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On ϵ-Constraint Based Methods for the Generation of Pareto Frontiers

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Abstract: Over the years, a number of methods have been proposed for the generation of uniform and globally optimal Pareto frontiers in multi-objective optimization problems. This has been the case irrespective of the problem definition. The most commonly applied methods are the normal constraint method and the normal boundary intersection method. The former suffers from the deficiency of an uneven Pareto set distribution in the case of vertical (or horizontal) sections in the Pareto frontier, whereas the latter suffers from a sparsely populated Pareto frontier when the optimization problem is numerically demanding (ill-conditioned). The method proposed in this paper, coupled with a simple Pareto filter, addresses these two deficiencies to generate a uniform, globally optimal, well-populated Pareto frontier for any feasible bi-objective optimization problem. A number of examples are provided to demonstrate the performance of the algorithm.

Key words: Pareto frontier, multiobjective optimization, scalarization methods, ϵ-constraint methods, design optimization.

1. Introduction

Often, engineers are faced with the challenge of designing industrial applications to meet a defined set of system requirements, while at the same time keeping the physical size and cost of the system to a minimum. This is inherently a bi-objective optimization problem, where the set of system requirements are referred to as *constraints* and the minimization of the physical size of the system and its cost are the *objectives* of the problem.

The bi-objective optimization problem is a specific case in generic multi-objective optimization in which any number of objectives can be defined. Multi-objective optimization originally grew out of three areas: economic equilibrium and welfare theories, game theory and pure mathematics [1]. As a consequence, many terms and fundamental ideas originate from these fields.

The optimization problem generally requires the minimization of an objective function subject to constraints, some of which are an inherent part of the original problem, while others may be introduced by way of the multi-objective problem formulation [2]. The design of a system with more than one objective is, in the literature, referred to as a multiple criteria decision making problem [3]. This type of decision and planning problem involves multiple conflicting objectives that need to be considered simultaneously.

Solving the multi-objective optimization problem does not lead to a single global solution. Owing to the competing nature of the objectives, it might be possible to obtain an infinite number of solutions where each unique solution assigns different priorities to the problem objectives. The solutions are known as Pareto points and constitute the Pareto optimal set. The definition for Pareto optimality as defined by Pareto [4] as follows:

Pareto Optimality: A point $x^* \in X$ is Pareto

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optimal if and only if there does not exist another point $x \in X$, such that $F(x) \leq F(x^*)$, and $F_i(x) < F_i x^*$ for at least one function.

It is then up to the decision maker (DM) to select a point from the Pareto optimal set. The generation of the entire or part of the Pareto optimal set is, however, computationally intensive and therefore its generation should be avoided unless absolutely necessary. This norm especially applies when the number of objectives is larger than two, leading to a three- or multi-dimensional Pareto surface. In such cases, the DM should specify preferences articulated in terms of goals or the relative importance of different objectives. Methods based on this principle are referred to as having *a priori* articulation of preferences [1]. Most of these methods incorporate parameters that can be set to reflect the DM preferences.

In general, it is difficult for a DM to define the preference function *a priori*. In this case, it can be more suitable to provide the DM with a palette of solutions through the Pareto optimal set, from which one is chosen. Methods that incorporate algorithms that generate the Pareto optimal set are referred to as having *a posteriori* articulation of preferences [1]. Clearly, however, *a posteriori* methods are computationally more intensive than *a priori* methods since a whole set of solutions needs to be generated as opposed to the latter in which only one solution is calculated.

The systematic generation of the Pareto frontier can be accomplished through one of two predominant techniques: scalarization, and vectorization methods. On the one hand, scalarization methods convert the Multi-Objective Optimization Problem (MOOP) into a series of parametric Single-Objective Optimization Problems (SOOPs), while on the other hand, vectorization methods tackle the MOOP directly [5].

Vectorization techniques such as genetic algorithms [6], are in general easy to implement and regarded as global optimization approaches. However, due to the stochastic nature of their search procedures, if the search space is not highly constrained, the optimization process will require a large number of objective function evaluations. As a consequence, the algorithm can be very computationally intensive.

Scalarization techniques, in general, do not suffer from the above drawback, since efficient deterministic optimization approaches can be exploited to find a (local) optimum for the different possibly large-scale SOOPs. The efficient solution of the SOOPs is crucial, because their number increases exponentially with the number of the objectives [7]. A number of scalarization methods have been proposed in the literature, such as the linear weighted sum, the normal constraint [2], the normal boundary intersection [8] and the physical programming [9] methods. A number of deficiencies in these methods were addressed by further work and modifications to the algorithms, which resulted in the modified [10] and enhanced [11] normal constraint methods, the normalized angular constraint method [12], the modified normal boundary intersection method [13] and the modified physical programming method [14].

This paper proposes an alternative method, herein referred to as the adaptive bisection ϵ-constraint method, for bi-criterion problems. The proposal uses the ϵ-constraint method as a basis and develops an algorithm that systematically manipulates the SOOP formulations to generate a diverse, well-distributed Pareto frontier even in the case of non-convex Pareto sets. The proposed method performs as well as the normal constraint, normal boundary intersection methods and their enhancements for bi-criterion problems with loose or no constraints. It is however superior in the case of tightly constrained problems when individual SOOPs are sometimes infeasible. In such cases, the algorithm searches in the vicinity of the infeasible problem space until feasible solutions are found, such that the Pareto frontier will remain densely populated. To the best knowledge of the authors, this approach has not been adopted in other methods, resulting in Pareto frontiers that are sparsely populated due to the lack of feasible solutions.

This paper is structured as follows: Section 2 presents the formulation of a general bi-objective optimization problem. In section 3, an overview of the most commonly applied scalarization methods is presented. Specifically, the linear weighted sum, normal boundary intersection and normal constraint methods are described with particular attention to their deficiencies in generating complete, well-distributed optimal Pareto frontiers. Section 4 presents the equidistant ϵ-constraint method, which is explained from a graphical perspective. The ensuing problem formulation is also presented and an analysis with respect to the scalarization methods surveyed is made. In section 5, the adaptive bisection ϵ-constraint method is presented, addressing the deficiencies in the equidistant ϵ-constraint method and the methods described is section 3. Section 6 describes the tests carried out on the methods described in sections 4 and 5 and their results. Four bi-criterion examples are considered, starting from a relatively simple test resulting in a convex Pareto frontier, then increasing the complexity with two further tests resulting in non-convex Pareto sets, and finally a tightly constrained optimization problem. An analysis ensues, which compares the proposed methods with the ones surveyed. A simple Pareto filter is finally coupled to the adaptive bisection ϵ-constraint method to address the generation of non-optimal Pareto points. This final step in the development is also put to test with the examples having non-convex Pareto optimal sets. This paper is concluded with section 7, where a conclusion and some remarks are made with respect to the work presented.

2. Bi-Objective Optimization Problem Formulation

The bi-objective optimization problem is defined as follows:

$$\min_x\{\mu_1 \mu_2\} \tag{1}$$

subject to

$$g_j(x) \leq 0, (1 \leq j \leq r) \tag{2}$$

$$h_k(x) = 0, (1 \leq k \leq s) \tag{3}$$

$$x_{li} \leq x_i \leq x_{ui}, (1 \leq i \leq n_x) \tag{4}$$

where x is the vector of decision variables to be optimized of dimension n_x, $g_j(x)$ and $h_k(x)$ are the $r$inequality and $s$equality constraints respectively, and x_{li} and x_{ui} are the lower and upper bound constraint vectors on x respectively, both of dimension n_x.

3. Overview of Scalarization Methods

The three most applied scalarization methods in the literature are the linear weighted sum (LWS), the normal boundary intersection (NBI) and the normal constraint (NC) methods. A brief overview of the individual methods is given in this section, followed by an analysis of the advantages and disadvantages of each with respect to each other.

3.1 Linear Weighted Sum (LWS) Method

The most common approach to multi-objective optimization is the LWS method. The technique is to reduce the MOOP into a SOOP by combining the multiple objectives into a single objective through the selection of a multiplicative weight for each criterion and summing them together:

$$U = \sum_{i=1}^{k} w_i F_i(x) \tag{5}$$

By modifying the weights, different points on the Pareto optimal set can be found. Unfortunately, varying the weights systematically may not necessarily result in an even distribution of Pareto optimal points. In such cases, a complete representation of the Pareto optimal set will not be achieved [15]. In most cases, the LWS is unable to capture the middle ground of the Pareto set, rendering it fairly useless as a means of studying the trade-off between conflicting objectives [8]. Moreover, it is impossible to obtain points on non-convex portions of the Pareto optimal set in the criterion space [1]. Even though non-convex Pareto optimal sets are rather uncommon, some examples are found in the literature [16-18]. This is a serious and significant limitation because in real world problems using simulation models and/or systems of partial differential equations it is not always easy to check for convexity. If this method is used for non-convex problems, therefore, the Pareto frontier generated will be incomplete, leaving the DM with a non-complete set of feasible solutions [3].

3.2 Normal Boundary Intersection (NBI) Method

The NBI method was developed by Das and Dennis [8] to address deficiencies in the LWS approach. An even distribution of Pareto optimal points for consistent weight variations, for both convex and non-convex Pareto sets, is possible though this approach. The approach is formulated as follows:

$$\min_{x \in X, \lambda} \lambda \quad (6)$$

subject to

$$\varphi w + \lambda n = F(x) - F^o \quad (7)$$

where φ is a $k \times k$ pay-off matrix in which the *i*th column is composed of the vector $F(x_i^*) - F^o$, and $F(x_i^*)$ is the vector of objective functions evaluated at the minimum of the *i*th objective function. The diagonal elements of φ are zeros, w is a vector of scalars such that $\sum_{i=1}^{k} w_i = 1$ and $w \geq 0$. $n = -\varphi e$, where $e \in R^k$ is a column vector of ones in the criterion space. As w is systematically modified, an even distribution of Pareto optimal points representing the complete Pareto set results, even with a non-convex Pareto optimal set.

It is worth noting that, unlike the LWS, the weights in the NBI method do not give the DM information on the relative importance of the objectives. Therefore, *a priori* articulation of preferences is not suitable for this method, as the selection of preferences would be hard for the DM to visualize. A weak point of the NBI method is that in highly nonlinear problems, it is hard to obtain optimal solutions due to equality constraints [19]. Yet, another problem is that, with this method, solutions of sub-problems need not be Pareto-optimal (not even locally), because the NBI approach aims at getting boundary points on the Pareto frontier. The resulting boundary points are a superset of the Pareto optimal set [13].

3.3 Normal Constraint (NC) Method

The NC method developed by Messac et al. [2] provides an alternative to the NBI method with some improvements. The method proceeds as follows. First, the utopia point, which is a combination of the individual minima of the objective functions, is found, and its components are used to normalize the objectives. A utopia hyperplane in the criterion space is thus formed from the vertices of the individual minima of the objective functions. A sample of evenly distributed points on the utopia hyperplane is determined from a linear combination of the vertices with evenly varied weights in the criterion space. The pareto points are generated by minimizing one of the normal objective functions with additional inequality constraints in succession using the weights generated from the evenly distributed points on the utopia hyperplane. It is relevant to note that in the NC method additional inequality constraints are added in the problem formulation as opposed to equality constraints in the NBI method.

As this approach may lead to non-optimal Pareto optimal solutions, the NC method is usually used with a Pareto filter. Furthermore, evenly spaced Pareto optimal points in the criterion space are produced if a normalization process is included in the algorithm. Failure to include this feature will lead to non-evenly distributed Pareto sets for differently scaled design objectives. The addition of the Pareto filter and the normalization process to the algorithm leads to what is referred to as the Normalized Normal Constraint (NNC) method. Even with these added features, a drawback of the NNC method is that, although in general it generates well-distributed solutions in the Pareto frontier, it loses consistency if the slope of this frontier is close to the horizonal [10].

4. The Equidistant ϵ-Constraint Method

The ϵ-constraint method was first proposed by Haimes et al. in 1971 [20]. In this method, one of the objective functions is selected to be optimized while the other(s) are converted into additional constraints, leading to a solution that can be proven to always be weakly Pareto optimal [3]. Systematic modification of the values of the objective functions forming the additional constraints leads to the generation of an evenly distributed Pareto frontier. A method for systematically modifying the additional constraints, referred to as the equidistant ϵ-constraint method, is being proposed and proceeds as follows.

The algorithm starts by obtaining the anchor points μ_1^* and μ_2^* of the BOOP, corresponding to the minimum values of each of the objective functions through solving the following two single-objective optimization problems:

$$\min_x\{\mu_1\mu_2\} \tag{8}$$

subject to

$$g_j(x) \le 0, (1 \le j \le r) \tag{9}$$

$$h_k(x) = 0, (1 \le k \le s) \tag{10}$$

$$x_{li} \le x_i \le x_{ui}, (1 \le i \le n_x) \tag{11}$$

The solution of Eq. (8) leads to the anchor points of the Pareto frontier being captured, ensuring that no part of the Pareto frontier fails to be considered in the optimization process. The intersection of the lines $\mu_1 = \mu_1^*$ and $\mu_2 = \mu_2^*$ defines the utopian point μ_u, which, albeit being an ideal solution, does not lie in the feasible region of the optimization problem (Fig. 1).

Defining d as the vertical distance between the utopian point μ_u and the anchor point μ_1^*, the line d is divided into $(n_p - 1)$ equidistant points of separation Δd (Fig. 2). The value of n_p is user specified and it determines the target number of points on the Pareto frontier.

The equidistant ϵ-constraint method transforms the BOOP into n_p SOOPs. The anchor points are determined through the solution of two single-objective problems, leaving $(n_p - 2)$ problems to be solved, which are formulated as follows:

$$\min_x \mu_1 \tag{12}$$

subject to

$$g_j(x) \le 0, (1 \le j \le r) \tag{13}$$

$$h_k(x) = 0, (1 \le k \le s) \tag{14}$$

$$x_{li} \le x_i \le x_{ui}, (1 \le i \le n_x) \tag{15}$$

with the additional constraint:

$$\mu_2 = \varepsilon_c \tag{16}$$

The additional constraint of Eq. (16) is unique for each problem formulation and is calculated as follows:

$$\varepsilon_c = \mu_2^* + \Delta_d(c-2) \tag{17}$$

where c is the problem formulation number having a range $[3, n_p]$.

The equidistant ϵ-constraint method is a very intuitive and simple method to apply to bi-criterion problems. However, it suffers from a significant number of deficiencies. Firstly, without proper scaling of the objective functions, a well-distributed spread of the Pareto frontier is not achieved. Nevertheless, it still performs better than the LWS method, as the whole frontier is covered, albeit with a varying density of points. Secondly, the solutions, although being weakly Pareto optimal, are not necessarily global

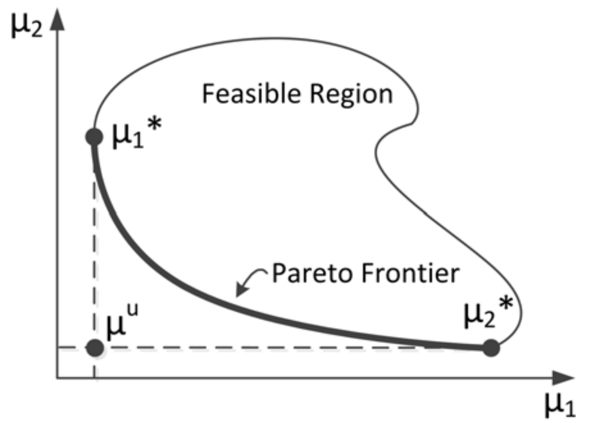


Fig. 1 Graphical representation of the design metric space for a BOOP.

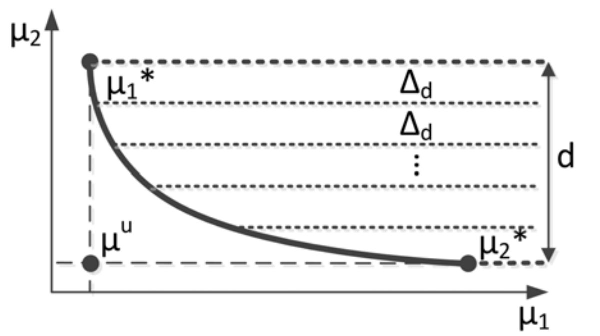


Fig. 2 A set of equidistant points on the Utopia line for a BOOP.

Pareto optimal points and this raises the need for a Pareto filter to dispose of non-global Pareto points. Another significant problem that is present in all the methods reviewed, including the NBI and NNC methods, is that if a specific SOOP formulation is infeasible, a potential point is lost from the Pareto frontier since no other SOOP is formulated to find an alternative solution in the Pareto optimal set. Thus, this method, together with the other reviewed methods, is not suitable for tightly constrained problems or Pareto frontiers with discontinuities when the DM requires a representation of the whole Pareto frontier. It is interesting, however, that this method is able to represent non-convex Pareto frontiers, unlike the LWS method.

5. The Adaptive Bisection ϵ-Constraint Method

The adaptive bisection ϵ-constraint method proposed in this work was developed to address the deficiencies in the equidistant ϵ-constraint method. It differs from the equidistant ϵ-constraint method in the way the additional constraint of Eq. (16) is calculated.

The concept behind the method is to first find the anchor points of the Pareto frontier. Then, the utopia line, which is the straight line joining the anchor points, is bisected to obtain a value ε_c for the additional constraint of Eq. (16) of the BOOP defined in Eq. (12).

The solution of the optimization problem will lead to an additional point on the Pareto frontier. Sometimes, this will lead to an infeasible problem that leads to no solution. In this case, the line joining the anchor points is subdivided into four sections and the constraint ε_c is set to the value of μ_2 at one-fourth the length of the line. If the problem is still infeasible, the value of μ_2 at three-quarters of the line joining the anchor points is then tried. The line will continue being bisected until a solution is found or a constant K set by the user is reached.

Once an additional point μ_3^* is found, the euclidean distance between the point and other points on the Pareto frontier is determined. The two points with minimal euclidean distance are then used to find an additional point on the Pareto frontier by using the method of line bisection as previously described.

This process is repeated until the number of Pareto points n_p requested by the user is found. A simplified flow diagram of the algorithm is illustrated in Fig. 3.

The adaptive bisection ϵ-constraint method, albeit slightly more complex to implement than the equidistant ϵ-constraint method, is still very intuitive and simple to implement and apply to bi-criterion problems. Nonetheless, the simplicity does not compromise the performance of the algorithm. As opposed to the first method proposed, this method does not suffer from scaling problems. It is also able to generate equally distributed Pareto frontiers for both non-convex and horizontal (or vertical) Pareto frontiers, which is not the case for the NNC method. Finally, the greatest feature is the adaptivity of the algorithm, which allows the reformulation of individual SOOPs in the case of infeasibility to ensure a dense Pareto frontier for even the most difficult of optimization problems.

6. Numerical Examples

In this section. four examples are used to generate Pareto frontiers with the proposed methods. First, the equidistant ϵ-constraint method is considered with inequality constraints, i.e., setting Eq. (16) to $\mu_2 \leq \varepsilon_c$. Such a formulation is less difficult than setting an equality constraint from an optimization point of view. Then, the ϵ-constraint method as formulated in section 4 is put into practice. Finally the adaptive bisection ϵ-constraint method is considered.

6.1 Example 1

The first example considered [2] results in a convex Pareto frontier, constituting both vertical and horizontal parts with the scales of the objective

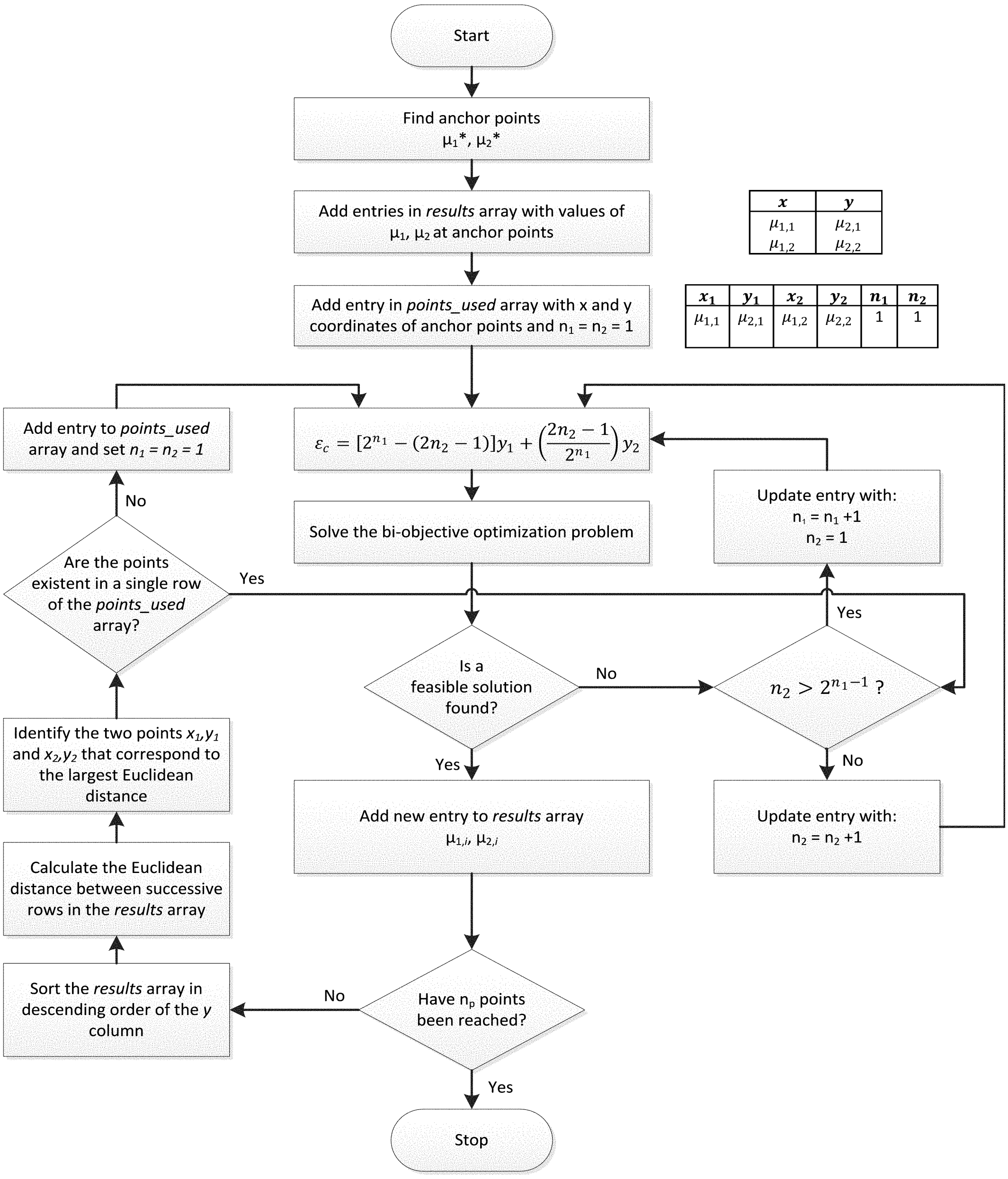


Fig. 3 Flow diagram of the adaptive bisection ϵ-constraint method.

functions having different orders of magnitude. The BOOP has two optimization variables, two equality constraints and an inequality constraint and it is defined as follows:

$$\min_x\{\mu_1 \mu_2\} \tag{18}$$

subject to

$$\mu_1 = x_1 \tag{19}$$

$$\mu_2 = x_2 \tag{20}$$

$$\left(\frac{x_1-20}{20}\right)^8 + \left(\frac{x_2-1}{1}\right)^8 \leq 1 \tag{21}$$

6.2 Example 2

The second example considered [2] generates a non-convex Pareto frontier. Similar to example 1, it has two optimization variables, two equality constraints and one inequality constraint:

$$\min_x\{\mu_1 \mu_2\} \tag{22}$$

subject to

$$\mu_1 = x_1 \tag{23}$$

$$\mu_2 = x_2 \tag{24}$$

$$5e^{-x_1} + 2e^{-0.5(x_1-3)^2} \leq x_2 \quad (25)$$

6.3 Example 3

An example with several noncontiguous complex parts is considered next [21]. The number of optimization variables increases drastically to thirty. However, no constraints are set. The BOOP problem is formulated as follows:

$$\min_x\{\mu_1\mu_2\} \quad (26)$$

subject to

$$\mu_1 = x_1 \quad (27)$$

$$\mu_2 = g\left[1 - \sqrt{\frac{x_1}{g}} - \frac{x_1}{g}\sin 10\pi x_1\right] \quad (28)$$

$$g = 1 + 9\left(\textstyle\sum_{i=2}^{n} x_i\right)/(n-1) \quad (29)$$

$$n = 30 \quad (30)$$

6.4 Example 4

Finally, a bi-objective problem which has particularly tight constraints [22], is put to the test. It is bi-objective, has two optimization variables, two equality constraints and two inequality constraints:

$$\min_x\{\mu_1\mu_2\} \quad (31)$$

subject to

$$\mu_1 = x_1 \quad (32)$$

$$\mu_2 = (1 + x_2)/x_1 \quad (33)$$

$$x_2 + 9x_1 \geq 6 \quad (34)$$

$$-x_2 + 9x_1 \geq 1 \quad (35)$$

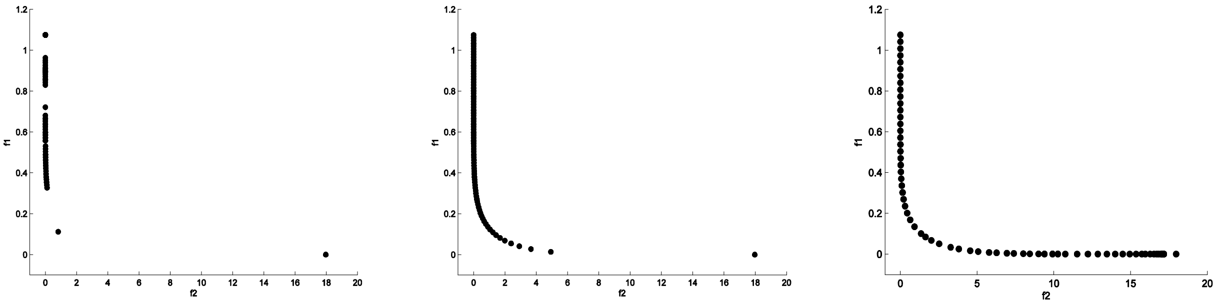
6.5 Results and Discussion

In the first algorithm, involving the equidistant ϵ-constraint method with inequality constraints, the inequality constraint replacing Eq. (16) made the optimization problems less constrained, resulting in less computational resources being required. Unfortunately, however, the downside is that for a number of individual BOOPs, the gradient-based optimizer proceeded with its search away from the potentially infeasible solutions at $\mu_2 = \varepsilon_c$. This resulted in repeated solutions at other points on the Pareto frontier where, from an initial stage, the solution will seem more feasible. Figs. 4a, 5a, 6a, and 7a demonstrate this fact. As a consequence, the Pareto frontiers generated are incomplete with a high density of points in only a few regions.

The equidistant ϵ-constraint method as formulated in section 4 performed better than the equidistant ϵ-constraint method with inequality constraints. Nonetheless, it still suffered from a number of deficiencies. The first can be observed in Fig. 4b, where the vertical part of the Pareto frontier is represented by equally distributed points. However, as the gradient of the Pareto frontier starts decreasing, the distribution of the points starts getting sparser. Clearly the scalarization approach is performing badly because the different scale magnitudes of the objective functions are different. Another deficiency is that a number of gaps appear in the Pareto frontier, shown clearly in Figs. 5b and 7b. These gaps occur because in the respective area, the optimizer finds a difficulty finding a feasible solution with the initial guess it is given. For tightly constrained problems such as that of example 4, this deficiency will lead to the situation where the DM does not have enough information to make an informed choice on the best solution for the problem at hand.

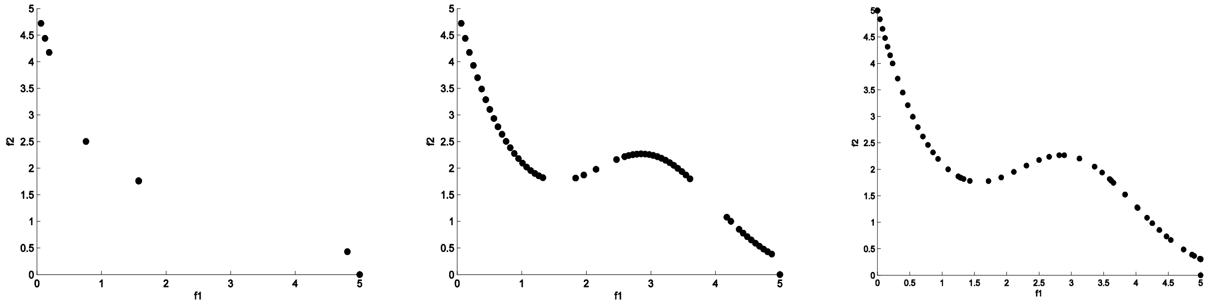
Finally, the adaptive bisection ϵ-constraint method proposed in this paper was tested. In the first example, the deficiency suffered by the previous two methods was resolved and a complete, evenly distributed Pareto frontier can be observed in Fig. 4c. The gaps in the Pareto frontiers generated by the equidistant ϵ-constraint methods have also been addressed by formulating problems in the vicinity of infeasible regions until a solution is found. The algorithm actively varies the constraints to obtain better Pareto diversity, even in the case of problems with high levels of infeasible solutions. As a result, all Pareto frontiers generated for the four examples considered are evenly distributed as illustrated in Figs. 4d, 5d, 6d, and 7d.

In all three methods, both convex and non-convex Pareto frontiers were captured. Unfortunately, one deficiency observed in all methods remains which is the generation of non-globally optimal Pareto solutions



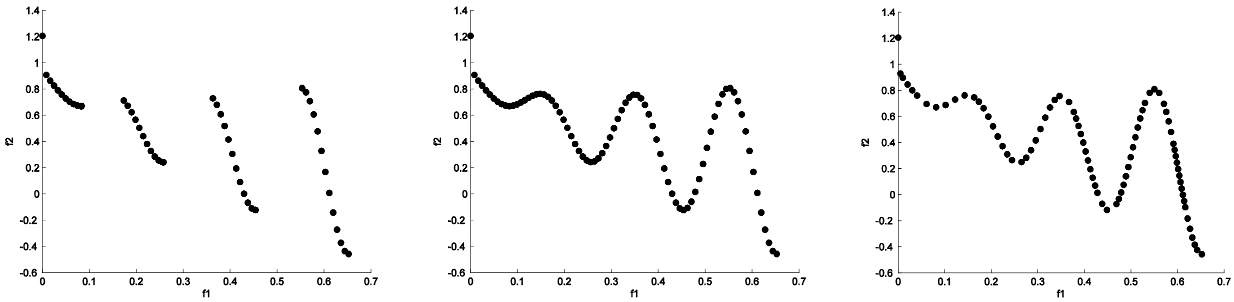
(a) (b) (c)

Fig. 4 Pareto frontiers for Example 1: (a) Equidistant ϵ-constraint method with inequality constraint; (b) Equidistant ϵ-constraint method and (c) Adaptive bisection ϵ-constraint method.



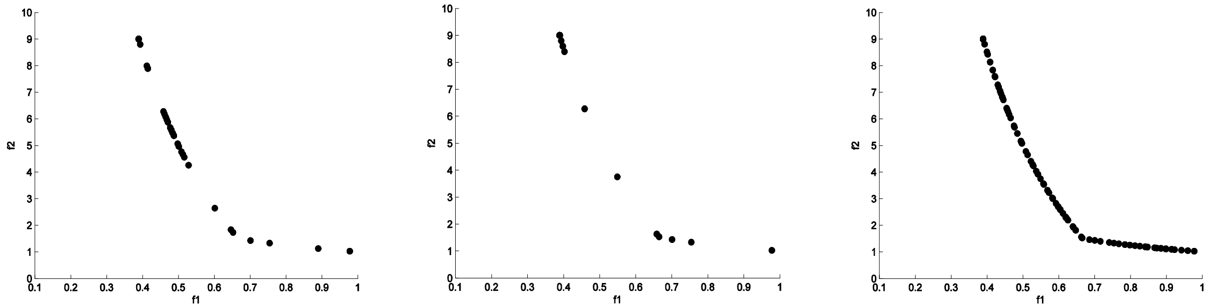
(a) (b) (c)

Fig. 5 Pareto frontiers for Example 2: (a) Equidistant ϵ-constraint method with inequality constraint; (b) Equidistant ϵ-constraint method and (c) Adaptive bisection ϵ-constraint method.



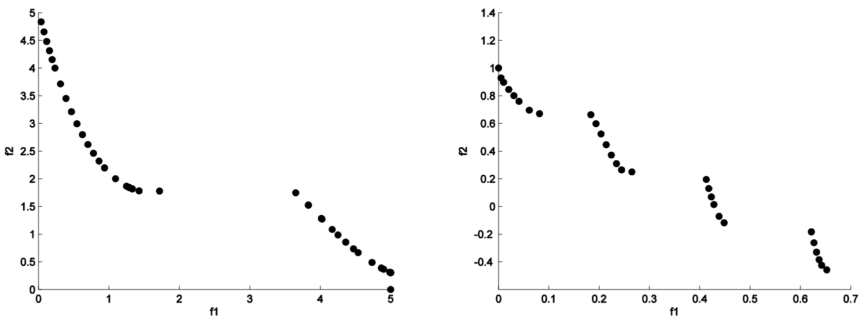
(a) (b) (c)

Fig. 6 Pareto frontiers for Example 3: (a) Equidistant ϵ-constraint method with inequality constraint; (b) Equidistant ϵ-constraint method and (c) Adaptive bisection ϵ-constraint method.



(a) (b) (c)

Fig. 7 Pareto frontiers for Example 4: (a) Equidistant ϵ-constraint method with inequality constraint; (b) Equidistant ϵ-constraint method and (c) Adaptive bisection ϵ-constraint method.



(a) (b)

Fig. 8 Non-dominated Pareto frontiers: (a) Example 2; (b) Example 3.

in the non-convex part of the Pareto frontier. As such, a Pareto filter similar to the one used with the NNC method [2], needs to be used in conjunction with any of the methods above to remove any sub-optimal solutions. Following Pareto filtering of the Pareto frontiers of examples 2 and 3, which have non-convex regions, the true non-dominated set of solutions are represented in the Pareto frontier as can be seen in Fig. 8.

7. Conclusions

This paper presented the adaptive bisection ϵ-constraint method, which adaptively generates the Pareto frontier for bi-objective optimization problems. It addresses the deficiencies in the other scalarization methods, namely the NC, NNC and NBI methods to generate well-distributed Pareto frontiers for various problems. The distribution of the points is uniform even for vertical (or horizontal) parts in the Pareto frontier, a feature which lacks in the NNC method. An important feature in the algorithm is its robustness when generating Pareto frontiers for ill-conditioned or numerically demanding optimization problems. In such cases, the NBI method fails to generate a complete Pareto frontier, in contrast to the adaptive bisection ϵ-constraint method. The final Pareto set may contain non-optimal solutions if the set problems are non-convex, in which case a simple Pareto filter can be used to remove non-optimal solutions, leaving only the Pareto optimal set.

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