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A Non-heuristic Approach to the General Two Water Jugs Problem

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Abstract: The two water jugs problem is a famous problem in recreational mathematics, problem-solving, artificial intelligence, neuroscience, computer programming and cognitive psychology. The methods of solutions are usually based on heuristics or search methods such as BFS (breadth first search) or DFS (depth first search), which could be time and memory consuming. In this paper, we present a non-heuristic approach to solve this problem, which can be modeled by the Diophantine equation $mx + ny = d$, where m, n denote the capacities of the jugs and d denotes the amount of water to be determined, with $0 < m < n$ and $0 < d < n$. By simple additions and subtractions only, the special solutions (x, y) can be found very easily by using the non-Heuristic approach, which correspond to the number of times of the water jugs being fully filled in the whole water pouring process. Also, a simple formula for determining an upper bound on the total number of pouring steps involved is derived, namely $2(m + n - 2)$, based on the method of linear congruence. Due to its simplicity and novelty, this approach is suitable for either hand calculation or computer programming. Some illustrative examples are provided.

Key words: Two water jugs problem, non-heuristic approach, Diophantine equation, problem-solving, artificial intelligence.

1. Introduction

The water jugs problem is a famous problem in problem-solving, geometry, recreational mathematics, discrete mathematics, computer programming, cognitive psychology, neuroscience and artificial intelligence [1-9], etc.. The problem says:

"You are at the side of a river. You have a 3 L jug and a 5 L jug. The jugs do not have markings to allow measuring smaller quantities. How can you use the jugs to measure 4 L of water? "

There are various ways to solve this problem, such as the working backwards approach [1], the billiards approach [2, 3], the diagraph approach [4], the search approach (e.g., BFS or DFS) [5, 9] and the methods of heuristics [6, 9-11]. However, they could be time and memory consuming sometimes. In this paper, we present a simple non-heuristic approach to solve the problem, which was introduced by Man [12]. A novel feature of this approach is that one can deduce the total amount of water (say V) in the jugs at each pouring step by simple additions or subtractions only and the actual pouring sequence can be easily determined by referring to the computed value of V. Due to its novelty and simplicity, this approach is quite suitable for either hand calculation or computer programming. However, unlike the common search methods adopted for solving this problem, no additional memory cost is needed for performing searching and branching by using this non-heuristic approach. Also, a formula for determining an upper bound on the total number of pouring steps involved is provided, which is a simple linear function of the capacities of the given water jugs.

The whole paper is organized as follows: in Section 2, the authors will present the non-heuristic approach

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for solving the general two water jugs problem and describe the mathematical background behind. In the Section 3, the authors will illustrate how to apply such a new approach with some illustrative examples. In the Section 4, the authors will discuss how to derive an upper bound on the total number of pouring steps involved, based on the method of linear congruence. Then, the authors will conclude with some final remarks in the last section.

2. A Non-heuristic Approach

A non-heuristic approach to solve the general two water jugs problem was introduced in Ref. [12], which can be used to handle the problem below:

"Let *m*, *n*, *d* be positive integers. You are at the side of a river. You have a m-liter jug and a n-liter jug, where 0 < m < n. The jugs do not have markings to allow measuring smaller quantities. How can you use the jugs to measure d (< n) liters of water? "

This problem can be modeled by means of a Diophantine equation, namely $mx + ny = d$, whose solvability is determined by the theorem below. A proof of this important result in number theory can be found in Ref. [10].

Theorem 2.1

The Diophantine equation $mx + ny = d$ is solvable if and only if the greatest common divisor of m and n, namely gcd (m, n), divides d.

For instance, the water jugs problem described in the introduction section is solvable since 4 is divisible by gcd (3, 5). However, if the jugs are replaced by a 3 L jug and a 9 L jug, then it will be insolvable since 4 is not divisible by gcd (3, 9). Now, let us assume that $mx + ny = d$ is solvable in the discussions below. Depending on which jug is chosen to be filled first, there are two possible solutions for the two water jugs problems, say M_1 and M_2, which can be determined by the integer sequences obtained by using the following algorithms.

Algorithm 2.1

Input: Integers m, n, d, where $0 < m < n$ and $d < n$.

Output: An integer sequence corresponding to a feasible solution (called M_1) for the general two water jugs problem, by filling the *m*-litre jug first.

Procedure:

Step 1. Initialize a dummy variable $k = 0$ for the integer sequence;

Step 2. If $k \neq d$, then repeat adding m to k and assign the result to k until $k = d$ or $k > n$;

Step 3. If $k > n$, then subtract n from k and assign the result to k;

Step 4. If $k = d$, then stop. Otherwise, repeat the steps from Step 2 to Step 4.

In this algorithm, the number of additions (say x_1) and subtractions (say y_1) involved will provide a solution to the Diophantine equation $mx + ny = d$, namely $x = x_1$, $y = -y_1$. The actual water pouring sequence can be determined easily by referring to the integer sequence obtained.

Algorithm 2.2

Input: Integers m, n, d, where $0 < m < n$ and $d < n$.

Output: An integer sequence corresponding to a feasible solution (called M_2) for the general two water jugs problem, by filling the *n*-litre jug first.

Procedure:

Step 1. Initialize a dummy variable $k = 0$ for the sequence;

Step 2. If $k \neq d$, then add n to k and assign the result to k;

Step 3. If $k > d$, then repeat subtracting m from k and assign the result to k until $k = d$ or $k < m$;

Step 4. If $k = d$, then stop. Otherwise, repeat the steps from Step 2 to Step 4.

In this algorithm, the number of subtractions (say x_2) and additions (say y_2) involved will provide a solution to the Diophantine equation $mx + ny = d$, namely $x = -x_2$, $y = y_2$. Again, the actual pouring sequence can be determined easily by referring to the integer sequence obtained.

3. Examples

We now illustrate how to apply the non-heuristic approach to solve the two water jugs problems below.

Example 3.1

There are a 3 L jug and a 5 L jug. We want to use them to measure 4 L of water, as described in the introduction section above. Using the above notations, we have $m = 3$, $n = 5$, $d = 4$ and the associated Diophantine equation is $3x + 5y = 4$. By applying Algorithm 2.1, we can obtain the following integer sequence for M_1:

$$\boxed{0} \xrightarrow[+3]{} \boxed{3} \xrightarrow[+3]{} \boxed{6} \xrightarrow[-5]{} \boxed{1} \xrightarrow[+3]{} \boxed{4}$$

The number of additions and subtractions involved are 3 and 1, respectively, so $x = 3$, $y = -1$ is a solution of the Diophantine equation $3x + 5y = 4$. Since the integers inside the boxes of the sequence refer to the total amount of water in the jugs at different stages, so we can work out the corresponding water pouring steps very easily. If we use a coordinate (x, y) to represent the amounts of water inside the 3-litre jug and the 5-litre jug in each pouring step, then the successive pouring steps for M_1 can be described as follows:

$(0, 0) \to (3, 0) \to (0, 3) \to (3, 3) \to (1, 5) \to (1, 0) \to (0, 1) \to (3, 1) \to (0, 4)$

Hence, the total number of water pouring steps involved in M_1 is 8. Similarly, we can obtain the following integer sequence for M_2 by applying Algorithm 2.2:

$$\boxed{0} \xrightarrow[+5]{} \boxed{5} \xrightarrow[-3]{} \boxed{2} \xrightarrow[+5]{} \boxed{7} \xrightarrow[-3]{} \boxed{4}$$

The number of additions and subtractions involved are 2 and 2, respectively, so $x = -2$, $y = 2$ is a solution of the Diophantine equation $3x + 5y = 4$. The corresponding water pouring steps for M_2 are as follows:

$(0, 0) \to (0, 5) \to (3, 2) \to (0, 2) \to (2, 0) \to (2, 5) \to (3, 4)$

Thus, the total number of water pouring steps involved in M_2 is 6. By comparing the number of steps in M_1 and M_2, we can see that M_2 is a more optimal solution to this water jug problem.

Example 3.2

There are a 3 L jug and a 7 L jug. We want to use them to measure 5 L of water. So, $m = 3$, $n = 7$, $d = 5$ and the associated Diophantine equation is $3x + 7y = 5$. By applying Algorithm 2.1, we can obtain the following integer sequence for M_1:

$$\boxed{0} \xrightarrow[+3]{} \boxed{3} \xrightarrow[+3]{} \boxed{6} \xrightarrow[+3]{} \boxed{9} \xrightarrow[-7]{} \boxed{2} \xrightarrow[+3]{} \boxed{5}$$

The number of additions and subtractions involved are 4 and 1, respectively, so $x = 4$, $y = -1$ is a solution of the Diophantine equation $3x + 7y = 5$. The corresponding water pouring steps for M_1 are described as follows:

$(0, 0) \to (3, 0) \to (0, 3) \to (3, 3) \to (0, 6) \to (3, 6) \to (2, 7) \to (2, 0) \to (0, 2) \to (3, 2) \to (0, 5)$

Thus, the total number of water pouring steps involved in M_1 is 10. Similarly, we can obtain the following integer sequence for M_2 by applying Algorithm 2.2:

$$\boxed{0} \xrightarrow[+7]{} \boxed{7} \xrightarrow[-3]{} \boxed{4} \xrightarrow[-3]{} \boxed{1} \xrightarrow[+7]{} \boxed{8} \xrightarrow[-3]{} \boxed{5}$$

The number of additions and subtractions involved are 2 and 3 respectively, so $x = -3$, $y = 2$ is a solution to the equation $3x + 7y = 5$. The corresponding water pouring steps for M_2 are as follows:

$(0, 0) \to (0, 7) \to (3, 4) \to (0, 4) \to (3, 1) \to (0, 1) \to (1, 0) \to (1, 7) \to (3, 5)$

Thus, the total number of water pouring steps involved in M_2 is 8. By comparing the number of steps in M_1 and M_2, we can see that M_2 is a more optimal solution to this water jug problem.

Example 3.3

There are a 5 L jug and a 8 L jug. We want to use them to measure 6 L of water. So, $m = 5$, $n = 8$, $d = 6$ and the associated Diophantine equation is $5x + 8y = 6$. By applying Algorithm 2.1, we can obtain the following integer sequence for M_1:

$$\boxed{0} \xrightarrow[+5]{} \boxed{5} \xrightarrow[+5]{} \boxed{10} \xrightarrow[-8]{} \boxed{2} \xrightarrow[+5]{} \boxed{7} \xrightarrow[+5]{} \boxed{12}$$

$$\xrightarrow[-8]{} \boxed{4} \xrightarrow[+5]{} \boxed{9} \xrightarrow[-8]{} \boxed{1} \xrightarrow[+5]{} \boxed{6}$$

The number of additions and subtractions involved are 6 and 3, respectively, so $x = 6$, $y = -3$ is a solution of the Diophantine equation $5x + 8y = 6$. The corresponding water pouring steps for M_1 are as follows:

$(0, 0) \to (5, 0) \to (0, 5) \to (5, 5) \to (2, 8) \to (2, 0) \to (0, 2) \to (5, 2) \to (0, 7) \to (5, 7) \to (4, 8) \to (4, 0) \to (0, 4) \to (5, 4) \to (1, 8) \to (1, 0) \to (0, 1) \to (5, 1) \to (0, 6)$

Thus, the total number of water pouring steps involved in M_1 is 18. Similarly, we can obtain the following integer sequence for M_2 by applying Algorithm 2.2:

$$\boxed{0} \xrightarrow[+8]{} \boxed{8} \xrightarrow[-5]{} \boxed{3} \xrightarrow[+8]{} \boxed{11} \xrightarrow[-5]{} \boxed{6}$$

The number of additions and subtractions involved are 2 and 2, respectively, so $x = -2$, $y = 2$ is a solution to the equation $5x + 8y = 6$. The corresponding water pouring steps for M_2 are as follows:

$(0, 0) \to (0, 8) \to (5, 3) \to (0, 3) \to (3, 0) \to (3, 8) \to (5, 6)$

Thus, the total number of water pouring steps involved in M_2 is 6. By comparing the number of steps in M_1 and M_2, we can see that M_2 is a more optimal solution to this water jug problem.

4. A Bound on the Number of Pouring Steps

Assume the Diophantine equation $mx + ny = d$ is solvable. By using linear congruence, the smallest positive integral solution, say (x', y'), can be found by solving the congruence equations below:

$$mx \equiv d \pmod{n}; \quad ny \equiv d \pmod{m}. \tag{1}$$

Hence, we have:

$$|x'| \le n - 1; \quad |y'| \le m - 1 \tag{2}$$

Let N be the total number of pouring steps involved in applying the non-heuristic approach described in this paper. Since each number in the integer sequence M_1 or M_2 denotes the total amount of water in the two jugs in a particular step and there are at most two water pouring steps associated with such a number as illustrated in the examples above, so we have:

$$N \le 2(|x'| + |y'|) \le 2(m + n - 2). \tag{3}$$

It is obvious that the value of N may not be the same for M_1 and M_2 in general, and N is strictly less than $2(m + n - 2)$ in most cases, as shown in the examples described in Section 3.

5. Conclusions

A simple non-heuristic approach for solving the general two water jugs problem is presented in this paper, which is suitable for either hand calculations or computer programming. The integer sequences for M_1 or M_2 (see Section 2 for their meanings) can be computed easily by simple additions and subtractions only, via the use of Algorithm 2.1 or Algorithm 2.2. In addition, an upper bound on the total number of water pouring steps involved can be obtained by simply substituting the values of m, n into the expression $2(m + n - 2)$. Unlike some common search methods, there is no additional memory cost required for doing searching and branching. Due to its simplicity and novelty, this non-heuristic approach is suitable for introduction to university students or researchers studying or doing researches in the areas of problem-solving, discrete mathematics, computer programming, neuroscience, cognitive psychology, artificial intelligence or recreational mathematics, etc.. The possibility of extension or modification of this new non-heuristic approach to tackle the more general k (> 2) water jugs problem will be an interesting and challenging research problem for further pursue.

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