# Early exercise European option and early termination
# American option pricing models

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**Abstract:** The maximum relative error between continuous-time American option pricing model and binomial tree model is very small. In order to improve the European and American options in trade course, the thesis tried to build early exercise European option and early termination American option pricing models. Firstly, the authors reviewed the characteristics of American option and European option, then there was compares between them. Base on continuous-time American option pricing model, this research analyzed the value of these options.

**Key words:** option pricing; early exercise European option pricing; early termination American option pricing

## 1. Introduction

It is a world puzzle for us to price American option because it has not fix exertion time, so we have to price American option using binomial tree (Cox, Ross & Rubinstein, 1979, CRR) and computer. A continuous-time American option pricing model had set up by Yan and Zhao (2008) and the model are similar to Black-Scholes (1973, BS). The maximum relative error between Yan-Zhao and binomial tree models is only 2.47%. The binomial tree model underestimated the value of American call options, overestimated the value of American put option than Yan-Zhao model.

European options can be exercised only on the expiration date which is limited in three to five days, so that option seller is easy at manipulating stock prices. In encourage plans with call option, companies call for employees to postpone exercise European options is a common phenomenon, so the thesis will be offered for the continuous-time early exercise European options pricing model that is similar to Black-Scholes model.

American options can be exercised at any time up to the expiration date, so most of the options traded on exchanges are American. Seller's margin accounts must have sufficient margin to prevent buyer to exercise American options at any time. Margin account funds increased the opportunity cost of American options' seller. If the American options' seller provides two sections of exercise times to the buyer, the seller can reduce the financial pressure. Therefore, the thesis proposes early termination American option pricing model that is similar to Yan-Zhao model.

In section 2, the authors describe the pricing models relationship between BS European option and Yan-Zhao American option, as well as the maximum relative error between Yan-Zhao and binomial tree models. Section 3 develops early exercise European option and pricing models. Section 4 proposes early termination American option and pricing models. Section 5 presents evident studies to compare European option to American option,

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early exercise European option and early termination American option.

## 2. The relationship between European and American options

There are some factors affecting the price of a stock option: $S_0$ is current stock price; $x$ is exercise or strike price of option; $T$ is time to expiration of option; $S_T$ is stock price at option maturity; $r$ is continuously compounded risk-free rate of interest for $T$ years; $\sigma$ is volatility per annum; $c_E$ is price of European call option per share; $p_E$ is price of European put option per share; $c_A$ is price of American call option per share; $p_A$ is price of American put option per share.

The Black-Scholes models for the prices of European calls and puts on non-dividend-paying stocks are:

$$c_E = S_0 N(d_1) - Xe^{-rT}N(d_2)$$

$$p_E = Xe^{-rT}N(-d_2) - S_0 N(-d_1)$$

where,

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{1}{2}x^2}\, dx$$

$$d_{1,2} = \frac{\ln(S_t / X) + (r \pm \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

The American option holder's choice opportunity is bigger than the European option holder's, so the American options' value should be bigger than the European option. In other words, when early exercise American options, investors believe that American options' income should be greater than European options', otherwise, investors will not early exercise American options. Because the money has time value, if early exercise American options, investors can withdraw funds invest in risk-free rate is $r$ of the assets, the value of the American option should be the European option value of $e^{rT}$ times.

$$c_A = c_E e^{rT}$$

$$p_A = p_E e^{rT}$$

or,

$$c_A = S_0 e^{rT} N(d_1) - XN(d_2)$$

$$p_A = XN(-d_2) - S_0 e^{rT} N(-d_1)$$

To test the continuous-time American option pricing model effectiveness, the following introduce two numbers case (Hull, 2001).

(1) American call option on index future

Consider an American call option on index futures. The current index future price is 300, the exercise price is 300, the risk-free interest rate is 8% per year, and the volatility of index is 30% per year, the life of option is four months. The life of option divided into 4 time steps to construct binomial tree, the estimated value of the option is 19.16. With 50 time steps DerivaGem gives a value of 20.18, and with 100 time steps DerivaGem gives a value of 20.22.

In this case, $F_0 = S_0 e^{rT} = 300$, $X = 300, r = 0.08, \sigma = 0.3, T = 4/12$. Continuous-time American option pricing model give a value of 20.72. The maximum relative error between Continuous-time American option pricing model and binomial tree model is only 2.47%. The binomial tree model underestimated the value of American call options than Yan-Zhao model.

(2) American put option on non-dividend-paying stock

Considering an American put option on non-dividend-paying stock, when the current stock price is 50, the exercise price is 50, the risk-free interest rate is 10% per annum, and the volatility of stock is 40% per annum, the life of option is five months. The life of option is divided into 5 time steps to construct binomial tree, the estimated value of the option is 4.49. With 30, 50 and 100 time steps DerivaGem gives values for the option of 4.263, 4.272 and 4.278, respectively.

This means $S_0 = 50$, $X = 50$, $r = 0.10$, $\sigma = 0.4$ and $T = 5/12$. Continuous-time American option pricing model give a value for the option of 4.25. The maximum relative error between continuous-time American option pricing model and binomial tree model is only -0.65%. The binomial tree model overestimated the value of American put option than Yan-Zhao model.

When the binary tree intervals tend to infinity, with continuous-time American option pricing model to calculate the value of the option is equal to the limit value of the binomial model. The following describes early exercise European option pricing model and early termination American pricing model.

## 3. Early exercise European option pricing model

The variable $T$ is the time to expiration of European option; $T_1$ is the early exercise start time of European option, and the relationship between $T$ and $T_1$ is $0 \le T_1 \le T$. The closed time length of early exercise European option is $T_1$, and the length of exercise time is $T - T_1$. The gains of early exercise European option may be invested in forward risk-free interest rate assets. Early exercise European call option and early exercise European put option pricing model are:

$$c_{Ee} = e^{-rT}E[\max(S_T - X, 0)]e^{r_f(T-T_1)}$$
$$p_{Ee} = e^{-rT}E[\max(X - S_T, 0)]e^{r_f(T-T_1)}$$

or,

$$c_{Ee} = c_E e^{r_f(T-T_1)}$$
$$p_{Ee} = p_E e^{r_f(T-T_1)}$$

or,

$$c_{Ee} = [S_t N(d_1) - Xe^{-rT}N(d_2)]e^{r_f(T-T_1)}$$
$$p_{Ee} = [Xe^{-rT}N(-d_2) - S_t N(-d_1)]e^{r_f(T-T_1)}$$

where we define: $c_{Ee}$ is the value of early exercise European call option; $p_{Ee}$ is the value of early exercise European put option; $r_f$ is continuously compounded forward risk-free interest rate for $T - T_1$ years after $T_1$ years.

The value of early exercise of the European option is equal to the value of deferred exercise American option.

## 4. Early termination American option pricing model

In this section, the authors will discuss the early termination American option pricing model. We define $T$ is time to expiration of American option; $T_1$ is American option exercise window termination time, and the relationship between $T$ and $T_1$ is $0 \le T_1 \le T$, in other words, the American option exercise in $T_1$ years, the time length is $T_1$ years. Another exercise time of American option is at a few days before the time to expiration. Early

termination American call and American put option pricing models are:

$$c_{Ae} = E[\max(S_T - X, 0)]e^{-r_f(T-T_1)}$$
$$p_{Ae} = E[\max(X - S_T, 0)]e^{-r_f(T-T_1)}$$

or,

$$c_{Ae} = c_A e^{-r_f(T-T_1)}$$
$$p_{Ae} = p_A e^{-r_f(T-T_1)}$$

or,

$$c_{Ae} = [S_t e^{rT} N(d_1) - X N(d_2)]e^{-r_f(T-T_1)}$$
$$p_{Ae} = [X N(-d_2) - S_t e^{rT} N(-d_1)]e^{-r_f(T-T_1)}$$

We define $c_{Ae}$ is the value of early termination American call option; $p_{Ae}$ is the value of early termination American put option.

## 5. Evident studies

Consider a European option and an American option on non-dividend-paying stock when the life of options is two years. One year after the European option is exercised until expiration, and one year after the exercise of the American option is stopped until expiration. The current stock price is 20, the exercise price of two options is 20, the risk-free interest rate for two years is 2.79% per annum, the risk-free interest rate for one year is 2.25% per annum, and the volatility of stock is 30% per annum.

In this case, $T = 2$, $T_1 = 1$, $r = 2.79\%$, $\sigma = 30\%$, $S_t = 20$, $X = 20$. According to the term structure of interest rate theory, one year latter one year time forward interest rate is:

$$r_f = \frac{(1 + 0.0279)^2}{1 + 0.0225} - 1 = 3.33\%$$

The following we compute the values of European option, American option, early exercise European option and the early termination American option. The above parameters are substituted into the option pricing models. We obtain four values of call and put options. The results are listed in Table 1.

**Table 1　Comparative table of four option values**

| Option name | Call options | Put options |
| --- | --- | --- |
| European options | 3.58 | 3.03 |
| American options | 3.78 | 3.20 |
| Early exercise European options | 3.70 (3.66) | 3.13 (3.10) |
| Early termination American options | 3.66 (3.70) | 3.10 (3.13) |

If you think that one year after the one year forward rate should be equal to one year spot interest rate now, the value of new option numbers in brackets. The authors believe that with forward interest rate is reasonable, because the money has time value, the value of early exercise European option should be greater than the value of early termination American option.

## 6. Conclusion

If the same underlying asset, start time and expiration time of option, the value of American option is greater than the value of European option. If the same early exercise time of European option and early termination time of American option, the value of early exercise European option is greater than the value of early termination American option. Early exercise European option value and early termination American option value are greater than the value of European option, less than the value of American option. Early exercise European option value is equal to the deferred exercise American option value.

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