{(j)} + \mu_2^{(j)}), \quad A_2 =$$

$$\sum_{j,k=1}^m (\mu_1^{(k)} \mu_2^{(j)}) + \sum_{j \neq k}^m (\mu_1^{(k)} \mu_1^{(j)}) +$$

$$\sum_{j \neq k}^m (\mu_2^{(k)} \mu_2^{(j)}) - \sum_{j=1}^m (\mu_1^{(j)} + \mu_2^{(j)}) p_{m+1}$$

$$2! A_3 =$$

$$\sum_{j,k,s=1}^m (\mu_1^{(k)} \mu_2^{(j)} \mu_1^{(s)}) + \sum_{j \neq k \neq s}^m (\mu_2^{(k)} \mu_1^{(j)} \mu_2^{(s)}) +$$

$$p_{m+1} [\sum_{j,k=1}^m (\mu_1^{(k)} \mu_2^{(j)}) +$$

$$\sum_{j \neq k}^m (\mu_1^{(k)} \mu_1^{(j)}) + \sum_{j,k=1}^m (\mu_2^{(k)} \mu_2^{(j)})],$$

$$\dots (2m)! A_{2m+1} = p_{m+1} (\prod_{j=1}^m \mu_1^{(j)} \mu_2^{(j)}) \quad (49)$$

where $\mu_1^{(j)}, \mu_2^{(j)} (1 \leq j \leq m)$ - the roots of the algebraic equations (45).

So, we proof the following confirmation

Theorem 13. *Let in integral equation (48) parameters A_j with the roots of the characteristic equations (45) and number p_{m+1} connected by means of formulas (49). Then problem found the solution of the integral equation (1) at $n = 2m + 1$, or integral equation (48) reduce to problem finding solution integral equation*

$$\prod_{j=1}^m T_{p_j, q_j}(\Pi_{m+1}(\varphi)) = f(x). \quad (50)$$

In integral equation (50) introduce the new unknown functions

$$m+1(x) = \Pi_{m+1}(\varphi), \quad m(x) = T_{p_m, q_m}(m+1),$$

$$m-1(x) = T_{p_{m-1}, q_{m-1}}(m), \quad \psi_{m-2}(x) =$$

$$T_{p_{m-2}, q_{m-2}}(m-1) \dots, \quad \psi_2(x) = T_{p_1, q_1}(1) \quad (51)$$

$$T_{p_1, q_1}(1) = f(x), \quad (52)$$

the problem funding solution integral equation (50), we reduce to problem found solution splitting system m integral equations type (I) and one integral equation type (II).

In particular if all the roots of characteristic

equation (45) real, equal, negative and

$|p_{m+1}| > |p_m| > |p_{m-1}| > |p_{m-2}| \dots > |p_1|$, (53)
function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_{m+1}}], \quad \delta_{m+1} > |p_{m+1}|, \quad \text{at } x \rightarrow a, \quad (54)$$

then the solution of the integral equation (1) at $n = 2m + 1$ is given by formula

$$\varphi(x) = (\Pi_{p_{m+1}}^{C_{m+1}})^{-1} [\prod_{j=1}^m (T_{p_{m-j+1}, q_{m-j+1}}^{1C_1^{m-j+1}, C_2^{m-j+1}})^{-1} (f)], \quad (55)$$

where $C_1^{m-j+1}, C_2^{m-j+1} (1 \leq j \leq m), C_{m+1}$ – arbitrary constants,

$$(\Pi_{p_{m+1}}^{C_{m+1}})^{-1}(\omega) \equiv (x-a)^{|p_{m+1}|} C_{m+1} + \omega(x) - p_{m+1} \int_a^x \left(\frac{x-a}{t-a}\right)^{|p_{m+1}|} \frac{\omega(t)}{t-a} dt. \quad (56)$$

So, we proof the following confirmation

Theorem 14. *Let in integral equation (1) $n = 2m + 1$, parameters $A_j (1 \leq j \leq n)$ with coefficients algebraic equations (45) connected by formula (49). Besides let all the roots of characteristic equation (45) real, equal and negative, function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior (54) and in (45) parameters $p_j (1 \leq j \leq m), p_{m+1}$ be satisfied condition (53). Then integral equation (I) in class of function $\varphi(x) \in C(\overline{\Gamma})$, vanishing in point $x = a$ is always solvable and its general solution contains $2m + 1$ arbitrary constants and is given by the formula (55), where $C_k^{m-j+1} (k = 1, 2, 1 \leq j \leq m), C_{m+1}$ – arbitrary constants.*

5. Boundary Value Problems at $n = 3$ and When the Roots of Algebraic Equation (10) Real and Different

In the case, when parameters $A_j (1 \leq j \leq 3)$ with constants p_1, q_1, p_3 in integral operators (5) and (5₁) among themselves connected by formula (6), then problem found solution integral equation (4) reduce to problem found solution integral equations (8)

and (9). When, the roots of the characteristic equation (10) real, different and positive, the solution of the integral equation (9) is given by formula (11).

If, fulfillment any conditions of the theorem 2 and $f(x) \in C(\overline{\Gamma})$, then $\varphi(x) \in C(\overline{\Gamma})$ also. In this case, from (11), when $\Pi_{p_3}(\varphi) = \omega(x) \in C(\overline{\Gamma})$, $\omega(a) = 0$ with asymptotic behavior

$$\omega(x) = o[(x-a)^{\delta_7}], \quad \delta_7 > |p_3| \quad \text{at } x \rightarrow a \quad (57)$$

we have

$$C_1 = \frac{1}{\lambda_2 - \lambda_1} \left\{ (x-a)^{-\lambda_1} \left\{ [\lambda_2 \Pi_{p_3}(\varphi) - D_x^a (\Pi_{p_3}(\varphi))] \right\} \right\}_{x=a}, \quad (58)$$

$$C_2 = \frac{1}{\lambda_2 - \lambda_1} \left\{ (x-a)^{-\lambda_2} \left\{ [\lambda_1 \Pi_{p_3}(\varphi) - D_x^a (\Pi_{p_3}(\varphi))] \right\} \right\}_{x=a}, \quad (59)$$

where $D_x^a = (x-a) \frac{d}{dx}$.

From equality $\Pi_{p_3}(\varphi) = \omega(x)$, at $p_3 < 0$, we found (formula 12)

$$\begin{aligned} \varphi(x) &= (x-a)^{|p_3|} C_3 + \omega(x) \\ &- p_3 \int_a^x \left(\frac{x-a}{t-a}\right)^{|p_3|} \frac{\omega(t)}{t-a} dt \\ &\equiv (\Pi_{p_3}^{C_3})^{-1}(\omega). \end{aligned} \quad (60)$$

If $\omega(x)$ satisfy condition (57), then from (60) we found

$$[(x-a)^{p_3} \varphi(x)]_{x=a} = C_3 \quad (61)$$

So, if fulfillment any conditions theorem 2, then valued arbitrary constants $C_j (1 \leq j \leq 3)$, by valued $\varphi(x)$ and its first order derivative in point $x = a$ found by formulas (58), (59), (61).

In the case, when $p_1 < 0, q_1 < 0, D_1 = p_1^2 - 4q_1 > 0$, $\lambda_1 = \frac{|p_1| + \sqrt{D_1}}{2}$, $\lambda_2 = \frac{|p_1| - \sqrt{D_1}}{2}$ ($\lambda_1 > 0, \lambda_2 < 0$) solution integral equation (4) or (7) is

given by formula

$$\begin{aligned} \omega(x) &= (x-a)^{\lambda_1} C_4 + f(x) - \\ &- \frac{1}{\sqrt{D_1}} \int_a^x \left[\lambda_2^2 \left(\frac{t-a}{x-a} \right)^{|\lambda_2|} - \lambda_1^2 \left(\frac{x-a}{t-a} \right)^{\lambda_1} \right] \frac{f(t)}{t-a} dt \equiv \\ &\left(T_{p_1, q_1}^{C_4} \right)^{-1} (f), \end{aligned} \quad (62)$$

If $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$, with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_8}], \quad \delta_8 > \lambda_1 \text{ at } x \rightarrow a.$$

From here we found

$$[(x-a)^{-\lambda_1} \Pi_{p_3}(\varphi)]_{x=a} = C_4. \quad (63)$$

If $p_1 < 0$, then valued C_3 by valued $\varphi(x)$ in point $x = a$, found by formula (61).

In the case, when $p_1 > 0$, $q_1 > 0$, $D_1 > 0$, then

$$\lambda_1 = \frac{-p_1 + \sqrt{D_1}}{2} < 0, \quad \lambda_2 = \frac{-p_1 - \sqrt{D_1}}{2} < 0. \text{ In this case}$$

equation (4) or (7) have unique solution and its solution is given by formula

$$\begin{aligned} \omega(x) &= f(x) - \frac{1}{\sqrt{D_1}} \int_a^x \left[\lambda_2^2 \left(\frac{t-a}{x-a} \right)^{|\lambda_2|} \right. \\ &\quad \left. - \lambda_1^2 \left(\frac{t-a}{x-a} \right)^{|\lambda_1|} \right] \frac{f(t)}{t-a} dt \\ &\equiv \left(T_{p_1, q_1} \right)^{-1} (f). \end{aligned} \quad (64)$$

On the bases of higher mentioned properties the solution integral equation (4) or (7) in the case, when the roots of algebraic equation (28) real, different, for integral equation (4) or (7) stand and investigation the following boundary value problems.

Problem R_1 . Is required the solution of integral equation (4) or (7) from class $C(\overline{\Gamma})$ at $p_1 < 0$, $q_1 > 0$, $p_3 < 0$, $D_1 > 0$ ($A_1 < 0$, $A_2 > 0$, $A_3 < 0$) by boundary conditions

$$\begin{cases} [(x-a)^{p_3} \varphi(x)]_{x=a} = E_1 \\ \left\{ (x-a)^{-\lambda_1} \left\{ [\lambda_2 \Pi_{p_3}(\varphi) - D_x^a (\Pi_{p_3}(\varphi))] \right\} \right\}_{x=a} = E_2, \\ \left\{ (x-a)^{-\lambda_2} \left\{ [\lambda_1 \Pi_{p_3}(\varphi) - D_x^a (\Pi_{p_3}(\varphi))] \right\} \right\}_{x=a} = E_3 \end{cases} \quad (R_1)$$

where $E_j (1 \leq j \leq 3)$ -are given constants.

Problem R_2 . Is required the solution of integral equation (4) or (7) from class $C(\overline{\Gamma})$ at $p_1 < 0$, $q_1 < 0$, $p_3 < 0$, $D_1 > 0$ ($A_1 < 0$, A_2 - have arbitrary symbol, $A_3 > 0$) by boundary conditions

$$\begin{cases} [(x-a)^{p_3} \varphi(x)]_{x=a} = E_4 \\ \left\{ [(x-a)^{-\lambda_1} \Pi_{p_3}(\varphi)]_{x=a} = E_5 \right\}, \end{cases} \quad (R_2)$$

where $E_j (4 \leq j \leq 5)$ -are given constants.

Problem R_3 . Is required the solution of integral equation (4) or (7) from class $C(\overline{\Gamma})$ at $p_1 > 0$, $q_1 > 0$, $p_3 < 0$, $D_1 > 0$ (A_1 - have arbitrary symbol, A_2 - have arbitrary symbol, $A_3 < 0$) by boundary conditions

$$[(x-a)^{p_3} \varphi(x)]_{x=a} = E_6, \quad (R_3)$$

where E_6 -are given constant.

Solution problem R_1 . Let fulfillment any condition of the theorem 2. Then use integral representation (11), (12) (or 14), properties (58), (59), (63) and conditions (R_1), we found $C_1 = -\frac{E_2}{\sqrt{D_1}}$,

$$C_2 = -\frac{E_3}{\sqrt{D_1}}, \quad C_3 = E_1. \text{ Substituting this valued } C_1, C_2,$$

C_3 in formula (14), we found solution problem.

In the following form,

$$\varphi(x) = \left(\Pi_{p_3}^{E_1} \right)^{-1} \left[\left(T_{p_1, p_2}^{-\frac{E_2}{\sqrt{D_1}} - \frac{E_3}{\sqrt{D_1}}} \right)^{-1} (f) \right]. \quad (65)$$

So, we proof the following confirmation

Theorem 15. Let in integral equation (4) or (7) function $f(x)$ satisfy any conditions theorem 2. Then problem R_1 has a unique solution which is given by formula (65).

Solution problem R_2 . Let in integral equation (4) or (7) parameters p_1 , q_1 , p_3 , D_1 satisfied any conditions problem R_2 . Then use integral representation (62) and (60), properties (61) and (63), condition (R_2), we have, $C_4 = E_5$, $C_3 = E_4$. Substituting this valued C_4, C_3 in formulas (62) and (60), we found

$$\varphi(x) = \left(\Pi_{p_3}^{E_4} \right)^{-1} \left[\left(T_{p_1, q_1}^{E_5} \right)^{-1} (f) \right]. \quad (66)$$

So, we proof the following confirmation

Theorem 16. Let, in integral equation (4) or (7) parameters p_1, q_1, p_3 with constants, A_1, A_2, A_3 connection by formula (6). Let $p_1 < 0, q_1 < 0, p_3 < 0, D_1 > 0$ ($A_1 < 0, A_2$ – have arbitrary symbol, $A_3 > 0, \lambda_1 > 0, \lambda_2 < 0$), $\lambda_1 > |p_3|$. Function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_8}], \delta_8 > \lambda_1, \text{ at } x \rightarrow a. \quad (67)$$

Then, Problem R_2 has a unique solution which is given by formula (66).

Solution problem R_3 . Let $p_1 > 0, q_1 > 0, p_3 < 0, D_1 > 0$ ($A_1 > 0, A_2$ – arbitrary, $A_3 < 0, \lambda_1 < 0, \lambda_2 < 0$). In this case, if solution integral equation (9) exist, then its we will be represented in form (64). Solution (64) exist, if

$$f(x) \in C(\overline{\Gamma}), f(a) = 0, \text{ with asymptotic behavior } f(x) = o[(x-a)^\varepsilon] \varepsilon > 0, \text{ at } x \rightarrow a. \quad (68)$$

Then solution integral equation (4) or (7) is given by formula

$$\varphi(x) = (\Pi_{p_3}^{C_5})^{-1} [(T_{p_1, q_1})^{-1}(f)]. \quad (69)$$

where C_5 arbitrary constant.

The solution of the type (69) exist, if function $\omega(x) = (T_{p_1, p_2})^{-1}(f)$ be satisfied the following conditions: $\omega(x) \in C(\overline{\Gamma}), \omega(a) = 0$, with asymptotic behavior

$$\omega(x) = o[(x-a)^{\delta_{10}}], \delta_{10} > |p_3| \text{ at } x \rightarrow a.$$

If function $f(x) \in C(\overline{\Gamma}), f(a) = 0$, with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_{10}}], \delta_{10} > |p_3| \text{ at } x \rightarrow a, \quad (70)$$

then solution types (69) exist. Now use formula (69) properties (61) and conditions (R_3), we found, $C_5 = E_6$. Substitute obtained value C_5 to formula (69), we found solution problem R_3 in following form

$$\varphi(x) = (\Pi_{p_3}^{E_6})^{-1} [(T_{p_1, q_1})^{-1}(f)]. \quad (71)$$

So, we proof the following confirmation

Theorem 17. Let in integral equation (4) or (7)

parameters p_1, q_1, p_3 with constants, A_1, A_2, A_3 connection by formula (6). Let $p_1 > 0, q_1 > 0, p_3 < 0, D_1 > 0$ (A_1 – have arbitrary symbol, A_2 – have arbitrary symbol, $A_3 < 0, \lambda_1 < 0, \lambda_2 < 0$). Function $f(x)$ be satisfied condition (70). Then Problem R_3 has a unique solution which is given by formula (71).

Remark 5. Problems similar to problems (R_1) – (R_3) stand and investigation in the possibility cases: 1) $p_1 < 0, q_1 > 0, p_3 > 0, D_1 > 0$; 2) $p_1 < 0, q_1 < 0, p_3 > 0, D_1 > 0$.

6. Boundary Value Problems at $n = 3$ and When the Roots of Algebraic Equation (10) Real and Equal

When, the roots of the characteristic equation (10) real, equal and positive, the solution of the integral equation (9) is given by formula (16).

If, fulfillment any conditions of the theorem 2 and $f(x) \in C(\overline{\Gamma})$, then $\varphi(x) \in C(\overline{\Gamma})$ also. In this case, from (16), when $\Pi_{p_3}(\varphi) = \omega(x) \in C(\overline{\Gamma}), \omega(a) = 0$ with asymptotic behavior

$$\omega(x) = o[(x-a)^{\delta_{11}}], \delta_{11} > \frac{|p_1|}{2} \text{ at } x \rightarrow a. \quad (72)$$

we have

$$C_4 = \left\{ \ln(x-a)(x-a)^{\frac{p_1}{2}} \left\{ \frac{p_1}{2} \Pi_{p_3}(\varphi) + D_x^a (\Pi_{p_3}(\varphi)) \right\} \right\}_{x=a}, \quad (73)$$

$$C_5 = - \left\{ (x-a)^{\frac{p_1}{2}} \left\{ \frac{p_1}{2} \Pi_{p_3}(\varphi) + D_x^a (\Pi_{p_3}(\varphi)) \right\} \right\}_{x=a}, \quad (74)$$

In this case stand the following boundary value problems

Problem R_4 . Is required the solution of integral equation (4) or (7) from class $C(\overline{\Gamma})$ at $p_1 < 0, 4q_1 = p_1^2, p_3 < 0$ ($A_1 < 0, A_2 > 0, A_3 < 0$) by boundary conditions

$$\left\{ \begin{array}{l} [(x-a)^{p_3} \varphi(x)]_{x=a} = E_7 \\ \left\{ \ln(x-a)(x-a)^{\frac{p_1}{2}} \left\{ \frac{p_1}{2} \Pi_{p_3}(\varphi) + D_x^a \left(\Pi_{p_3}(\varphi) \right) \right\} \right\}_{x=a} = E_8, \\ \left\{ (x-a)^{\frac{p_1}{2}} \left\{ \frac{p_1}{2} \Pi_{p_3}(\varphi) + D_x^a \left(\Pi_{p_3}(\varphi) \right) \right\} \right\}_{x=a} = E_9 \end{array} \right. \quad (R_4)$$

where $E_j (7 \leq j \leq 9)$ -are given constants.

Problem R_5 . Is required the solution of integral equation (4) or (7) from class $C(\bar{\Gamma})$ at $p_1 > 0$, $4q_1 = p_1^2$, $p_3 < 0$ (A_1 - have arbitrary symbol, A_2 - have arbitrary symbol, $A_3 < 0$) by boundary conditions

$$[(x-a)^{p_3} \varphi(x)]_{x=a} = E_{10}, \quad (R_5)$$

where E_{10} -are given constant.

Solution problem R_4 . Let $p_1 < 0$, $4q_1 = p_1^2$, $p_3 < 0$, $(A_1 < 0, A_2 > 0, A_3 < 0, \lambda_1 = \lambda_2 = -\frac{p_1}{2} < 0)$. In this case, if solution integral equation (9)

exist, then its we will be represented in form (16). Solution type (16) exist, if

$$f(x) = o[(x-a)^{\delta_{11}}], \delta_{11} > \frac{|p_1|}{2} \text{ at } x \rightarrow a \quad (72.1)$$

Then solution integral equation (4) or (7) is given by formula

$$\varphi(x) = (\Pi_{p_3}^{C_6})^{-1} \left[(T_{p_1, q_1}^{2, C_4, C_5})^{-1} (f) \right]. \quad (75)$$

The solution of the type (75) exist, if function

$\omega(x) = (T_{p_1, q_1}^{2, C_4, C_5})^{-1} (f)$ be satisfied the following conditions: $\omega(x) \in C(\bar{\Gamma})$, $\omega(a) = 0$, with asymptotic behavior

$$\omega(x) = o[(x-a)^{\delta_{12}}], \delta_{12} > |p_3|, \text{ at } x \rightarrow a.$$

If function $f(x) \in C(\bar{\Gamma})$, $f(a) = 0$, with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_{13}}], \delta_{13} > \frac{|p_1|}{2} - \varepsilon, \text{ at } x \rightarrow a, \quad (76)$$

then solution types (75) exist. Now use formula (75) properties (73), (74) and conditions (R_4) , we found, $C_6 = E_7$, $C_4 = E_8$, $C_5 = E_9$. Substitute obtained value C_4 , C_5 , C_6 to formula (75), we found solution problem R_4 in following form

$$\varphi(x) = (\Pi_{p_3}^{E_7})^{-1} \left[(T_{p_1, q_1}^{2, E_8, E_9})^{-1} (f) \right]. \quad (77)$$

So, we proof

Theorem 18. Let in integral equation (4) or (7) parameters p_1 , q_1 , p_3 with constants, A_1 , A_2 , A_3 connection by formula (6). Let $p_1 < 0$, $4q_1 = p_1^2$, $p_3 < 0$ ($A_1 < 0$, A_2 - have arbitrary symbol,

$A_3 < 0$, $\lambda_1 = \lambda_2 = \frac{|p_1|}{2}$). Function $f(x) \in C(\bar{\Gamma})$, $f(a) = 0$ with asymptotic behavior (76). Then, a Problem R_4 has a unique solution which is given by formula (77).

Solution problem R_5 . Let $p_1 > 0$, $4q_1 = p_1^2$, $p_3 < 0$, (A_1 - have arbitrary symbol, A_2 - have

arbitrary symbol, $A_3 < 0$, $\lambda_1 = \lambda_2 = -\frac{p_1}{2} < 0$). In this case, if solution integral equation (9) exist, then its we will be represented in form

$$\omega(x) = f(x) - \int_a^x \left(\frac{t-a}{x-a} \right)^{\frac{p_1}{2}} [p_1 - q_1 \ln \left(\frac{x-a}{t-a} \right)] \frac{f(t)}{t-a} dt \equiv (T_{p_1, q_1}^2)^{-1} (f). \quad (78)$$

Solution type (78) exist, if $f(x) \in C(\bar{\Gamma})$, $f(a) = 0$, with asymptotic behavior

$$f(x) = o[(x-a)^\varepsilon], \varepsilon > 0 \text{ at } x \rightarrow a.$$

But for solvability integral equation $\Pi_{p_3}(\varphi) = \omega(x)$, necessary $\omega(x) \in C(\bar{\Gamma})$, $\omega(a) = 0$, with asymptotic behavior

$$\omega(x) = o[(x-a)^{\delta_{14}}], \delta_{14} > |p_3| \text{ at } x \rightarrow a. \quad (79)$$

Asymptotic behavior (79) possibly, if $(x) \in C(\bar{\Gamma})$, $f(a) = 0$, with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_{14}}], \delta_{14} > |p_3|, \text{ at } x \rightarrow a. \quad (80)$$

So, we proof

Theorem 19. Let in integral equation (4) or (7)

parameters p_1, q_1, p_3 with constants, A_1, A_2, A_3 connection by formula (6). Let $p_1 > 0, 4q_1 = p_1^2, p_3 < 0$ (A_1 – have arbitrary symbol, A_2 –

have arbitrary symbol, $A_3 < 0, \lambda_1 = \lambda_2 = -\frac{p_1}{2}$).

Function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior (80). Then, Problem **R₅** has a unique solution which is given by formula)

$$\varphi(x) = (\Pi_{p_3}^{E_{10}})^{-1} [(T_{p_1, q_1}^2)^{-1}(f)].$$

7. Boundary Value Problems at $n = 3$ and When the Roots of Algebraic Equation (10) Complex and Conjugate

When, the roots of the characteristic equation (10) complex and conjugate and $p_1 < 0$, the solution of the integral equation (9) is given by formula (34).

If, fulfillment any conditions of the theorem 2 and $f(x) \in C'(\overline{\Gamma})$, then $\varphi(x) \in C'(\overline{\Gamma})$ also. In this case, from (16), when $\Pi_{p_3}(\varphi) = \omega(x) \in C'(\overline{\Gamma})$, $\omega(a) = 0$ with asymptotic behavior

$$\omega(x) = o[(x-a)^{\delta_{11}}], \delta_{11} > \frac{|p_1|}{2} \text{ at } x \rightarrow a \quad (81)$$

we have

$$\begin{cases} [(x-a)^{p_3}\varphi(x)]_{x=a} = E_{11} \\ \left\{ \left((x-a)^{\frac{p_1}{2}} \left\{ \left(\frac{|p_1|}{2} \sin[B\ln(x-a)] + B\cos[B\ln(x-a)] \right) \Pi_{p_3}(\varphi) + \sin[B\ln(x-a)] D_x^a (\Pi_{p_3}(\varphi)) \right\} \right) \right\}_{x=a} = E_{12} \\ \left\{ \left((x-a)^{\frac{p_1}{2}} \left\{ -\left[\frac{|p_1|}{2} \cos[B\ln(x-a)] - B\sin[B\ln(x-a)] \right] \Pi_{p_3}(\varphi) + \cos[B\ln(x-a)] D_x^a (\Pi_{p_3}(\varphi)) \right\} \right) \right\}_{x=a} = E_{13} \end{cases}, \quad (R_6)$$

where $E_j (11 \leq j \leq 13)$ -are given constants.

Problem R₇. Is required the solution of integral equation (4) or (7) from class $C(\overline{\Gamma})$ at $p_1 > 0, 4q_1 - p_1^2 > 0, p_3 < 0$ (A_1 – have arbitrary symbol, A_2 – have arbitrary symbol, $A_3 < 0$) by boundary conditions

$$[(x-a)^{p_3}\varphi(x)]_{x=a} = E_{14}, \quad (R_7)$$

where E_{10} -are given constant.

Solution problem R₆. Let $p_1 < 0, 4q_1 - p_1^2 > 0, p_3 < 0$,

$$C_7 = \frac{1}{B} \left\{ (x-a)^{\frac{p_1}{2}} \left\{ \left(\frac{|p_1|}{2} \sin[B\ln(x-a)] + B\cos[B\ln(x-a)] \right) \Pi_{p_3}(\varphi) + \sin[B\ln(x-a)] D_x^a (\Pi_{p_3}(\varphi)) \right\} \right\}_{x=a} \quad (82)$$

$$C_8 = \frac{1}{B} \left\{ (x-a)^{\frac{p_1}{2}} \left\{ -\left[\frac{|p_1|}{2} \cos[B\ln(x-a)] - B\sin[B\ln(x-a)] \right] \Pi_{p_3}(\varphi) + \cos[B\ln(x-a)] D_x^a (\Pi_{p_3}(\varphi)) \right\} \right\}_{x=a} \quad (83)$$

In this case stand the following boundary value problems

Problem R₆. Is required the solution of integral equation (4) or (7) from class $C(\overline{\Gamma})$ at $p_1 < 0, 4q_1 - p_1^2 > 0, p_3 < 0$ ($A_1 < 0, A_2$ – have arbitrary symbol, $A_3 < 0$) by boundary conditions

($A_1 < 0, A_2$ – have arbitrary symbol, $A_3 < 0$,

$\lambda_1 = A + iB, \lambda_2 = A - iB$, $A = -\frac{p_1}{2}$, $B = \frac{\sqrt{p_1}}{2}$,

$D_1 = 4q_1 - p_1^2$). In this case, if solution integral equation (9) exist, then its we will be represented in form (20). Solution type (20) exist, if

$f(x) \in C(\overline{\Gamma})$, $f(a) = 0$, with asymptotic behavior (21).

Then solution integral equation (4) or (7) is given by formula (22), that is by formula

$$\varphi(x) = (\Pi_{p_3}^{C_9})^{-1} \left[\left(T_{p_1, q_1}^{3, C_7, C_8} \right)^{-1} (f) \right]. \quad (84)$$

The solution of the type (84) exist, if function

$\omega(x) = \left(T_{p_1, q_1}^{3, C_7, C_8} \right)^{-1} (f)$ be satisfied the following conditions: $\omega(x) \in C(\overline{\Gamma})$, $\omega(a) = 0$, with asymptotic behavior

$$\omega(x) = o[(x-a)^{\delta_{12}}], \delta_{12} > |p_3|, \text{ at } x \rightarrow a.$$

If, function $f(x) \in C(\overline{\Gamma})$, $\omega(a) = 0$, with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_{13}}] \delta_{13} > \frac{|p_1|}{2} - \varepsilon, \text{ at } x \rightarrow a, \quad (85)$$

then solution types (84) exist. Now use formula (84) properties (82), (83) and conditions (R_6), we found,

$$C_9 = E_{11}, C_7 = \frac{E_{12}}{B}, C_8 = \frac{E_{13}}{B}.$$

Substitute obtained value C_7, C_8, C_9 to formula (84), we found solution problem R_6 in following form

$$\varphi(x) = (\Pi_{p_3}^{E_{11}})^{-1} \left[\left(T_{p_1, q_1}^{\frac{E_{12}}{B}, \frac{E_{13}}{B}} \right)^{-1} (f) \right]. \quad (86)$$

So, we proof

Theorem 20. Let, in integral equation (4) or (7) parameters p_1, q_1, p_3 with constants, A_1, A_2, A_3 connection by formula (6). Let $p_1 < 0, 4q_1 - p_1^2 > 0, p_3 < 0$ ($A_1 < 0, A_2$ – have arbitrary symbol, $A_3 < 0$). Function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior (85). Then, the problem R_6 has a unique solution which is given by formula (86).

Solution problem R_7 . Let $p_1 > 0, 4q_1 - p_1^2 > 0, p_3 < 0$ (A_1 – have arbitrary symbol, A_2 – have arbitrary symbol, $A_3 < 0$). In this case, if solution integral equation (9) exist, then its we will be represented in form

$$\begin{aligned} \omega(x) = f(x) - 2 \int_a^x \left(\frac{t-a}{x-a} \right)^{\frac{p_1}{2}} \left[-A \cos \left[B \ln \left(\frac{x-a}{t-a} \right) \right] \right. \\ \left. + B \sin \left[B \ln \left(\frac{x-a}{t-a} \right) \right] \right] \frac{f(t)}{t-a} dt \equiv \\ \equiv \left(T_{p_1, q_1}^3 \right)^{-1} (f). \end{aligned} \quad (87)$$

Solution type (87) exist, if $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$, with asymptotic behavior

$$f(x) = o[(x-a)^\varepsilon], \varepsilon > 0 \text{ at } x \rightarrow a.$$

But for solvability solution integral equation $\Pi_{p_3}(\varphi) = \omega(x)$, necessary $\omega(x) \in C(\overline{\Gamma})$, $\omega(a) = 0$, with asymptotic behavior

$$\omega(x) = o[(x-a)^{\delta_{13}}], \delta_{13} > |p_3| \text{ at } x \rightarrow a. \quad (88)$$

Asymptotic behavior (88) is possibly, if $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$, with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_{14}}], \delta_{14} > |p_3|, \text{ at } x \rightarrow a. \quad (89)$$

So, we proof

Theorem 21. Let, in integral equation (4) or (7) parameters p_1, q_1, p_3 with constants, A_1, A_2, A_3 connection by formula (6). Let $p_1 > 0, 4q_1 - p_1^2 > 0, p_3 < 0$ (A_1 – have arbitrary symbol, A_2 – have arbitrary symbol, $A_3 < 0$). Function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior (89). Then, Problem R_7 has a unique solution which is given by formula

$$\varphi(x) = (\Pi_{p_3}^{E_{14}})^{-1} \left[\left(T_{p_1, q_1}^3 \right)^{-1} (f) \right].$$

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