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Characterization of Negative Exponential Distribution Through Expectation of Function of Order Statistics

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For characterization of negative exponential distribution one needs any arbitrary non constant function only in place of approaches such as identical distributions, absolute continuity, constancy of regression of order statistics, continuity and linear regression of order statistics, non-degeneracy etc. available in the literature. Recently Bhatt characterized negative exponential distribution through expectation of non constant function of random variable. Attempt is made to extend the characterization of negative exponential distribution through expectation of any arbitrary non constant function of order statistics.

Introduction

Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with absolutely continuous distribution F and $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the set of corresponding order statistics. For $n = 2$ Fisz (1958) assumed that random variables X_1 and X_2, were positive and used ratio instead of difference and proved that X_1 and X_2 have negative exponential distribution if and only if $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent. Tanis (1964) made Fisz's result arbitrary for any finite n, that is, $X_{(1)}$ and $\sum_{i=2}^{n}\left(X_{(i)} - X_{(1)}\right)$ are independent under the same setup (identical distributions and absolute continuity) and the same method (derive results for exponent of negative of assumed positive variables using ratio instead of difference).

Under the same setup assuming independence of positive difference of adjacent order statistics; $\left(X_{(m+1)} - X_{(m)}\right)$ and $X_{(m)}$; $m = 2, 3, \ldots, n$, Rogers (1963) showed that if the regression of $\left(X_{(m+1)} - X_{(m)}\right)$ on $X_{(m)}$ is constant then distribution F is negative exponential. replacing Fisz's assumptions of absolute continuity and Rogers's assumptions of constancy of regression of $\left(X_{(m+1)} - X_{(m)}\right)$ on $X_{(m)}$ by continuity and linear regression of $X_{(m+1)}$ on $X_{(m)}$, Fergusion (1967) generalizd the Roger's result and noted that it is not known which distribution would be characterized if non-adjacent order statistics were considered. Closely related to Fisz'sresults, Fergusion (1964, 1965) and Crawford (1966) considered stronger assumptions of independence and non-degeneracy for X_1, X_2 in place of identical distribution and absolute continuity, derived characterization for negative exponential distribution and geometric distribution. Srivastave (1967) added one more characterization to existing set of characterization of negative

Nagaraja (1977, 1988) obtained characterization results based on linear regression of two adjacent record valuesand noted that Fergusion's remark for non-adjacent order statistics holds for record values too. Khan (2009) characterized family of continuous probability distributions through the difference of two conditional expectations, conditioned on anon-adjacent order statistics which include negative exponential distribution.

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Using identity of distribution and equality of expectation of function of a random variable, Bhatt (2013) derived characterization of negative exponential distribution with probability density function (pdf)

$$f_X(x,\theta) = \begin{cases} e^{-(x-\theta)}\,;\ \theta < x < \infty; \\ \quad 0\,;\ \text{otherwise}, \end{cases} \tag{1.1}$$

where $-\infty \le a < b \le \infty$ are known constants, e^{-x} is positive absolutely continuous function and e^{θ} is everywhere differentiable function. Since derivative of e^{θ} being positive and since range is truncated by θ from left e^{-b} = 0.

Aim of this research note is to extend Bhatt's result for expectation of order statistics. In section 2 negative exponential distribution with pdf (1.1) is characterized by using function of first order statistic. Section 3 is devoted for illustrativeexamples.

Characterization

Theorem 2.1: Let $X_1, X_2, \dots, X_n$ be a random sample of size n from distribution function F. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the corresponding set of order statistics. Assume that F is continuous on the interval (a,b), where $-\infty \le a < b \le \infty$. Let $\phi(X_{1:n})$ and $g(X_{1:n})$ be two distinct differentiable and intregrable functions of first order statistic; $X_{1:n}$, on the interval (a,b), where $-\infty \le a < b \le \infty$ and moreover $g(X_{1:n})$ be non-constant of $X_{1:n}$. Then

$$E\left[g(X_{1:n}) - \left(\frac{1}{n}\right)\frac{d}{dX_{1:n}}g(X_{1:n})\right] = g(\theta) \tag{2.1}$$

is the necessary and sufficient condition for pdf $f(x,\theta)$ of F to be $f(x,\theta)$ defined in (1.1).

Proof. Given $f(x,\theta)$ defined in (1.1), for necessity of (2.1) if $\phi(X_{1:n})$ is such that $g(\theta) = E[\phi(X_{1:n})]$ where $g(\theta)$ is differentiable function then using $f(x_{1:n},\theta)$; pdf of first order statistics one gets

$$g(\theta) = \int_\theta^b \phi(x_{1:n})\, f(x_{1:n},\theta)\, dx_{1:n} \tag{2.2}$$

Differentiating with respect to θ on both sides of (2.2), replacing $X_{1:n}$ for θ and simplifying the result one gets

$$\phi(X_{1:n}) = g(X_{1:n}) - \left(\frac{1}{n}\right)\frac{d}{dX_{1:n}}g(X_{1:n}) \tag{2.3}$$

which establishes necessity of (2.1). Conversely given (2.1), if $k(x_{1:n},\theta)$ is pdf of first order statistics; $X_{1:n}$ of random variable X then and one gets $g(\theta)$ as

$$g(\theta) = \int_\theta^b \left[g(x_{1:n}) - \left(\frac{1}{\text{n}}\right)\frac{d}{dx_{1:n}}g(x_{1:n})\right] k(x_{1:n},\theta)\, dx_{1:n} \tag{2.4}$$

Since $e^{nx_{1:n}}$ is increasing integrable and differentiable function on the interval (a,b) with $e^{-b} = 0$ the following identity holds

$$g(\theta) \equiv -e^{n\theta} \int_{\theta}^{b} \left[\frac{d}{dx_{1:n}}\{e^{-nx_{1:n}} g(x_{1:n})\}\right] dx_{1:n} \tag{2.5}$$

Differentiating integrand of (2.5), $\{e^{-nx_{1:n}} g(x_{1:n})\}$ with respect to $x_{1:n}$ one gets

$$g(\theta) \equiv -e^{n\theta} \int_{\theta}^{b} \left[g(x_{1:n}) \frac{d}{dx_{1:n}} e^{-nx_{1:n}} + e^{-nx_{1:n}} \frac{d}{dx_{1:n}} g(x_{1:n})\right] dx_{1:n} \tag{2.6}$$

Taking $\left\{-\frac{d}{dx_{1:n}}(e^{-nx_{1:n}})\right\}$ as one factor, (2.6) reduces to

$$g(\theta) \equiv -e^{n\theta} \int_{\theta}^{b} \left[g(x_{1:n}) + \frac{e^{-nx_{1:n}}}{\frac{d}{dx_{1:n}} e^{-nx_{1:n}}} \frac{d}{dx_{1:n}} g(x_{1:n})\right] \frac{d}{dx_{1:n}} e^{-nx_{1:n}}\, dx_{1:n} \tag{2.7}$$

and after simplification one gets $g(\theta)$ as

$$g(\theta) \equiv \int_{\theta}^{b} \left[g(x_{1:n}) + \frac{e^{-nx_{1:n}}}{\frac{d}{dx_{1:n}} e^{-nx_{1:n}}} \frac{d}{dx_{1:n}} g(x_{1:n})\right] \left\{-e^{n\theta} \frac{d}{dx_{1:n}} e^{-nx_{1:n}}\right\} dx_{1:n} \tag{2.8}$$

Substituting $\frac{d}{dx_{1:n}} e^{-nx_{1:n}}$ in (2.8) one gets

$$g(\theta) \equiv \int_{\theta}^{b} \left[g(x_{1:n}) - \left(\frac{1}{n}\right) \frac{d}{dx_{1:n}} g(x_{1:n})\right] \left\{\frac{1}{n} e^{-n(x_{1:n}-\theta)}\right\} dx_{1:n} \tag{2.9}$$

From (2.4) and (2.9) by uniqueness theorem

$$k(x_{1:n}, \theta) = \frac{1}{n} e^{-n(x_{1:n}-\theta)};\ a < \theta < x_{1:n} < b \tag{2.10}$$

Since $e^{nx_{1:n}}$ be increasing integrable and differentiable function on the interval (a, b) where $-\infty \leq a < b \leq \infty$ with $e^{-b} = 0$ and integrating (2.10) over the interval (θ, b) on both sides, one gets

$1 = \int_{\theta}^{b} k(x_{1:n};\ \theta)\, dx_{1:n}$.

Hence from (2.10) $[k(x_{1:n}; \theta)]_{n=1}$ reduces to $f(x; \theta)$ defined in (1.1) which establishes sufficiency of (2.1).

Remark 2.1

Using $\phi(X_{1:n})$ derived in (2.3), the $f(x, \theta)$ given in (1.1) can be determined by

$$M(X_{1:n}) = \frac{\frac{d}{dX_{1:n}} g(X_{1:n})}{\phi(X_{1:n}) - g(X_{1:n})} \tag{2.11}$$

and pdf is given by

$$f(x, \theta) = -\left[\frac{\frac{d}{dX_{1:n}}(T(X_{1:n}))}{T(\theta)}\right]_{n=1} \tag{2.12}$$

where $T(X_{1:n})$ is decreasing function for $-\infty \leq a < b \leq \infty$ with $T(b) = 0$ such that it satisfies

$$M(X_{1:n}) = \frac{d}{dX_{1:n}}\left[log\big(T(X_{1:n})\big)\right] \tag{2.13}$$

Remark 2.2 The theorem 2.1 for function of first order statistics also holds for random variable X when $(n = 1)$.

Illustrative Examples

Example. 3.1 Using method described in the remark characterization of negative exponential distribution through expectation of non constant function of order statistics $g(X_{1:n})$ as the uniformly minimum variance unbiased (UMVU) estimator $\hat{g}(\theta)$ and maximum likelihood estimator (MLE) $\tilde{g}(\theta)$ of $g(\theta)$ such as $\mu_1'(\theta)$, mean; $\mu_r'(\theta)$, r^{th} moment; e^{θ}, $e^{-\theta}$; $Q_p(\theta)$, p^{th} quantile; $F(t)$, distribution function; $\bar{F}(t)$, reliability function; $\boldsymbol{\lambda}$(t), hazard rat one gets $[\phi(X_{1:n}) - g(X_{1:n})]$ as given below.

$g(\theta)$	$g(X_{1:n})$	$\phi(X_{1:n}) - g(X_{1:n})$
$\mu_1'(\theta)$	$\widehat{\mu_1'(\theta)} = X_{1:n} + 1 - \frac{1}{n}$	$-\frac{1}{n}$
	$\widetilde{\mu_1'(\theta)} = X_{1:n} + 1$	$-\frac{1}{n}$
$\mu_r'(\theta)$	$\widehat{\mu_r'(\theta)} = \left[\begin{array}{c}\sum_{i=0}^{r}\frac{r!}{(r-i)!}x_{1:n}^{r-i}\\ -\frac{1}{n}\sum_{i=0}^{r}\frac{r!}{(r-i-1)!}x_{1:n}^{r-i-1}\end{array}\right]$	$-\frac{1}{n}\left[\begin{array}{c}\frac{1}{n}\sum_{i=0}^{r}\frac{r!\,(r-i-1)}{(r-i-2)!}x_{1:n}^{r-i-2}\\ -\sum_{i=0}^{r}\frac{r!}{(r-i-1)!}x_{1:n}^{r-i-1}\end{array}\right]$
	$\widetilde{\mu_r'(\theta)} = \sum_{i=0}^{r}\frac{r!}{(r-i)!}x_{1:n}^{r-i}$	$-\frac{1}{n}\sum_{i=0}^{r}\frac{r!}{(r-i-1)!}x_{1:n}^{r-i-1}$
e^{θ}	$\widehat{e^{\theta}} = e^{X_{1:n}}\left(1-\frac{1}{n}\right)$	$-e^{-X_{1:n}}\left(1-\frac{1}{n}\right)\frac{1}{n}$
	$\widetilde{e^{\theta}} = e^{X_{1:n}}$	$-\frac{e^{X_{1:n}}}{n}$
$e^{-\theta}$	$\widehat{e^{-\theta}} = e^{-X_{1:n}}\left(1+\frac{1}{n}\right)$	$e^{-X_{1:n}}\left(1+\frac{1}{n}\right)\frac{1}{n}$
	$\widetilde{e^{-\theta}} = e^{-X_{1:n}}$	$\frac{e^{-X_{1:n}}}{n}$
$Q_p(\theta)$	$\widehat{Q_p(\theta)} = -log(1-P) + X_{1:n} - \frac{1}{n}$	$-\frac{1}{n}$
	$\widetilde{Q_p(\theta)} = -log(1-P) + X_{1:n}$	$-\frac{1}{n}$
$F(t)$	$\hat{F}(t) = \left\{1 - e^{-(t-X_{1:n})}\left[1-\frac{1}{n}\right]\right\}$	$\frac{1}{n}\left[1-\frac{1}{n}\right]e^{-(t-X_{1:n})}$
	$\tilde{F}(t) = 1 - e^{-(t-X_{1:n})}$	$-\frac{1}{n}e^{-(t-X_{1:n})}$
$\bar{F}(t)$	$\hat{\bar{F}}(t) = e^{-(t-X_{1:n})}\left[1-\frac{1}{n}\right]$	$-\frac{1}{n}\left[1-\frac{1}{n}\right]e^{-(t-X_{1:n})}$
	$\tilde{\bar{F}}(t) = e^{-(t-X_{1:n})}$	$-\frac{1}{n}e^{-(t-X_{1:n})}$

Defining $M(X_{1:n})$ given in (2.11) and using $T(X_{1:n})$ as appeared in (2.13), the pdf (1.1) is characterized by (2.12).

Example. 3.2 In context of remark 2.2 the pdf $f(x,\theta)$ defined in (1.1) can be characterized through non constant functions of random variable X when $(n = 1)$

$$g_i(\theta) = \begin{cases} \theta + 1; for i = 1, Mean, \\ \sum_{k=0}^{r} \frac{r!}{(r-k)!}\theta^{r-k};\ for i = 2, \text{r}^{\text{th}} \text{ raw moment}, \\ e^{\theta};\ for i = 3, \\ e^{-\theta}; for i = 4, \\ -\log(1-p) + \theta; for i = 5, \text{p}^{\text{th}} \text{ quantile}, \\ 1 - e^{-(t-\theta)}; for i = 6, \text{distribution function at t}, \\ \mathrm{e}^{-(\mathrm{t}-\theta)};for i = 7, \text{reliability function at t}, \\ 1;\ for i = 8, \text{hazard Function}, \end{cases}$$

by using

$$[\phi_i(X) - g_i(X)] = \begin{cases} -1; for i = 1, \text{Mean}, \\ -\sum_{k=0}^{r-1} \frac{r!}{(r-k-1)!} X^{r-k-1};\ for i = 2, r^{th} \text{ raw moment}, \\ -e^{x}; for i = 3, \\ e^{-x}; for i = 4, \\ -1; for i = 5, p^{th} \text{ quantile}, \\ e^{-(t-x)};\ for i = 6, \text{distribution function at } t, \\ -e^{-(t-x)}; for i = 7, \text{reliability function at } t, \end{cases}$$

and defining $M(X)$ given in (2.11) and using $T(X)$ as appeared in (2.13), for (2.12).

Note that as Hazard function for negative exponential distribution being constant it can not characterize pdf $f(x,\theta)$ given in (1.1).

Refference

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