Existence of Positive Solutions to Semilinear Elliptic Systems Involving Concave and Convex Nonlinearities

Mohamed El Mokhtar ould El Mokhtar

Departement of Mathematics, Qassim University, BO 6644, Buraidah 51452, Kingdom of Saudi Arabia

Abstract: In this paper, we studied the combined effect of concave and convex nonlinearities on the number of positive solutions for a semilinear elliptic system. We prove the existence of at least four positive solutions for a semilinear elliptic system involving concave and convex nonlinearities by using the Nehari manifold and the center mass function.

Key words: Elliptic systems, concave and convex nonlinearities, Nehari manifold, maximum principle.

1. Introduction

This paper deals with the existence and nonexistence of positive solutions to the following problem:

\[-\Delta u + u = (\alpha + 1)u^\alpha v^{\beta + 1} + \mu(\alpha' + 1)u^{\alpha'} v^{\beta + 1} \quad \text{in } \Omega\]

\[-\Delta v + v = (\beta + 1)u^{\alpha + 1}v^\beta + \mu(\beta' + 1)u^{\alpha + 1}v^{\beta'} \quad \text{in } \Omega\]

\[u > 0, v > 0 \quad \text{in } \Omega\]

\[u = v = 0 \quad \text{on } \partial \Omega\]

where, \(\Omega\) is a bounded regular domain in \(\mathbb{R}^N\) (\(N \geq 3\)) containing 0 in its interior, \(1 < p = \alpha + \beta + 1 < 2^* - 1, 2^* = 2N/(N - 2)\) is the Sobolev critical exponent, \(0 \leq q = \alpha' + \beta' + 1 < 1\) and \(\mu\) is a real parameter.

Semilinear scalar elliptic equations with concave and convex nonlinearities are widely studied; we refer the readers to [2, 3, 7, 13, 14, 21] etc.. For the semilinear elliptic systems, we refer to Ahammou [1, 5, 6, 9, 12, 15, 17, 19]. The model type is written as follows:

\[
\begin{cases}
-\Delta u + u = hu^t + \mu gu^t & \text{in } \Omega \\
0 < u \in H^1_0(\Omega)
\end{cases}
\] (2)

When \(h \equiv g \equiv 1\), Eq. (1) can be regarded as a perturbation problem of the following equation:

\[
\begin{cases}
-\Delta u + u = u^t & \text{in } \Omega \\
0 < u \in H^1_0(\Omega)
\end{cases}
\] (3)

The number of positive solutions of Eq. (3) is affected by the shape of the domain \(\Omega\), which has been the focus of a great deal of research in recent years. For example, we quote the works of Byeon [8], Dancer [10], Damascelli, L.et al [11] and Wu [22].

Considering the multiplicity of positive solutions to problem Eq. (1), we extended this method in this paper. We consider the Sobolev spaces \(H(\Omega) = H^1_0(\Omega) \times H^1_0(\Omega)\) with respect to the norm

\[\|(u, v)\| = (\|u\|^2 + \|v\|^2)^{1/2}\]

where,

\[\|u\| = \|u\|_{H^1_0} = \left(\int_\Omega |\nabla u|^2 + u^2\right)^{1/2}\]

is a standard norm in \(H^1_0(\Omega)\).

Since our approach is variational, we define the functional \(I_\mu\) on \(H(\Omega)\) by

\[I_\mu(u, v) := (1/2)\|(u, v)\|^2 - \int_\Omega u^{\alpha+1} v^{\beta+1} dx - \mu \int_\Omega u^{\alpha+1} v^{\beta+1} dx\]

Let

\[C_{(\alpha, \beta)} = \inf_{u \in H^1_0(\Omega)} \left(\int_\Omega |u|^2 dx\right)^{1/(\alpha+1)}\]

and

\[C_{(\beta, \beta)} = \inf_{v \in H^1_0(\Omega)} \left(\int_\Omega |v|^2 dx\right)^{1/(\beta+1)}\]
2. Preliminaries

2.1 Nehari Manifold

Considering the Nehari manifold:
\[ \mathcal{M}_\mu(\Omega) = \left\{ (u, v) \in \mathcal{H}(\Omega) \setminus \{0, 0\} \mid \langle I_{\mu}(u, v), (u, v) \rangle = 0 \right\} \]

Define
\[ \phi_{\mu}(u, v) = \langle I_{\mu}(u, v), (u, v) \rangle \]
\[ = \| (u, v) \|^2 - (p + 1) \int_\Omega u^{\alpha+1} v^{\beta+1} dx - \mu(q+1) \int_\Omega u^{\alpha+1} v^{\beta+1} dx \]

Then, for \((u, v) \in \mathcal{M}_\mu(\Omega)\)
\[ \left\langle \phi'_{\mu}(u, v), (u, v) \right\rangle = 2\| (u, v) \|^2 - (p + 1)^2 \int_\Omega u^{\alpha+1} v^{\beta+1} dx - \mu(q+1)^2 \int_\Omega u^{\alpha+1} v^{\beta+1} dx = (1 - p)\| (u, v) \|^2 - \mu(q-p)(1+q) \int_\Omega u^{\alpha+1} v^{\beta+1} dx \]
\[ = (1 - q)\| (u, v) \|^2 - (p + 1)(p-q) \int_\Omega u^{\alpha+1} v^{\beta+1} dx. \quad (4) \]

Now, as in Tarantello [18], we split \(\mathcal{M}_\mu(\Omega)\) into three parts:
\[ \mathcal{M}_\mu^0(\Omega) = \left\{ (u, v) \in \mathcal{M}_\mu(\Omega) : \phi_{\mu}(u, v), (u, v) > 0 \right\} \]
\[ \mathcal{M}_\mu^0(\Omega) = \left\{ (u, v) \in \mathcal{M}_\mu(\Omega) : \phi_{\mu}(u, v), (u, v) = 0 \right\} \]
\[ \mathcal{M}_\mu^0(\Omega) = \left\{ (u, v) \in \mathcal{M}_\mu(\Omega) : \phi_{\mu}(u, v), (u, v) < 0 \right\} \]

It is well known that \(I_{\mu}\) is of class \(C^1\) in \(\mathcal{H}(\Omega)\), and the solutions of Eq. (1) are the critical points of \(I_{\mu}\), which is not bounded below on \(\mathcal{H}(\Omega)\). Note that \(\mathcal{M}_\mu(\Omega)\) contains every nontrivial solution of the problem Eq. (1). Moreover, we have the following results:

Let
\[ \mu_0 := \left( \frac{(p-1)}{(p+1)} \right) \left( \frac{(1-q)}{(p-q)(p+1)} \right)^{\frac{1}{2}} \mathcal{L}(\Omega)^{\frac{(p+1)}{(p-1)}} \]
\[ (K(\alpha, \beta)C_{(\alpha, \beta)})^{\frac{1}{2}} \mathcal{L}(\Omega)^{\frac{(p+1)}{(p-1)}} \]

where, \(\mathcal{L}(\Omega)\) is the Lebesgue measure of domain \(\Omega\).

Lemma 1 We have \(\mathcal{M}_\mu^0(\Omega) = \emptyset\) for all \(\mu \in (0, \mu_0)\).

Proof Let us reason by contradiction. Suppose \(\mathcal{M}_\mu^0(\Omega) \neq \emptyset\), let \(u \in \mathcal{M}_\mu^0(\Omega)\), by Eq. (5), we have
where

$$\Omega$$

For each $$(u, v)$$ by Eq. (6) and the Holder inequality, we obtain

$$\| (u, v) \| \geq \left[ \frac{(p-1)}{(p-q)(p+1)} \right]^{1/(q-1)} | \Omega |^{1/(q-1)} \cdot | \Omega |^{1/(p-1)}$$

(7)

by Eq. (6) and the Holder inequality, we obtain

$$\| (u, v) \| \geq \mu^{1/(q-1)} \left[ \frac{(p-1)}{(p-q)(p+1)} \right]^{1/(q-1)} | \Omega |^{1/(q-1)}$$

(8)

From Eqs. (7) and (8), we get $\mu \geq \mu_0$ which contradicts the fact that $\mu \in (0, \mu_0)$.

By Lemma 1, we can write $\mathcal{M}_\mu(\Omega) = \mathcal{M}_\mu^+(\Omega) \cup \mathcal{M}_\mu^-(\Omega)$, for $\mu \in (0, \mu_0)$. Define

$$a_\mu = \inf_{(u,v) \in \mathcal{M}_\mu(\Omega)} I_\mu(\Omega), \quad a_\mu = \inf_{(u,v) \in \mathcal{M}_\mu(\Omega)} I_\mu(\Omega)$$

The following Lemma shows that the minimizers on $\mathcal{M}_\mu(\Omega)$ are usually critically points for $I_\mu$.

**Lemma 2** [22] Suppose that $$(u_0, v_0)$$ is a local minimizer for $I_\mu$ on $\mathcal{M}_\mu(\Omega)$. Then for $\mu \in (0, \mu_0)$, $$(u_0, v_0)$$ is a critical point of $I_\mu$.

For each $$(u, v) \in \mathcal{H}(\Omega) \setminus \{ (0, 0) \}$$, we write

$$t_m := \max_{(u,v) \in \mathcal{M}_\mu(\Omega)} I_\mu(\Omega)$$

$$t_m = \left[ \frac{(1-q)|u(v)|^{(1-q)}}{(p-q)(p+1)} \right]^{1/(q-1)} \cdot \Omega \cdot \Omega \cdot \Omega$$

Let

$$\| (u, v) \|^2 = \int_\Omega u^{\alpha+1} v^{\beta+1} \, dx$$

and

$$b := (p+1)(p-q) \cdot | \Omega |^{1/(q-1)}$$

(9)

Lemma 3 For each $\mu \in (0, \mu_1)$ and $$(u, v) \in \mathcal{H}(\Omega) \setminus \{ (0, 0) \}$$,

(i) there is an unique $t = (t^u, t^v) > t_m > 0$ such that $$(t^u, t^v) \in \mathcal{M}_\mu(\Omega)$$

and

$$I_\mu(t^u, t^v) = \max_{t \geq t_m} I_\mu(tu, tv)$$

(ii) $$(t^u, t^v)$$ is a critical point for nonzero $$(u, v)$$;

$$\mathcal{M}_\mu(\Omega) = \left\{ (u, v) \in \mathcal{H}(\Omega) \setminus \{ (0, 0) \} \right\}$$

(iii) there is an unique $$(u, v) \in \mathcal{H}(\Omega) \setminus \{ (0, 0) \}$$ such that

$$(t^u, t^v) \in \mathcal{M}_\mu(\Omega)$$

and

$$I_\mu(t^u, t^v) = \min_{0 \leq t \leq t_m} I_\mu(tu, tv)$$

Proof Similar to the proof of Lemma 5 in Wu [22].

For $c > 0$, we define

$$F_0(c, u, v) = \left( \frac{1}{2} \right) \| (u, v) \|^2 - \int_\Omega cu^{\alpha+1} v^{\beta+1} \, dx$$

$$\mathcal{M}_0(\Omega) = \left\{ (u, v) \in \mathcal{H}(\Omega) \setminus \{ (0, 0) \} \right\}$$

(iv) there is an unique $$(u, v) \in \mathcal{H}(\Omega) \setminus \{ (0, 0) \}$$ such that

$$(t^u, t^v) \in \mathcal{M}_\mu(\Omega)$$

and

$$I_\mu(t^u, t^v) = \min_{0 \leq t \leq t_m} I_\mu(tu, tv)$$

Proof Similar to the proof of Lemma 5 in Wu [22].

For $c > 0$, we define

$$F_0(c, u, v) = \left( \frac{1}{2} \right) \| (u, v) \|^2 - \int_\Omega cu^{\alpha+1} v^{\beta+1} \, dx$$

$$\mathcal{M}_0(\Omega) = \left\{ (u, v) \in \mathcal{H}(\Omega) \setminus \{ (0, 0) \} \right\}$$

Note that $t_m = F_0$ for $c = 1$ and for each $$(u, v) \in \mathcal{M}_\mu(\Omega)$$, there is an unique $$(u, v) \in \mathcal{M}_0(\Omega)$$. Furthermore, we have the following result.

**Lemma 4** For each $$(u, v) \in \mathcal{M}_\mu(\Omega)$$,

(i) there is an unique $$(u, v) \in \mathcal{M}_\mu(\Omega)$$ such that $$(u, v) \in \mathcal{M}_\mu(\Omega)$$

and

$$\max_{\tau \geq 0} F_0(t\mu, tv) = F_0(S^\tau((u, v)))$$

$$= \left[ \frac{1}{2} \right] \| (u, v) \|^2 - \int_\Omega cu^{\alpha+1} v^{\beta+1} \, dx$$

$$\| (u, v) \|^2 = \frac{1}{2} \left( \int_\Omega cu^{\alpha+1} v^{\beta+1} \, dx \right)$$

(ii) for $\tau \in (0, 1)$,

$$I_\mu(u, v) \geq (1 - \mu \theta)^{\alpha+1} I_0(S^\tau((u, v)))$$

$$- \left[ \mu \left( \frac{1}{2} \right) \right] \theta^{\alpha+1}$$
(K(\alpha, \beta) C_{(\alpha, \beta)})^{\frac{2(p+1)}{2-q}} \| \Omega \|^{\frac{2(p-q)}{2-q(1+q)}}

and

I_\mu(u, v) \leq (1 + \mu \theta) \int_\Omega (S(u, v)(u, v)) + \mu \left( \frac{1 - q}{2(q + 1)} \right) \int_\Omega \theta \frac{1}{u^{1-q}}

(K(\alpha, \beta) C_{(\alpha, \beta)})^{\frac{2(p+1)}{2-q}} \| \Omega \|^{\frac{2(p-q)}{2-q(1+q)}}

Proof (i) we have that

I_\mu^0(tu, tv) =

(t^{\theta/2} \| (u, v) \|^{\theta - t^{(q+1)}} \int_\Omega cu^{\theta/2} + \int C \| u^{\theta/2} \|^2 dx

Let g(t) = at^{\theta} - bt^{(q+1)} with a = (1/2)

\| (u, v) \|^2 b = \int_\Omega cu^{\theta/2} + \int C \| u^{\theta/2} \|^2 dx. Furthermore, we have that g attains its maximum at

t_m = (ap/b(q + 1))^{1/(q - p)} = S^*(u, v) > 0. Thus,

I_\mu^0(S^*(u, v)(u, v)) = g(t_m)

= \left( \frac{1}{2} \| (u, v) \|^2 - \frac{\| (u, v) \|^{(\theta - p)}}{\int_\Omega cu^{\theta/2} + \int C \| u^{\theta/2} \|^2 dx} \right)^{2/(\theta - p)}

(ii) for each (u, v) \in M^\mu(\Omega), let \epsilon = 1/(1 - \mu \theta),

S^\epsilon = S^* (u, v) > 0 and S^\epsilon_u > 0 such that

S^\epsilon (u, v) \in M^\mu(\Omega) and S^\epsilon_u (u, v) \in M^\mu_0(\Omega)

Then, for each \theta \in (0, 1), we have

\int_\Omega S^\epsilon u^{\theta/2} + \int C \| u^{\theta/2} \|^2 dx \leq

(K(\alpha, \beta) C_{(\alpha, \beta)})^{\theta/(p+1)} \| S^\epsilon (u, v) \|^q + \mu \left( \frac{1 - q}{2} \right) \int_\Omega \theta \frac{1}{u^{1-q}} \| S^\epsilon (u, v) \|^2

Thus, by (i) and Lemma 3 (i), we obtain (ii).

3. Existence Result

3.1 Existence of a Local Minimum in M^\mu(\Omega)

Let \mu_2 = (p-1)(q+1)/2(p-q).

Lemma 5 For \mu \in (0, \mu_2], we have

(i) for each (u, v) \in M^\mu(\Omega), I_\mu(u, v) < 0. In particular

\alpha_\mu(\Omega) \leq \alpha_\mu^*(\Omega) < 0;

(ii) I_\mu is coercive and bounded from below on M^\mu(\Omega).

Proof (i) for each (u, v) \in M^\mu(\Omega), we have

\mu \int_\Omega u^{\theta/2} + \int C \| u^{\theta/2} \|^2 dx >

\int_\Omega u^{\theta/2} + \int C \| u^{\theta/2} \|^2 dx

Thus one from of deduces the result (i).

(ii) for (u, v) \in M^\mu(\Omega), by the Holder and Young inequalities:

I_\mu(u, v) = \left( \frac{(p-1)}{(2p+1)} \right) \| (u, v) \|^2 -

\mu \left( \frac{(p-q)}{(p+1)(q+1)} \right) \int_\Omega u^{\theta/2} + \int C \| u^{\theta/2} \|^2

\int_\Omega u^{\theta/2} + \int C \| u^{\theta/2} \|^2 \left[ \frac{(p-1)}{2} \right]

Thus, I_\mu is coercive and bounded from below on M^\mu(\Omega) for all \mu \in (0, \mu_2].

Now, we establish the existence of a local minimum.

Theorem 2 Let \mu_3 = \min \{\mu_0, \mu_1, \mu_2\}, for \mu \in (0, \mu_3], the functional I_\mu has an unique minimizer (u_{min}, v_{min}) in M^\mu(\Omega) which satisfies

(i) I_\mu((u_{min}, v_{min})) = \alpha_\mu(\Omega) = \alpha_\mu^*(\Omega); (ii) (u_{min}, v_{min}) is positive solution of Eq. (1); (iii) I_\mu((u_{min}, v_{min})) goes to 0 as tends to 0.

Proof there exists a minimizing sequence (u_n, v_n) for I_\mu on M^\mu(\Omega) such that

I_\mu((u_n, v_n)) = \alpha_\mu(\Omega) + o(1) and I_\mu((u_n, v_n)) = o(1) in H^\prime(\Omega) (dual of H(\Omega))

as n tends to \infty.

By Lemma 5 and the compact imbedding theorem, there exists a subsequence still denoted by
may assume that \((u_{\min}, v_{\min})\) is a positive solution of our problem Eq. (1).

By Lemma 5, we have

\[ 0 > I_{\mu}((u_{\min}, v_{\min})) \geq -\mu[(p-1)(p-q)/2(q+1)](K(\alpha, \beta)C(\alpha, \beta))^{q+1} \]

Thus, we obtain that

\[ I_{\mu}((u_{\min}, v_{\min})) \rightarrow 0 \text{ as } \mu \rightarrow 0. \]

3.2 Existence of Two Positive Solutions in \(\mathcal{M}_{\mu}^+(\Theta_t)\)

In this section, we consider the filtration of \(\mathcal{M}_{\mu}^+(\Theta_t)\) and we will prove that Eq. (1) has two positive solutions for \(\mu\) sufficiently small in \(\Theta_t\).

For that, we need the following notations.

\[ S^+_l = \{(x,y) \in S / y > l\} \]

and

\[ S^-_l = \{(x,y) \in S / y < l\} \]

For positive number \(\delta^*\), let

\[ \mathcal{M}_0^+(\delta, \Theta_t) = \{ (u,v) \in \mathcal{M}_0(\Theta_t)/I_0(u,v) \leq \alpha_0(S) + \delta \} \]

and

\[ \mathcal{M}_0^+(\delta, \Theta_t) = \\{(u,v) \in \mathcal{M}_0(\Theta_t)/\int_{S^{\delta^*} \cap \Theta_t} u^{q+1}v^{\beta+1}dx < \}

\[ ((p+1)/(p-1))\alpha_0(S) \} \]

\[ \mathcal{M}_0^+(\delta, \Theta_t) = \\{(u,v) \in \mathcal{M}_0(\Theta_t)/\int_{S^{\delta^*} \cap \Theta_t} u^{q+1}v^{\beta+1}dx < \}

\[ ((p+1)/(p-1))\alpha_0(S) \} \]

**Lemma 6** There exists \(\delta_0, t_0 > 0\) such that for \(t > t_0\), we have

(i) \(\mathcal{M}_0^+(\delta_0, \Theta_t) \neq \emptyset\); 

(ii) \(\mathcal{M}_0^+(\delta_0, \Theta_t) \cap \mathcal{M}_0^+(\delta_0, \Theta_t) = \emptyset\); 

(iii) \(\mathcal{M}_0(\delta_0, \Theta_t) = \mathcal{M}_0^+(\delta_0, \Theta_t) \cup \mathcal{M}_0^+(\delta_0, \Theta_t)\)

**Proof** Similar as in [21].

Furthermore, Eq. (1) has two positive solutions \(v_0^+\) such that

\[ v_0^+ \in \mathcal{M}_0^+(\delta_0, \Theta_t) \]

and
I_0(v^+) = \inf_{v \in M_\mu(\bar{\Theta}, 0)} I_0(v) < a_0(S_0^+ \cap \Theta_I) < a_0(S) + \delta_0. 

Since \( a_0(S_0^+ \cap \Theta_I) < a_0(S) + \delta_0 \), we can choose a positive number \( \tilde{\delta} < \delta_0 \) such that \( a_0(S_0^+ \cap \Theta_I) < a_0(S) + \tilde{\delta} \). Moreover, we consider the filtration of the manifold \( M_\mu(\bar{\Theta}, I) \) as follows:

\[
N_\mu(\tilde{\delta}, \Theta_I) = \{ (u, v) \in M_\mu(\Theta_I) \mid \mathcal{I}(u) \leq a_0(S) + \tilde{\delta} \}.
\]

Then we have the following result.

**Lemma 7** Let \( \mu_3 \) as in Theorem 3, then there exists \( \mu_4 < \mu_3 \) such that for \( \mu \in (0, \mu_4) \), \( N_\mu(\tilde{\delta}, \Theta_I) \) are nonempty sets. Furthermore,

\[
N_\mu(\tilde{\delta}, \Theta_I) = \{ (u, v) \in N_\mu(\bar{\Theta}, \Theta_I) \mid \int \mathcal{I}^p(u^{\delta+1}, v^{\delta+1}) \leq \left( \frac{p + 1}{p - 1} \right) a_0(S) \}.
\]

Proof We only need to prove the case “+” since \( S_0^+ \cap \Theta_I \) is bounded domain. Thus, Eq. (1) in \( S_0^+ \cap \Theta_I \) has a positive solution \( (u^+, v^+) \) such that \( I_0(u^+, v^+) = a_0(S_0^+ \cap \Theta_I) \) and \( (u^+, v^+) = (0, 0) \) in \( S_0^+ \cap \Theta_I \). Let \( (u_{\min}, v_{\min}) \) be a positive solution of Eq. (1) in \( \Theta_I \) as in Theorem 3. Then, for \( v, w > 0 \), we have

\[
I_\mu((u_{\min}, u_{\min}) + l(u^+, v^+)) < I_\mu((u_{\min}, u_{\min}) + I_0(l(u^+, v^+))
\]

\[
- \int_{\Theta_I} \left\{ \int_0^{\mathcal{I}(u^+, u^+)} \left( \int (u_{\min}, v_{\min}) \right) + s \right\} (v + w)^k (v + w)^k \, ds \right\}.
\]

Thus, \( I_\mu((u_{\min}, u_{\min}) + l(u^+, v^+)) < I_\mu((u_{\min}, u_{\min}) + I_0(l(u^+, v^+))) \), and there exists \( l_0 > 0 \) such that \( \sup l_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) = I_\mu((u_{\min}, v_{\min}) + I_0(l(u^+, v^+))) \).

Since \( I_0(l(u^+, v^+)) \to -\infty \) as \( l \to +\infty \), there exists \( l_0 > 0 \) such that

\[
\sup l_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) = \sup (u_{\min}, v_{\min}) + l(u^+, v^+).
\]

Thus, \( g_1(l) = I_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) \) for \( l \geq 0 \). For the continuity of \( g_1 \), given \( \epsilon > 0 \), there exists \( 0 < l_1 < l_0 \) such that

\[
g_1(l) < g_1(0) + I_0((u^+, v^+)) \leq \epsilon \text{ for } l \leq l_1.
\]

Then,

\[
\sup l_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) = I_\mu((u_{\min}, v_{\min}) + I_0((u^+, v^+))).
\]

Now, we only need to show that \( \sup l_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) < I_\mu((u_{\min}, v_{\min}) + l_0((u^+, v^+))). \) Let \( g_2(l) = \int (u^+, v^+) \leq 0 \). Then

\[
g_2(l) = l \mathcal{I}(u^+, v^+)^2 - \int_{\Theta_I} (u^+) (v^+) p+1 \, dx
\]

and there is an unique \( \tilde{\gamma} \) such that \( g_2'(\tilde{\gamma}) = 0 \) and \( g_2''(\tilde{\gamma}) < 0 \). Thus, \( g_2 \) has an absolute maximum at \( \tilde{\gamma} = 1 \). Therefore,

\[
\sup l_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) = I_0((u^+, v^+)) + a_0(S_0^+ \cap \Theta_I).
\]

By Eqs. (8) and (9) we obtain

\[
\sup l_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) < I_\mu((u_{\min}, v_{\min}) + a_0(S_0^+ \cap \Theta_I)).
\]

Thus,

\[
\sup l_\mu((u_{\min}, v_{\min}) + l(u^+, v^+)) < I_\mu((u_{\min}, v_{\min}) + a_0(S_0^+ \cap \Theta_I)) \leq I_\mu((u_{\min}, v_{\min}) + a_0(S) + \tilde{\delta}).
\]

Next, we prove that there exists an unique \( \Omega_3 \) such that for \( \mu \in (0, \mu_4) \),

\[
\int_{S_0^+ \cap \Theta_I} |u_{\min} + l(u^+)^{p+1} | v_{\min} + l(u^+)^{\beta+1} \, dx < ((p + 1)/(p - 1)) a_0(S).
\]
Let
\[ A_1 = \left\{ (u, v) \in \mathcal{H}(\Omega) \setminus \{0\} \right\} / \]
\[
\frac{1}{\| (u, v) \|} - t \left( \frac{(u, v)}{\| (u, v) \|} \right) > 1 \}
\cup \{0\} \]
\[ A_2 = \left\{ (u, v) \in \mathcal{H}(\Omega) \setminus \{0\} / \right\}
\[
\frac{1}{\| (u, v) \|} - t \left( \frac{(u, v)}{\| (u, v) \|} \right) < 1 \}
\]
Then \( \mathcal{M}_\mu(\Theta_t) \) disconnects \( \mathcal{H}(\Theta_t) \) in two connected components \( A_1, A_2 \) and
\[ \mathcal{H}(\Theta_t) \cap \mathcal{M}_\mu(\Theta_t) = A_1 \cup A_2. \]

For each \((u, v) \in \mathcal{M}_\mu(\Theta_t)\), we have \( 1 < t_m \text{im}(u, v) \). Since 
\[ t \left( \frac{(u, v)}{\| (u, v) \|} \right) = 1 \frac{1}{\| (u, v) \|} - t \left( \frac{(u, v)}{\| (u, v) \|} \right) \]
then \( \mathcal{M}_\mu(\Theta_t) \subset A_1 \). In particular, \((u_0, v_0) \in A_1 \). We claim that
there exists \( s_0 > 0 \) such that 
\[ ((u_{\min}, u_{\min}) + s_0 (u^*, v^*)) \in A_2. \]
Firstly, we find a constant \( c > 0 \), such that 
\[ 0 < t \left( \frac{(u_{\min}, v_{\min}) + (u^*, v^*)}{\| (u_{\min}, v_{\min}) + (u^*, v^*) \|} \right) < c \]
for all \( 1 \geq 0 \).
Otherwise, there exists a sequence \((l_n)\), such that 
\[ l_n \rightarrow \infty \]
and 
\[ t \left( \frac{(u_{\min}, v_{\min}) + (u^*, v^*)}{\| (u_{\min}, v_{\min}) + (u^*, v^*) \|} \right) \rightarrow \infty \text{ as } n \rightarrow \infty. \]
Let \((u_n, v_n) = \frac{(u_{\min}, v_{\min}) + (u^*, v^*)}{\| (u_{\min}, v_{\min}) + (u^*, v^*) \|} \), since \( t \left( \frac{(u_n, v_n)}{\| (u_n, v_n) \|} \right) (u_n, v_n) \in \mathcal{M}_\mu(\Theta_t) \subset \mathcal{M}_\mu(\Theta_t) \).
by the Lebesgue dominated convergence theorem,
\[ \int_{\Theta_t} (u_n)^{p+1} (v_n)^{p+1} dx \rightarrow \]
\[ \| (u^*, v^*) \|^{-(p+1)} \int_{\Theta_t} (u^*)^{p+1} (v^*)^{p+1} dx \text{ as } n \rightarrow \infty. \]
Hence, we have 
\[ I_\mu(t \left( \frac{(u_n, v_n)}{\| (u_n, v_n) \|} \right) (u_n, v_n)) \rightarrow \infty \text{ as } n \rightarrow \infty, \]
this contradicts to the fact that \( I_\mu \) is bounded below on \( \mathcal{M}_\mu(\Theta_t) \). Let 
\[ l_0 = \| (u^*, v^*) \|^{-1} \left| c^2 - \| (u^*, v^*) \|^{-2} \right| \frac{1}{2} + 1. \]
Then,
\[ \| (u_{\min}, v_{\min}) + l_0 (u^*, v^*) \|^2 > c^2 \]
\[ = \left( \frac{(u_{\min}, v_{\min}) + (u^*, v^*)}{\| (u_{\min}, v_{\min}) + (u^*, v^*) \|} \right)^2, \]
that is 
\[ ((u_{\min}, v_{\min}) + l_0 (u^*, v^*)) \in A_2. \]
Define a path \( \gamma(s) = ((u_{\min}, v_{\min}) + s_0 l_0 (u^*, v^*)) \text{ for } s \in [0,1], \)
then \( \gamma(0) = (u_{\min}, v_{\min}) \in A_t, \gamma(1) = (u_{\min}, v_{\min}) + s_0 l_0 (u^*, v^*) \in A_2 \),
and there exists \( s_0 \in (0,1) \) such that 
\[ ((u_{\min}, v_{\min}) + s_0 l_0 (u^*, v^*)) \in \mathcal{M}_\mu(\Theta_t) \].
By (12), we obtain that:
\[ I_\mu((u_{\min}, v_{\min}) + s_0 l_0 (u^*, v^*)) < I_\mu((u_{\min}, v_{\min}) + s_0 l_0 (u^*, v^*)) < \]
\[ I_\mu((u_{\min}, v_{\min}) + a_0(S) + \delta). \]
Thus,
\[ ((u_{\min}, v_{\min}) + s_0 l_0 (u^*, v^*)) \in \mathcal{N}_\mu(S, \Theta_t) \].
Moreover,
\[ \int_{S_{\Theta_t}^+ \Theta_t} (u^*)^{p+1} (v^*)^{p+1} dx = 0 \]
and
\[ \| (u_{\min}, v_{\min}) \| \rightarrow 0 \text{ as } \mu \rightarrow 0. \]
Thus, there exists \( \mu_4 < \mu_3 \) such that for 
\( \mu \in (0, \mu_4) \)
\[ \int_{S_{\Theta_t}^+ \Theta_t} |u_{\min} + s_0 l_0 u^*|^{a+1} v_{\min} + s_0 l_0 v^*|^{p+1} dx \]
\[ = \int_{S_{\Theta_t}^+ \Theta_t} |u_{\min}|^{a+1} v_{\min} |^{p+1} dx < \left( \frac{p + 1}{p - 1} \right) a_0(S) \]
This implies that \( \mathcal{N}_\mu(S, \Theta_t) \) are nonempty sets for all \( \mu \in (0, \mu_4) \).
Note that \( a_0(S) < a_0(\Theta_t) \) for all \( t > 0 \) and for each \( u \in \mathcal{M}_\mu(\Theta_t) \), there is an unique \( S(u, v) > 0 \) such that 
\( S(u, v)(u, v) \in \mathcal{M}_\mu(\Theta_t) \). Moreover, we have the following result.

**Lemma 8** There exists \( \mu_5 < \mu_4 \) such that for \( \mu \in (0, \mu_5) \), we have
(i) \( 1 < S_{\Theta_t}^{(p+1)} \left( \frac{a_0(S)}{a_0(\Theta_t)} \right) \) for all \((u, v) \in \mathcal{M}_\mu(\Theta_t) \)
(ii) \( \int_{\Theta_t} |u|^{a+1} |v|^{p+1} dx \]
\[ \left( \frac{p + 1}{p - 1} \right) a_0(S) \text{ for all } (u, v) \in \mathcal{M}_\mu(\Theta_t) \]
**Proof** (i) For \((u, v) \in \mathcal{M}_\mu(\Theta_t) \), we have
\[ \| (u, v) \|^2 - \int_{\Theta_t} u^{a+1} v^{p+1} dx - \]
\[ \mu \int_{\Omega} u^{a+1} v^{p+1} dx = 0 \phantom{<} (13) \]
and
\[(1 - p)\|(u,v)\|^2 < (p - q)\int_{\bar{\Omega}} u^{a+1} v^{\beta+1} \ dx \tag{14}\]

Thus, there is an unique \(S_{(u,v)} > 0\) such that \(S_{(u,v)}(u,v) \in \mathcal{M}_0(\Theta_t)\) and so
\[S_{(u,v)}^2 \|(u,v)\|^2 = S^{(p+1)}(p+1)\int_{\bar{\Omega}} u^{a+1} v^{\beta+1} \ dx.\]

Then by Eq. (13) and the Holder inequality we get
\[1 < S_{(u,v)}^{(p-1)} \leq \int_{\bar{\Omega}} u^{a+1} v^{\beta+1} \ dx \leq 1 + \mu \int_{\bar{\Omega}} u^{a+1} v^{\beta+1} \ dx \equiv \mu - \frac{1}{q - p - 1} \int_{\bar{\Omega}} \left(\frac{p - q - 1}{1 - q}\right) S_{(u,v)}^{(p+1)}(p+1) \ dx \tag{15}\]

Then there exists \(\mu_s < \mu_t\) such that for \(\mu \in (0, \mu_s)\), we have
\[1 < S_{(u,v)}^{(p+1)} \leq S_{(u,v)}^{(p+1)} \leq \frac{a_0(\Theta_t)}{\alpha_0(S)} \tag{16}\]

(ii) Since
\[\int_{\bar{\Omega}} u^{a+1} v^{\beta+1} \ dx > \left(\frac{2(p+1)}{p - 1}\right) \frac{S_{(u,v)}^{(p+1)}(p+1)}{\alpha_0(S)} \alpha_0(\Theta_t).\]

By (i), we can conclude that
\[\int_{\bar{\Omega}} u^{a+1} v^{\beta+1} \ dx > \left(\frac{2(p+1)}{p - 1}\right) \alpha_0(S) \tag{17}\]

This completes the proof.

**Lemma 9** There exists \(\mu_s < \mu_{s_0}\) such that for \(\mu \in (0, \mu_s)\), we have
\[\begin{align*}
(i) & \quad \mathcal{N}_{\mu}^- (\bar{\Omega}, \Theta_t) \neq \emptyset; \\
(ii) & \quad \mathcal{N}_{\mu}^- (\bar{\Omega}, \Theta_t) \cap \mathcal{N}_{\mu}^- (\bar{\Omega}, \Theta_t) = \emptyset; \\
(iii) & \quad \mathcal{N}_{\mu}^- (\bar{\Omega}, \Theta_t) = \mathcal{N}_{\mu}^- (\bar{\Omega}, \Theta_t) \cup \mathcal{N}_{\mu}^- (\bar{\Omega}, \Theta_t). \\
\end{align*}\]

**Proof** Let \((u, v) \in \mathcal{N}_{\mu}(\bar{\Omega}, \Theta_t)\), then by Lemma 4, there is an unique \(S_{(u,v)} > 0\) such that \(S_{(u,v)}(u,v) \in \mathcal{M}_0(\Theta_t)\) and
\[I_0(S_{(u,v)}(u,v)) \leq (1 - \mu \theta) F_{\mu}^{\frac{p+1}{p-1}} [a_0(S) + \delta + \mu \left(\frac{1 - q}{2(q+1)}\right) \frac{S_{(u,v)}^{(p+1)}(p+1)}{\alpha_0(S)}] \]

Since \(\delta < \delta_0\), we can conclude that for each \(\theta \in (0,1)\) there exists \(\mu_s \leq \mu_s\) so that for \(\mu \in (0, \mu_s)\), we obtain
\[I_0(S_{(u,v)}(u,v)) \leq a_0(S) + \delta_0 \text{ for all } (u,v) \in \mathcal{N}_{\mu}(\bar{\Omega}, \Theta_t) \tag{16}\]

By Eq.(16) and Lemma 6, for each \((u,v) \in \mathcal{N}_{\mu}(\bar{\Omega}, \Theta_t)\) there is either \(S_{(u,v)}(u,v) \in \mathcal{M}_0(\bar{\Omega}, \Theta_t)\) or \(S_{(u,v)}(u,v) \in \mathcal{M}_0(\bar{\Omega}, \Theta_t)\). Without loss generality, we may assume that \(S_{(u,v)}(u,v) \in \mathcal{M}_0(\bar{\Omega}, \Theta_t)\). Then by Lemma 8
\[\int_{\bar{\Omega}} u^{a+1} v^{\beta+1} \ dx < \left(\frac{p+1}{p-1}\right) S_{(u,v)}^{(p+1)}(p+1) a_0(S) \]

Thus, \((u,v) \in \mathcal{N}_{\mu}(\bar{\Omega}, \Theta_t)\).

To complete the proof of Lemma 9, it remains to be shown that
\[\mathcal{N}_{\mu}^- (\bar{\Omega}, \Theta_t) \cap \mathcal{N}_{\mu}^- (\bar{\Omega}, \Theta_t) = \emptyset.\]

Suppose that there exists \((u_0, v_0) \in \mathcal{N}_{\mu}(\bar{\Omega}, \Theta_t)\) such that
\[\int_{S_{(u_0,v_0)}(u,v)} u^{a+1} v^{\beta+1} \ dx < \left(\frac{p+1}{p-1}\right) a_0(S).\]

It implies
\[2 \left(\frac{p+1}{p-1}\right) a(\Theta_t) \leq \int_{\bar{\Omega}} u^{a+1} v^{\beta+1} \ dx < \left(\frac{p+1}{p-1}\right) a_0(S) \]

which is a contradiction.

Now, we have the following result

**Lemma 10** \(\mathcal{N}_{\mu}^- (\bar{\Omega}, \Theta_t)\) are closed.

**Proof** We only need to prove the case “−”.

Supposing that \((u_0, v_0)\) is a limit point of \(\mathcal{N}_{\mu}^- (\bar{\Omega}, \Theta_t)\). Then
\[\int_{S_{(u_0,v_0)}(u,v)} u^{a+1} v^{\beta+1} \ dx \leq \left(\frac{p+1}{p-1}\right) a_0(S)\]
and

$$I_\mu(u_0, v_0) \leq \alpha_0(S) + \delta.$$

That implies \((u_0, v_0) \in \mathcal{N}_\mu(\tilde{\delta}, \Theta_t)\). Since

$$\mathcal{N}_\mu(\tilde{\delta}, \Theta_t) = \mathcal{N}_\mu^+(\tilde{\delta}, \Theta_t) \cup \mathcal{N}_\mu^-\tilde{\delta}, \Theta_t),$$

and

$$\mathcal{N}_\mu^-(\tilde{\delta}, \Theta_t) \cap \mathcal{N}_\mu^+(\tilde{\delta}, \Theta_t) = \emptyset.$$

Hence, if

$$\int_{S_{\tilde{\delta}, \Theta_t}} u^{a+1} v^{\beta+1} dx = \left(\frac{p+1}{p-1}\right) \alpha_0(S),$$

then \((u_0, v_0) \in \mathcal{N}_\mu^+(\tilde{\delta}, \Theta_t)\). By Lemma 8,

$$2\left(\frac{p+1}{p-1}\right) \alpha_0(S) < \int_{\Theta_t} u_0^{a+1} v_0^{\beta+1} dx \leq \int_{S_{\tilde{\delta}, \Theta_t}} u_0^{a+1} v_0^{\beta+1} dx + \int_{S_{\tilde{\delta}, \Theta_t}} u_0^{a+1} v_0^{\beta+1} dx$$

$$< 2\left(\frac{p+1}{p-1}\right) \alpha_0(S),$$

which is a contradiction. Thus, \(\mathcal{N}_\mu^+(\tilde{\delta}, \Theta_t)\) are closed.

Now, we consider the minimization problems in \(\mathcal{N}_\mu^+(\tilde{\delta}, \Theta_t)\) for \(I_\mu,\)

$$\sigma^\pm(\tilde{\delta}, \Theta_t) = \inf_{(u,v)\in \mathcal{N}_\mu^+(\tilde{\delta}, \Theta_t)} I_\mu(u,v).$$

Then we have the following result.

**Lemma 11** For each \(\mu \in (0, \mu_*]\), Eq. (1) has two positive solutions \((u_0^\pm, v_0^\pm)\) such that \((u_0^+, v_0^+) \in \mathcal{N}_\mu^+(\tilde{\delta}, \Theta_t)\) and

$$I_\mu((u_0^+, v_0^+)) = \sigma^+(\tilde{\delta}, \Theta_t).$$

**Proof** Similar to the proof in Wu [21], there exists minimizing sequences

\((u_n^\pm, v_n^\pm)\) for \(I_\mu\) on \(\mathcal{N}_\mu^+(\tilde{\delta}, \Theta_t)\) such that

$$I_\mu((u_n^+, v_n^+)) = \sigma^+(\tilde{\delta}, \Theta_t) + o(1)$$

and

$$I_\mu((u_n^-, v_n^+)) = o(1) \text{ in } \mathcal{H}(\Theta_t), \text{ as } n \to \infty.$$

By Lemma 5 and the compact imbedding theorem, there exist subsequences still denoted by

\((u_n^\pm, v_n^\pm)\) and \(u_0^\pm \in \mathcal{N}_\mu^+(\tilde{\delta}, \Theta_t)\) such that

\((u_n^+, v_n^+) \rightharpoonup (u_0^+, v_0^+)\) weakly in \(\mathcal{H}(\Theta_t);\)

\((u_n^-, v_n^+) \to (u_0^-, v_0^+)\) strongly in \(L^{p+1}(\Theta_t);\)

\((u_n^-, v_n^+) \to (u_0^-, v_0^+)\) strongly in \(L^{q+1}(\Theta_t);\)

Now, we show that \((u_n^-, v_n^+)\) converges to \((u_0^-, v_0^+)\) strongly in \(\mathcal{H}(\Theta_t).\) Suppose otherwise.

By the lower semi-continuity of the norm, then either

$$\| (u_0^+, v_0^+) \| < \liminf_{n \to \infty} \| (u_n^+, v_n^+) \|$$

and so

$$\| (u_0^+, v_0^+) \|^2 - 2\int_{\Omega} (u_0^+) a^{1+1} (v_0^+) ^{\beta+1} dx - \mu \int_{\Omega} (u_0^+) a^{1+1} (v_0^+) ^{\beta+1} dx < 0.$$

We get a contradiction with the fact that \((u_0^+, v_0^+) \in \mathcal{N}_\mu^+(\tilde{\delta}, \Theta_t).\) Hence, \((u_n^+, v_n^+)\) converge to \((u_0^+, v_0^+)\) strongly in \(\mathcal{H}(\Theta_t).\) This implies

$$I_\mu((u_n^+, v_n^+)) \to I_\mu((u_0^+, v_0^+)) = \sigma^+(\tilde{\delta}, \Theta_t) \text{ as } n \to \infty.$$

Since \(I_\mu((u_0^+, v_0^+)) = I_\mu((u_0^+, |v_0^+|))\) and \((u_0^+, |v_0^+|) \in \mathcal{N}_\mu^+(\tilde{\delta}, \Theta_t),\) then by Lemma 2 and the maximum principle, we may assume that \((u_0^+, v_0^+)\) are positive solutions of our problem Eq. (1).

**Proof of Theorem 1**

Let \((u, v)\) be the ground state solution of Eq. (1) in the infinite strip \(S,\) then \(I_\mu((\tilde{u}, \tilde{v})) = \alpha_0(S).\)

Without loss of generality, we shall assume that \(0 \in \omega\) and there exists \(r_0 > 0\) such that \(B^{N-1}(0, r_0) \subset \omega\) and \(\omega \subset B^{N-1}(0, r_0/2).\) Let \(\tilde{h} = (0, h) \in \mathbb{R}^{N-1} \times \mathbb{R}\) with \(|h| = 1/2\) and let \(\varphi : \mathcal{R}^N \to [0, 1]\) be a \(C^\infty\) cut-off function such that \(0 \leq \varphi \leq 1\) and

$$\varphi(z) = \begin{cases} 0, & z \in B^N(0, r_0/2) \cup S_+^\perp \cup S_-, \\ 1, & z \in S_{+t+1,-t-1} \setminus B^N(0, r_0) \end{cases}$$

and \(v_t(z) = \varphi(z) \tilde{u}(z - \tilde{h})\) for \(z \in S.\) Then \(v_t \in \mathcal{H}(\Theta_t)\) for all \(t > t_0.\) Moreover, for each \(t > t_0,\) there exists \(S_{t, \mu} > 0\) such that \((u, v)_{t, \mu} \in M_{\mu}(\Theta_t).\) For \(\mu \geq 0,\) defining the following

$$I_t = \{-t, t\},$$

$$V_{\mu} = \{ (u, v) \in M_{\mu}(\Theta_t) \mid (u, v) \geq 0 \},$$

$$\Gamma_\mu(\Theta_t) = \{ \gamma \in C[I_t, V_{\mu}] \cap (u, v) \geq 0 \},$$

$$\beta_\mu(\Theta_t) = \inf_{\gamma \in \Gamma_\mu(\Theta_t)} \max_{s \in I_t} I(\gamma)(s)$$

By Lemma 4.6 in Wu [21], there exists \(d_0 > 0\) such that
Existence of Positive Solutions to Semilinear Elliptic Systems Involving Concave and Convex Nonlinearities

\[
\inf_{(u,v) \in M^+_\mu(\delta_0, \Theta_1)} I_0((u,v)) < d_0 \leq \beta_0(\Theta_1) \quad (17)
\]

Similar to the argument of Theorem 4.1 in Wu [21] and by Lemmas 4 and 8, for any \( \theta \in (0,1) \), we have

\[
\beta_\mu(\Theta_1) \geq (1 - \theta \mu) \left( \frac{q+1}{q} \right) \beta_0(\Theta_1) - \\
\frac{(1-q)}{2(q+1)} \left| \frac{\Theta_1}{\Theta_0} \right|^{(q+1)} (K_{\alpha,\beta}(\alpha,\beta))^{\frac{2(q+1)}{1-q}}
\]

and

\[
\beta_\mu(\Theta_1) \leq (1 + \theta \mu) \left( \frac{q+1}{q} \right) \beta_0(\Theta_1) + \\
\frac{(1-q)}{2(q+1)} \left| \frac{\Theta_1}{\Theta_0} \right|^{(q+1)} (K_{\alpha,\beta}(\alpha,\beta))^{\frac{2(q+1)}{1-q}} \cdot (19)
\]

Furthermore, we have the following result.

**Lemma 12** For each \( \delta > 0 \), there exists \( \Lambda \leq \mu^* \) so that for \( \mu \in (0, \Lambda) \), we have

\[
\sigma^\pm(\delta, \Theta_1) < \inf_{(u,v) \in M^+_\mu(\delta_0, \Theta_1)} I_0((u,v)) < \beta_0(\Theta_1) - \delta < \beta_\mu(\Theta_1).
\]

**Proof** By Eqs. (17)-(19), we have

\[
I_0((u,v)) < \beta_0(\Theta_1) - \delta < \beta_\mu(\Theta_1).
\]

By the compact imbedding theorem, the Holder inequality and the maximum principle, we can conclude that Eq. (1) has a positive solution \((\hat{u}, \hat{v}) \in M^-_\mu(\Theta_1)\) such that

\[
I_\mu((\hat{u}, \hat{v})) = \beta_\mu(\Theta_1).
\]

By Lemmas 3, 11 and 13 we obtain that Eq. (1) has four positive solutions \((u_{\min}, v_{\min}), (u^+_0, v^+_0), (u^-_0, v^-_0)\) and \((\hat{u}, \hat{v})\). Since

\[
\mathcal{M}_\mu(\Theta_1) \cap \mathcal{N}^-_\mu(\delta, \Theta_1) = \emptyset, \mathcal{N}^-_\mu(\delta, \Theta_1) \cap \mathcal{N}^-_\mu(\delta, \Theta_1) = \emptyset
\]

and

\[
I_\mu((\hat{u}, \hat{v})) = \beta_\mu(\Theta_1) > \sigma^\pm(\delta, \Theta_1),
\]

this implies that \((u_{\min}, v_{\min}), (u^+_0, v^+_0), (u^-_0, v^-_0)\) and \((\hat{u}, \hat{v})\) are distinct.

**4. Conclusions**

Drawing on the work of Wu (22), this work shows that there exists at least four positive solutions to our system, and by splitting twice in the Nehari manifold: \(\mathcal{M}_\mu(\Theta_1)\) in which \(\mathcal{M}^+_\mu(\Theta_1)\) and \(\mathcal{M}^-_\mu(\Theta_1)\) and which provides a positive solution first in \(\mathcal{M}^+_\mu(\Theta_1)\) and \(\mathcal{M}^-_\mu(\Theta_1)\) in which \(\mathcal{N}^-_\mu(\delta, \Theta_1)\) and \(\mathcal{N}^-_\mu(\delta, \Theta_1)\) provides two positive solutions. In the end, using the compact imbedding theorem, the Holder inequality and the maximum principle, we prove the existence of a fourth positive solution in \(\mathcal{M}^+_\mu(\Theta_1)\), so we can continue this subdivision, for example \(\mathcal{N}^-_\mu(\delta, \Theta_1)\), to find other positive solutions.

**References**


Existence of Positive Solutions to Semilinear Elliptic Systems Involving Concave and Convex Nonlinearities.

Equation With Concave and Convex Nonlinearities.”


