Journal of Mathematics and System Science 4 (2014) 411-420

The Scope of the Structural Completeness in the Class of all Over-Systems of the Classical Functional Calculus with Identity

T. J. Stepień, L. T. Stepień

The Pedagogical University of Cracow, ul. Podchorazych 2, 30 - 084 Krakow, Poland

Received: February25, 2014 / Accepted: March 23, 2014 / Published: June 25, 2014.

Abstract: This paper is the continuation of the paper [13]. Namely, in [13], the scope of the structural completeness in the class of all over-systems of the classical predicate calculus, has been established. In this paper we establish the scope of the structural completeness in the class of all over-systems of the classical functional calculus with identity.

Key words: Structural completeness

1 Preliminaries

Let $\rightarrow$, $\sim$, $\vee$, $\wedge$, $\equiv$ denote the connectives of implication, negation, disjunction, conjunction and equivalence, respectively. Let here and later $i, j, k, n \in \mathcal{N}$, where $\mathcal{N} = \{1, 2, \dots\}$. $X \subseteq Y$ denotes that X is a subset of the set Y. $X \subset Y$ denotes that $X \subseteq Y$ and $Y \neq X$. $\emptyset$ denotes the empty set.

By $At_0 = \{p_1^1, p_1^2, \dots, p_1^k, p_2^k, \dots\}$ we denote the set of all propositional variables. Hence, S_0 is the set of all well-formed formulas, which are built in the usual manner by propositional variables and by means of logical connectives. $\mathfrak{M}_2$ denotes the well-known classical two-valued matrix. Z_2 denotes the set of all formulas valid in the classical two-valued matrix $\mathfrak{M}_2$ (see [6], [9], [15]).

Symbols $x_1, x_2, \dots$ are individual variables. V denotes the set of all individual variables. Symbols P_k^n are n-ary predicate letters. The set of all atomic formulas of the form $P_k^n(x_1, \dots, x_n)$, is denoted by At_1. The symbols $\bigwedge_{x_i}, \bigvee_{x_i}$ are quantifiers. The set S_1 of all well-formed formulas, is constructed in the usual way, by the symbols listed above.

Now we describe the set S_I of all well-formed formulas for a functional calculus with identity. Symbols $x_1, x_2, \dots$ are individual variables. V_I denotes the set of all individual variables. Symbols $a_1, a_2, \dots$ are individual constants. U_I denotes the set of all individual constants. Symbols f_k^n are n-ary functional letters. Now we define the set T_I of all terms. Namely, $U_I \cup V_I \subseteq T_I$ and if $t_1, \dots, t_n \in T_I$, then $f_k^n(t_1, \dots, t_n) \in T_I$. We assume here that P_1^2 denotes the predicate letter of identity. We use also the symbol I, as the predicate letter of identity. Namely, we write sometimes $I(t_k, t_n)$, instead of $P_1^2(t_k, t_n)$

In the next paper, we will establish the scope of the structural completeness in the class of all over-systems of Peano's Arithmetic System.

Corresponding author: L. T. Stepień, The Pedagogical University of Cracow, Kraków, Poland. E-mail: sfstepie@cyf-kr.edu.pl, http://www.ltstepien.up.krakow.pl.

and we write sometimes $I(t_k, t_n)$, instead of $t_k = t_n$. The set of all atomic formulas of the form $P_k^n(t_1, ..., t_n)$, is denoted by At_I. $\wedge_{x_i}, \vee_{x_i}$ are quantifiers. The set S_I of all well-formed formulas, is constructed in the usual manner, by the symbols listed above. $P_r^I(\phi)$ denotes the set of all predicate letters occuring in ϕ, where $\phi \in S_I$. $P_r^I(X)$ denotes the set of all predicate letters occuring in formulas, which belong to the set X, where $X \subseteq S_I$. $U_I(\phi)$ denotes the set of all individual constants occuring in ϕ, where $\phi \in S_I$. $vf_I(\phi)$ denotes the set of all free variables occuring in ϕ, where $\phi \in S_I$. $T_I(\phi)$ denotes the set of all terms occuring in ϕ, where $\phi \in S_I$. $\bar{S}_I = \{\phi \in S_I : vf_I(\phi) = \emptyset\}$. Next $\wedge \alpha = \wedge_{x_1}, \ldots, \wedge_{x_k} \alpha$, if $vf_I(\alpha) = \{x_1, \ldots, x_k\}$ and $\vee \alpha = \vee_{x_1}, \ldots, \vee_{x_k} \alpha$, if $vf_I(\alpha) = \{x_1, \ldots, x_k\}$. Hence, $\wedge \alpha = \alpha$, if $vf_I(\alpha) = \emptyset$ and $\vee \alpha = \alpha$, if $vf_I(\alpha) = \emptyset$. $C_I(\alpha)$ denotes the set of all connectives occuring in α, where $\alpha \in S_I$. $P_r^I(S_I)$ denotes the set of all predicate letters, occuring in S_I.

S_I^* is the set of all well-formed formulas, which are in prenex-conjunctive normal form, where $S_I^* \subseteq S_I$ (see [6] pp. 85 - 97, [2] pp. 186 - 194 and [1] pp. 35 - 42 and pp. 130 - 132). We use $\Rightarrow, \neg, \Leftrightarrow, \&, V, \forall, \exists$ as metalogical symbols. $S^I = \{\phi \in S_I : P_r^I(\phi) = \{I\}\}$, $\bar{S}^I = \{\phi \in S^I : vf_I(\phi) = \emptyset\}$, $S_C^I = \{\phi \in S^I : \sim \notin C_I(\phi)\}$, $\bar{S}_C^I = \{\phi \in S_C^I : vf_I(\phi) = \emptyset\}$. $R_{S_ı}$ denotes the set of all rules over $S_ı (ı \in \{0,1,I\})$ (see [8] p.37). For any $X \subseteq S_ı$ and $R \subseteq R_{S_ı}$, $Cn(R, X)$ is the smallest subset of $S_ı$, containing X and closed under the rules of $R \subseteq R_{S_ı}$ $(ı \in \{0,1,I\})$. Next, the couple $\langle R, X\rangle$ is called a system, whenever $X \subseteq S_ı$ and $R \subseteq R_{S_ı}$ $(ı \in \{0,1,I\})$. By $r_*^ı (ı \in \{0,1\})$ we denote the rule of substitution for propositional and predicate letters, respectively. Namely, $\langle\{\alpha\}, \beta\rangle \in r_*^ı \Leftrightarrow h^e(\alpha) = \beta$, for some endomorphism $h^e : S_ı \longrightarrow S_ı$, which is an extension of the function $e : At_ı \longrightarrow S_ı, e \in \varepsilon_*^ı (ı \in \{0,1\})$. Next, r_*^I denotes the rule of substitution for predicate letters in functional calculus with identity. Namely, $\langle\{\alpha\}, \beta\rangle \in r_*^I \Leftrightarrow h^e(\alpha) = \beta$, for some endomorphism $h^e : S_I \longrightarrow S_I$, which is an extension of the function $e : At_I \longrightarrow S_I, e \in \varepsilon_*^I$. Of course $e(I(t_1, t_2)) = I(t_1, t_2)$ (for details see [5], [7], [8], see also [10], [12]).

Next, here and later, r_0 denotes Modus Ponens for propositional calculus, r_0^1 denotes Modus Ponens for predicate calculus and r_0^I denotes Modus Ponens for functional calculus with identity, respectively.

Next, r_+^1 denotes the generalization rule for predicate calculus and r_+^I denotes the generalization rule for functional calculus with identity. Thus,

$R_0 = \{r_0\}, R_{0+}^1 = \{r_0^1, r_+^1\}$ and $R_{0+}^I = \{r_0^I, r_+^I\}$.

L_2 denotes the set of all formulas valid in the classical predicate calculus and L_2^I denotes the set of all formulas valid in the classical functional calculus with identity.

Now we define the function $ı : S_I \longrightarrow S_0$ as follows:

$(a_1) ı(P_k^n(t_1, ..., t_n) = p_k^n (p_k^n \in At_0)$,

$(a_2) ı(I(t_k, t_n)) = p_1 (p_1 \in At_0)$,

$(a_3) ı(\phi \rightarrow \psi) = ı(\phi) \rightarrow ı(\psi)$,

$(a_4) ı(\phi \vee \psi) = ı(\phi) \vee ı(\psi)$,

$(a_5) ı(\phi \wedge \psi) = ı(\phi) \wedge ı(\psi)$,

$(a_6) ı(\phi \equiv \psi) = ı(\phi) \equiv ı(\psi)$,

$(a_7) ı(\sim\phi) = \sim ı(\phi)$,

$(a_8) ı(\wedge_{x_k} \phi) = ı(\phi)$,

$(a_9)\mathrm{i}(\bigvee_{x_k}\phi) = \mathrm{i}(\phi)$.

Let $\phi \in S_I$ and $\alpha \in At_I$ and let $v: At_0 \longrightarrow |\mathfrak{M}_2|$ be arbitrary, but fixed valuation in the matrix $\mathfrak{M}_2$ such that $h^v(\mathrm{i}(\phi)) = 1$.

Then:

(I) $$e_\phi(\alpha) = \begin{cases} \bigwedge\phi \wedge \alpha, & \text{if } v(\mathrm{i}(\alpha)) = 0 \\ \bigwedge\phi \to \alpha, & \text{if } v(\mathrm{i}(\alpha)) = 1 \\ \alpha, & \text{if } \alpha \in S^I \end{cases}$$

Now we repeat the well-known properties of the operation of consequence. Let $X \subseteq S_I$ and $R \subseteq R_{S_I}$. Thus,

$(c_1) X \subseteq Cn(R, X)$,

$(c_2) X \subseteq Y \Rightarrow Cn(R, X) \subseteq Cn(R, Y)$,

$(c_3)\ R \subseteq R' \Rightarrow Cn(R, X) \subseteq Cn(R', X)$,

$(c_4) Cn(R, Cn(R, X)) \subseteq Cn(R, X)$,

$(c_5) Cn(R, X) = \bigcup\{Cn(R, Y): Y \in Fin(X)\}$,

Where $Y \in Fin(X)$ denotes that Y is the finite subset of X.

Now we repeat the well-known definitions of the permissible rule, the derivable rule and the structural rule (see [5], [8]). Let $X \subseteq S_I$ and $R \subseteq R_{S_I}$. Thus,

$$r \in Perm(R, X) \text{ iff}$$
$$(\forall \Pi \subseteq S_I)(\forall \phi \in S_I)[\langle \Pi, \phi\rangle \in r \,\&\, \Pi \subseteq Cn(R, X) \Rightarrow \phi \in Cn(R, X)]$$

$$r \in Der(R, X) \text{ iff}$$
$$(\forall \Pi \subseteq S_I)(\forall \phi \in S_I)[\langle \Pi, \phi\rangle \in r \Rightarrow \phi \in Cn(R, X \cup \Pi)]$$

$$r \in Struct_{S_I} \text{ iff}$$
$$(\forall \Pi \subseteq S_I)(\forall \phi \in S_I)(\forall e \in \varepsilon_*^I)[\langle \Pi, \phi\rangle \in r \Rightarrow \langle h^e(\Pi), h^e(\phi)\rangle \in r].$$

Next,

$$\langle R, X\rangle \in SCpl_{S_I} \text{ iff}$$
$$Struct_{S_I} \cap Perm(R, X) \subseteq Der(R, X),$$
$$Z_2^I = \{\phi \in S_I: \mathrm{i}(\phi) \in Z_2\},$$
$$\bar{Z}_2^I = \{\phi \in Z_2^I : vf_I(\phi) = \emptyset\}.$$

2 The Well-Known Theorems

It is well-known fact that on the ground of the classical functional calculus with identity, the following theorems are valid (see [2] and [8]):

THEOREM 1. *Let* $\alpha \in \bar{S}_I$ *and* $X \subseteq S_I$. *Then,* $\beta \in Cn(R_{0+}^I, L_2^I \cup X \cup \{\alpha\}) \Rightarrow \alpha \to \beta \in Cn(R_{0+}^I, L_2^I \cup X)$.

THEOREM 2. *Let* $\alpha, \beta, \delta, \phi, \psi \in S_I$ *and* $Q_1, \ldots, Q_n \in \{\bigwedge_{x_1}, \ldots, \bigwedge_{x_i}, \bigvee_{x_{i+1}}, \ldots, \bigvee_{x_n}\}$.

Then the following formulas are valid on the ground of the classical functional calculus with identity:

(II) $\bigwedge\phi \to \phi$

(III) $\sim\sim\phi \equiv \phi$

(IV) $(\phi \to \delta) \to \{(\phi \to \psi) \to [\phi \to (\delta \wedge \psi)]\}$

(V) $\sim(\phi \wedge \psi) \equiv (\phi \to \sim\psi)$

(VI) $Q_1 \ldots Q_k(\phi \to \psi) \equiv (\phi \to Q_1 \ldots Q_k\psi)$, if $x_1, \ldots, x_k \notin vf_I(\phi)$

(VII) $\bigwedge_{x_k}(\phi \to \psi) \equiv (\bigvee_{x_k}\phi \to \psi)$, if $x_k \notin vf_I(\psi)$

(VIII) $\sim \bigvee_{x_k} \sim\phi \equiv \sim\sim \bigwedge_{x_k}(\phi)$

(IX) $\phi \to \bigvee_{x_k}\phi$

(X) $\alpha \to (\beta \to (\alpha \wedge \beta))$

(XI) $\alpha \to (\alpha \vee \beta)$

(XII) $\alpha \to (\beta \vee \alpha)$

(XIII) $(\alpha \to \beta) \to [\alpha \to (\beta \vee \delta)]$

(XIV) $(\alpha \to \delta) \to [\alpha \to (\beta \vee \delta)]$.

THEOREM 3. $Cn(R_{0+}^I \cup \{r_*^I\}, L_2^I) = L_2^I$.

3 The Scope of the Structural Completeness in the Class of all Over-Systems of the Classical Functional Calculus with Identity

Lemma 1. *Let* $X \subseteq S_I$, $Cn(R^I_{0+}, L^I_2 \cup X) = Z_3$ *and*$(\forall\alpha \in \bar{Z}^I_2)[\alpha \in Z_3 \vee \sim\alpha \in Z_3]$. *Then,*

$(\forall\beta_0 \in \bar{S}^I)[\beta_0 \in Z_3 \vee \sim\beta_0 \in Z_3]$.

Proof. At first, we assume that

$$X \subseteq S_I, (1)$$

$$Cn(R^I_{0+}, L^I_2 \cup X) = Z_3, (2)$$

$$(\forall\alpha \in \bar{Z}^I_2)[\alpha \in Z_3 \vee \sim\alpha \in Z_3], (3)$$

$$\beta_0 \in \bar{S}^I. (4)$$

From (2), by the definition of the set L^I_2, it follows that

$$(\forall t_1 \in T_I)[\wedge I(t_1, t_1) \in Z_3]. (5)$$

At first, we consider the following cases:

$(a_1)(\forall v: At_0 \longrightarrow |\mathfrak{M}_2|)[h^v(\mathrm{i}(\beta_0)) = 1]$

or

$(a_2)(\forall v: At_0 \longrightarrow |\mathfrak{M}_2|)[h^v(\mathrm{i}(\beta_0)) = 0]$.

Hence, from (1) - (4), it follows that

$$\beta_0 \in Z_3 \text{ or } \sim\beta_0 \in Z_3. (6)$$

Next, we must consider the following cases:

$(b_1)(\exists v_1: At_0 \longrightarrow |\mathfrak{M}_2|)[h^{v_1}(\mathrm{i}(\beta_0)) = 1]$

and

$(b_2)(\exists v_2: At_0 \longrightarrow |\mathfrak{M}_2|)[h^{v_2}(\mathrm{i}(\beta_0)) = 0]$.

In the cases (b_1) and (b_2), from (4) and (5), it follows that

$$(c_1)(\forall v: At_0 \longrightarrow |\mathfrak{M}_2|)[h^v(\mathrm{i}(\alpha_0)) = 1 \Rightarrow h^v(\mathrm{i}(\beta_0)) = 1]$$

or

$$(c_2)(\exists v: At_0 \longrightarrow |\mathfrak{M}_2|)[h^v(\mathrm{i}(\alpha_0)) = 1 \;\&\; h^v(\mathrm{i}(\beta_0)) = 0],$$

where

$$\alpha_0 = \wedge I(t_1, t_1). (7)$$

Of course, in (c_1), from (1) - (5), one can obtain the following cases:

$(I_1) \sim(\alpha_0 \to \beta_0) \in Z_3$

or

$(I_2) \alpha_0 \to \beta_0 \in Z_3$.

In the case (I_1), from (1), (2), one can obtain that

$$\sim\beta_0 \in Z_3. (8)$$

In the case (I_2), from (1), (2), (5) and (7), it follows that

$$\beta_0 \in Z_3. (9)$$

Now, we consider the case (c_2). Hence, from (4), (5) and (7), one can obtain that

$$(\forall v: At_0 \longrightarrow |\mathfrak{M}_2|)[h^v(\mathrm{i}(\alpha_0)) = 1 \Rightarrow h^v(\mathrm{i}(\sim\beta_0)) = 1]. (10)$$

From (1) - (5) and (10), it follows that we have the following cases:

$(I_3) \sim(\alpha_0 \to \sim\beta_0) \in Z_3$

or

$(I_4)\ \alpha_0 \to \sim\beta_0 \in Z_3$.

In the case (I_3), from (1), (2), it follows that

$$\beta_0 \in Z_3. (11)$$

In the case (I_4), from (1), (2), (5) and (7), it follows that

$$\sim\beta_0 \in Z_3. (12)$$

what completes the proof.

□

Lemma 2. *Let* $\alpha_1 \in S^*_I, X \subseteq S_I, Cn(R^I_{0+}, L^I_2 \cup X) = Z_3, \bar{S}^I_C \subseteq Z_3,\ t_1, t_2 \in T_I$.

Then,

$$\sim\alpha_1 \notin \bar{Z}^I_2 \& \alpha_1 \neq \sim I(t_1, t_2) \Rightarrow h^{e_{\alpha_1}}(\alpha_1) \in Z_3.$$

Proof. Let

$$\alpha_1 \in S^*_I, (1)$$

$$X \subseteq S_I, (2)$$

$$Cn(R^I_{0+}, L^I_2 \cup X) = Z_3, (3)$$

$$\bar{S}^I_C \subseteq Z_3,\ \ t_1, t_2 \in T_I. (4)$$

Suppose that

$$\neg[\sim\alpha_1 \notin \bar{Z}^I_2 \& \alpha_1 \neq \sim I(t_1, t_2) \ \Rightarrow$$

$h^{e_{\alpha 1}}(\alpha_1) \in Z_3]$.(5)

Hence,

$\sim\alpha_1 \notin \bar{Z}_2^I$,(6)

$\alpha_1 \neq \sim I(t_1, t_2)$,(7)

$h^{e_{\alpha 1}}(\alpha_1) \notin Z_3$.(8)

From (6), it follows that

$(\exists v: At_0 \longrightarrow |\mathfrak{M}_2|)\ [h^v(\mathrm{i}(\alpha_1)) = 1]$.(9)

From (1), (7), one can obtain the following cases:

(1.1) $\alpha_1 = I(t_1, t_2)$

or

(1.2) $\alpha_1 = P_k^n(t_1, \dots, t_n)$

or

(1.3) $\alpha_1 = \sim P_k^n(t_1, \dots, t_n)$

or

(1.4) $\alpha_1 = \phi_1 \vee \phi_2$

or

(1.5) $\alpha_1 = \phi_1 \wedge \phi_2$

or

(1.6)$\alpha_1 = Q_1 \dots Q_k \phi_1$,

where

$n, k \in \mathcal{N}$(10)

and

$Q_1, \dots, Q_k \in \{\Lambda_{x_1}, \dots, \Lambda_{x_i}, \mathrm{V}_{x_{i+1}}, \dots, \mathrm{V}_{x_k}\}$.(11)

In (1.1), from (3), (4), (7), (9) and by (I), one can obtain that

$h^{e_{\alpha 1}}(\alpha_1) \in Z_3$.(12)

In (1.2) and in (1.3), from (3) and (9), by (I), it follows that

$h^{e_{\alpha 1}}(\alpha_1) \in Z_3$.(13)

In (1.4) one can assume inductively that

$(a_1) h^{e_{\alpha 1}}(\phi_1) \in Z_3$

or

$(a_2) h^{e_{\alpha 1}}(\phi_2) \in Z_3$.

From the properties of (I) and from **THEOREM 2**, it follows that

$h^{e_{\alpha 1}}(\phi_1 \vee \phi_2) = h^{e_{\alpha 1}}(\phi_1) \vee h^{e_{\alpha 1}}(\phi_2)$.(14)

Hence, from (3) and from (1.4), using **THEOREM 2** (XI) and **THEOREM 2** (XII), in (a_1) and in (a_2), we obtain that

$h^{e_{\alpha 1}}(\alpha_1) \in Z_3$.(15)

In (1.5) one can assume inductively that

$h^{e_{\alpha 1}}(\phi_1) \in Z_3$(16)

and

$h^{e_{\alpha 1}}(\phi_2) \in Z_3$.(17)

From the properties of (I) and from **THEOREM 2**, it follows that

$h^{e_{\alpha 1}}(\phi_1 \wedge \phi_2) = h^{e_{\alpha 1}}(\phi_1) \wedge h^{e_{\alpha 1}}(\phi_2)$.(18)

From (I), (3) and from (1.5), using **THEOREM 2** (X), (16) - (18), we get that

$h^{e_{\alpha 1}}(\alpha_1) \in Z_3$.(19)

In (1.6) one can assume inductively that

$h^{e_{\alpha 1}}(\phi_1) \in Z_3$.(20)

Hence, from (I), (3), (1.6) and using **THEOREM 2** (IX), one can obtain that

$h^{e_{\alpha 1}}(\alpha_1) \in Z_3$.(21)

Thus, in the cases (1.1) - (1.6), it follows that

$h^{e_{\alpha 1}}(\alpha_1) \in Z_3$,(22)

what contradicts (8).

□

Lemma 3. Let $\alpha_1, \beta_1 \in S_I, X \subseteq S_I, Cn(R_{0+}^I, L_2^I \cup X) = Z_3, Z_3 \subset S_I, (\forall \alpha \in \bar{Z}_2^I)[\alpha \in Z_3 \ \mathbb{V} \ \sim\alpha \in Z_3]$ and $(\forall e \in \varepsilon_*^I)[h^e(\alpha_1) \in Z_3 \ \Rightarrow \ h^e(\beta_1) \in Z_3]$.

Then,

$\Lambda\, \alpha_1 \rightarrow \beta_1 \in Z_3$.

Proof. At first we assume that

$\alpha_1, \beta_1 \in S_I$,(1)

$X \subseteq S_I$,(2)

$Cn(R_{0+}^I, L_2^I \cup X) = Z_3$,(3)

$Z_3 \subset S_I$,(4)

$(\forall \alpha \in \bar{Z}_2^I)[\alpha \in Z_3 \ \mathbb{V} \ \sim\alpha \in Z_3]$,(5)

$$(\forall e \in \varepsilon_*^I)[h^e(\alpha_1) \in Z_3 \Rightarrow h^e(\beta_1) \in Z_3].(6)$$

Suppose that

$$\wedge \alpha_1 \to \beta_1 \notin Z_3.(7)$$

From (1) - (7), by the well-known **THEOREM of Replacement** (see [2], pp. 192-194 and [1], pp.128-132), one can obtain

$$\alpha_1^*, \beta_1^* \in S_I^*,(8)$$

$$(\forall e \in \varepsilon_*^I)[h^e(\alpha_1^*) \in Z_3 \Rightarrow h^e(\beta_1^*) \in Z_3],(9)$$

$$\wedge \alpha_1^* \to \beta_1^* \notin Z_3.(10)$$

From (3), (9) and (10), it follows that

$$\wedge \alpha_1^*, \alpha_1^* \notin Z_3.(11)$$

From (3), by the definition of the set L_2^I, one can obtain that

$$(\forall k \in \mathcal{N})(\forall t_k \in T_I)[\wedge I(t_k, t_k) \in Z_3].(12)$$

From (2) - (5), by **Lemma 1**, it follows that

$$(\forall \beta_0 \in \bar{S}^I)[\beta_0 \in Z_3 \ \mathbb{V} \ \sim\beta_0 \in Z_3].(13)$$

Next, we consider the following cases:

(a_1) $(\exists t_1, t_2 \in T_I)[\wedge I(t_1, t_2) \notin Z_3]$

or

(a_2) $(\forall t_1, t_2 \in T_I)[\wedge I(t_1, t_2) \in Z_3]$.

In (a_1), from (13), it follows that

$$(\exists t_1, t_2 \in T_I)[\sim \wedge I(t_1, t_2) \in Z_3].(14)$$

By the definition of the set $\bar{Z}_2^I$, it follows that

$$(\forall t_1, t_2 \in T_I)(\forall t_k \in T_I)(\forall \delta \in \bar{S}_I)$$
$$[\wedge I(t_k, t_k) \to (\sim \wedge I(t_1, t_2) \to \delta) \in \bar{Z}_2^I].(15)$$

From (3), (5), (12), (14) and (15), it follows that

$$(\forall \delta \in \bar{S}_I)[\delta \in Z_3 \ \mathbb{V} \ \sim\delta \in Z_3].(16)$$

Hence, from (3) and (8), it follows that

(b_1) $\wedge\alpha_1^* \to \beta_1^* \in Z_3$

or

(b_2) $\alpha_1^* \in Z_3$.

Of course, (b_1) contradicts (10) and (b_2) contradicts (11).

Thus, the case (a_1) is excluded. In the case (a_2), it follows that

$$(\forall t_1, t_2 \in T_I)[\wedge I(t_1, t_2) \in Z_3].(17)$$

Hence,

$$\bar{S}_C^I \subseteq Z_3.(18)$$

From (3) and (10), it follows that

$$\sim \wedge \alpha_1^* \notin Z_3.(19)$$

Hence, from (3), (5), (9), (10) and (11), it follows that

$$\wedge \alpha_1^* \notin \bar{Z}_2^I(20)$$

and

$$\sim \wedge \alpha_1^* \notin \bar{Z}_2^I.(21)$$

Hence, by the definition of the set $\bar{Z}_2^I$, one can obtain that

$$\sim\alpha_1^* \notin \bar{Z}_2^I.(22)$$

From (22), by the definition of the set $\bar{Z}_2^I$, one can obtain that

$$(\exists v: At_0 \longrightarrow |\mathfrak{M}_2|)[h^v(\mathrm{i}(\alpha_1^*)) = 1].(23)$$

From (I), (2), (3), (8), (18), (22) and (23), by **Lemma 2**, one can obtain that

$$(\forall t_1, t_2 \in T_I)[\alpha_1^* \neq \sim I(t_1, t_2) \Rightarrow$$
$$h^{e_{\alpha_1^*}}(\alpha_1^*) \in Z_3].(24)$$

From (3) and (17), it follows that

$$(\forall \beta_1^* \in S_I^*)(\forall t_1, t_2 \in T_I)[\alpha_1^* = \sim I(t_1, t_2) \Rightarrow$$
$$\wedge \alpha_1^* \to \beta_1^* \in Z_3].(25)$$

Hence, from (3), (8), (10), (17) and (24), it follows that

$$h^{e_{\alpha_1^*}}(\alpha_1^*) \in Z_3.(26)$$

Hence, from (3) and (9), it follows that

$$h^{e_{\alpha_1^*}}(\beta_1^*) \in Z_3.(27)$$

From (3) and (8), it follows that

(1.7)$\beta_1^* = I(t_1, t_2)$

or

(1.8)$\beta_1^* = \sim I(t_1, t_2)$

or

(1.9) $\beta_1^* = P_k^n(t_1, \ldots, t_n)$

or

(1.10) $\beta_1^* = \sim P_k^n(t_1, \dots, t_n)$

or

(1.11) $\beta_1^* = \phi_1 \vee \phi_2$

or

(1.12) $\beta_1^* = \phi_1 \wedge \phi_2$

or

(1.14) $\beta_1^* = Q_1 \dots Q_k\, \phi$,

where $n, k \in \mathcal{N}$ and

$$Q_1, \dots, Q_k \in \{\bigwedge_{x_1}, \dots, \bigwedge_{x_i}, \bigvee_{x_{i+1}}, \dots, \bigvee_{x_k}\}.(28)$$

In (1.7) and (1.8), from (27), by (I), one can obtain that

$$\beta_1^* \in Z_3. (29)$$

Hence, from (3), it follows that

$$\bigwedge \alpha_1^* \to \beta_1^* \in Z_3.(30)$$

In (1.9) and (1.10), from (3) and (27), by (I), one can obtain that

$$\bigwedge \alpha_1^* \to \beta_1^* \in Z_3.(31)$$

In (1.11), from (I), (3) and (27), one can assume inductively that

(I_1) $\bigwedge \alpha_1^* \to \phi_1 \in Z_3$

or

(I_2) $\bigwedge \alpha_1^* \to \phi_2 \in Z_3$.

In (1.11) and (I_1), from (3), by **THEOREM 2** (XIII), one can obtain that

$$\bigwedge \alpha_1^* \to \beta_1^* \in Z_3.(32)$$

In (1.11) and (I_2), from (3), by **THEOREM 2** (XIV), one can obtain that

$$\bigwedge \alpha_1^* \to \beta_1^* \in Z_3.(33)$$

In (1.12), from (3) and (27), by (I), one can assume inductively that

$$\bigwedge \alpha_1^* \to \phi_1 \in Z_3(34)$$

and

$$\bigwedge \alpha_1^* \to \phi_2 \in Z_3.(35)$$

Thus, in (1.12), from (3), (34), (35) and by **THEOREM 2** (IV), it follows that

$$\bigwedge \alpha_1^* \to \beta_1^* \in Z_3.(36)$$

In (1.14), from (3), (27), by (I), one can assume inductively that

$$\bigwedge \alpha_1^* \to \phi \in Z_3.(37)$$

Hence, in (1.14), from (3), (28), **THEOREM 2** (VI) and **THEOREM 2** (IX), it follows that

$$\bigwedge \alpha_1^* \to Q_1 \dots Q_k \phi \in Z_3.(38)$$

Hence, from (1.14), it follows that

$$\bigwedge \alpha_1^* \to \beta_1^* \in Z_3.(39)$$

In consequence, in (1.7)-(1.14), one can obtain the contradiction with (10). This completes the proof. □

Lemma 4. Let $Cn(R_{0+}^I, L_2^I \cup X) = Z_3, Z_3 \subset S_I$.

Then, $\langle R_{0+}^I, L_2^I \cup X\rangle \in SCpl_{S_I} \Rightarrow (\forall \alpha \in \bar{Z}_2^I)[\alpha \in Z_3 \ \mathbb{V} \ \sim\alpha \in Z_3]$.

Proof. Let

$$Cn(R_{0+}^I, L_2^I \cup X) = Z_3,(1)$$

$$Z_3 \subset S_I,(2)$$

$$\langle R_{0+}^I, L_2^I \cup X\rangle \in SCpl_{S_I}.(3)$$

Suppose that

$$(\exists \alpha_1 \in \bar{Z}_2^I)[\alpha_1 \notin Z_3 \& \ \sim\alpha_1 \notin Z_3].(4)$$

Hence, let

$$A_1 = \{\alpha_1, \sim\alpha_1\},(5)$$

where

$$\alpha_1 \in \bar{Z}_2'.(6)$$

At last, suppose that

$$\neg(\forall e \in \varepsilon_*^I)(\forall \alpha_j \in A_1)(\exists \alpha_i \in A_1)\{[h^e(\sim\alpha_i \equiv (\sim\alpha_j \to \sim\alpha_i)) \in Z_3 \Rightarrow h^e(\sim\alpha_j) \in Z_3] \Rightarrow h^e(\sim\alpha_i) \in Z_3\}. \quad (7)$$

From (7) it follows that

$$(\exists e_1 \in \varepsilon_*^I)(\exists \alpha_j \in A_1)(\forall \alpha_i \in A_1)\{[h^{e_1}(\sim\alpha_i \equiv (\sim\alpha_j \to \sim\alpha_i)) \in Z_3 \Rightarrow h^{e_1}(\sim\alpha_j) \in Z_3] \ \& \ h^{e_1}(\sim\alpha_i) \notin Z_3\}. \quad (8)$$

Hence, from (5), it follows that

$(a_1)\alpha_j = \alpha_1$

or

$(a_2)\alpha_j = \sim\alpha_1$.

In the case (a_1), from (5) and (8), it follows that there exists $e_1 \in \varepsilon_*^I$ such that

$$\{h^{e_1}(\sim\alpha_1 \equiv (\sim\alpha_1 \to \sim\alpha_1)) \in Z_3 \Rightarrow h^{e_1}(\sim\alpha_1) \in Z_3\} \& \ h^{e_1}(\sim\alpha_1) \notin Z_3 \quad (9)$$

and

$$\{h^{e_1}(\alpha_1 \equiv (\sim\alpha_1 \to \alpha_1)) \in Z_3 \Rightarrow h^{e_1}(\sim\alpha_1) \in Z_3\} \& \ h^{e_1}(\alpha_1) \notin Z_3 \quad .(10)$$

From (1) and (9), it follows that

$$h^{e_1}(\sim\alpha_1) \notin Z_3 . (11)$$

From (1) and (10), it follows that

$$h^{e_1}(\sim\alpha_1) \in Z_3 \ ,(12)$$

what contradicts (11).

In the case (a_2), from (5) and (8), it follows that there exists $e_2 \in \varepsilon_*^I$ such that

$$\{h^{e_2}(\sim\alpha_1 \equiv (\alpha_1 \to \sim\alpha_1)) \in Z_3 \Rightarrow h^{e_2}(\alpha_1) \in Z_3\} \& \ h^{e_2}(\sim\alpha_1) \notin Z_3 \quad (13)$$

and

$$\{h^{e_2}(\alpha_1 \equiv (\alpha_1 \to \alpha_1)) \in Z_3 \Rightarrow h^{e_2}(\alpha_1) \in Z_3\} \& \ h^{e_2}(\alpha_1) \notin Z_3 \quad .(14)$$

From (1), (13), it follows that

$$h^{e_2}(\alpha_1) \in Z_3 . (15)$$

From (1), (14), it follows that

$$h^{e_2}(\alpha_1) \notin Z_3 \ ,(16)$$

what contradicts (15).

Thus,

$$(\forall e \in \varepsilon_*^I)(\forall \alpha_j \in A_1)(\exists \alpha_i \in A_1) \left\{\left[h^e\left(\sim\alpha_i \equiv (\sim\alpha_j \to \sim\alpha_i)\right) \in Z_3 \Rightarrow h^e(\sim\alpha_j) \in Z_3\right] \Rightarrow h^e(\sim\alpha_i) \in Z_3\right\}. \quad (17)$$

Hence, from (5), we obtain the following cases:

I) $\alpha_i = \alpha_j = \alpha_1$

II) $\alpha_i = \alpha_j = \sim\alpha_1$

III) $\alpha_i \neq \alpha_j \& \alpha_i = \alpha_1$

IV) $\alpha_i \neq \alpha_j \& \alpha_i = \sim\alpha_1$.

In the case I), from (1), (5) and (17), it follows that

$$(\forall e \in \varepsilon_*^I)\{[h^e(\sim\alpha_1 \equiv (\sim\alpha_1 \to \sim\alpha_1)) \in Z_3 \Rightarrow h^e(\sim\alpha_1) \in Z_3] \Rightarrow h^e(\sim\alpha_1) \in Z_3\} \quad .(18)$$

Hence,

$$(\forall e \in \varepsilon_*^I)\ [h^e(\sim\alpha_1) \in Z_3] . (19)$$

From (1) and (19), it follows that

$$\sim\alpha_1 \in Z_3 \ ,(20)$$

what, together with (6) and (17), contradicts (4).

In the case II), from (1), (5) and (17), it follows that

$$(\forall e \in \varepsilon_*^I)\{[h^e(\alpha_1 \equiv (\alpha_1 \to \alpha_1)) \in Z_3 \Rightarrow h^e(\alpha_1) \in Z_3] \Rightarrow h^e(\alpha_1) \in Z_3\} \quad .(21)$$

From (21) one can obtain that

$$(\forall e \in \varepsilon_*^I)\ [h^e(\alpha_1) \in Z_3] . (22)$$

Hence,

$$\alpha_1 \in Z_3 \ .(23)$$

Thus, (23) together with (6) and (17), contradicts (4).

In the case III), from (1), (5) and (17), it follows that

$$(\forall e \in \varepsilon_*^I)\{[h^e(\sim\alpha_1 \equiv (\alpha_1 \to \sim\alpha_1)) \in Z_3 \Rightarrow h^e(\alpha_1) \in Z_3] \Rightarrow h^e(\sim\alpha_1) \in Z_3\}. \quad (24)$$

Hence, from (1), it follows that

$$(\forall e \in \varepsilon_*^I)[\ h^e(\alpha_1) \in Z_3 \Rightarrow h^e(\sim\alpha_1) \in Z_3] . (25)$$

Let

$$r = \{\langle h^e(\alpha_1), h^e(\sim\alpha_1)\rangle : e \in \varepsilon_*^I\} .(26)$$

Hence, from (1) and (25), it follows that

$$r \in Struct_{S_I} \cap Perm(R_{0+}^I, L_2^I \cup X) . \quad (27)$$

Hence, from (1) and (3), it follows that

$$r \in Der(R_{0+}^I, L_2^I \cup X) . \quad (28)$$

From (1), (26) and (28), by **THEOREM 1**, it follows that

$$(\forall e \in \varepsilon_*^I)\,[h^e(\sim\alpha_1) \in Z_3].(29)$$

From (29) it follows that

$$\sim\alpha_1 \in Z_3 ,(30)$$

what, together with (6) and (17), contradicts (4).

In the case IV), from (1), (5) and (17), it follows that

$$(\forall e \in \varepsilon_*^I)\{[h^e(\alpha_1 \equiv (\sim\alpha_1 \to \alpha_1)) \in Z_3 \Rightarrow h^e(\sim\alpha_1) \in Z_3] \Rightarrow h^e(\alpha_1) \in Z_3\} \quad (31)$$

Hence, from (1), it follows that

$$(\forall e \in \varepsilon_*^I)[\, h^e(\sim\alpha_1) \in Z_3 \Rightarrow h^e(\alpha_1) \in Z_3] .(32)$$

Let

$$r = \{\langle h^e(\sim\alpha_1), h^e(\alpha_1)\rangle : e \in \varepsilon_*^I\} .(33)$$

From (1), (32) and (33), it follows that

$$r \in Struct_{S_I} \cap Perm(R_{0+}^I, L_2^I \cup X) .(34)$$

From (1), (3) and (34), it follows that

$$r \in Der(R_{0+}^I, L_2^I \cup X).(35)$$

Hence, from (1) and (33), by **THEOREM 1**, one can obtain that

$$(\forall e \in \varepsilon_*^I)\,[h^e(\alpha_1) \in Z_3].(36)$$

From (36), we obtain that

$$\alpha_1 \in Z_3,(37)$$

what, together with (6) and (17), contradicts (4). This completes the proof.

□

Finally (see also [11] and [14]):

Theorem. *Let* $X \subseteq S_I$ *and* $Cn(R_{0+}^I, L_2^I \cup X) = Z_3$. *Then,*

$$\langle R_{0+}^I, L_2^I \cup X\rangle \in SCpl_{S_I} \Leftrightarrow (\forall \alpha \in \bar{Z}_2^I)[\alpha \in Z_3 \quad \mathbb{V} \quad \sim\alpha \in Z_3].$$

Proof. By **Lemma 3** and **Lemma 4**.

□

Remark. *The notion of the structural rule in propositional calculus was defined in [3] by J. Los and R. Suszko.*

In [4] W. A. Pogorzelski introduced the notion of the structural completeness of the propositional calculus. In [5] W. A. Pogorzelski and T. Prucnal introduced the notion of the structural completeness of the predicate calculus (see also [7] and [8], p. 103).

References

[1] Yu. L. Ershov, E. A. Palyutin, *Mathematical Logic*, Nauka, Moscow; Engl. Transl., Mir Publishers, Moscow 1984.

[2] A. Grzegorczyk, *An Outline Of Mathematical Logic*, Pwn, Warszawa 1981. (In Polish)

[3] J. Los, R. Suszko, "Remark On Sentential Logic", *Proceedings Of The Koninklijke Nederlandse Akademie Van Weten*, Ser A, 61, (1958).

[4] W. A. Pogorzelski, "Structural Completeness Of The Propsitional Calculus", *Bulletin De L'Acadmie Polonaise Des Sciences. Srie Des Sciences Mathmatiques, Astronomiques Et Physiques*, Vol. 19, No. 5, (1971).

[5] W. A. Pogorzelski, T. Prucnal, "Structural Completeness Of The First-Order Predicate Calculus",*Zeitshrift Für Mathematische Logik Und Grundlagen Der Mathematik* Bd 21 (1975).

[6] W. A. Pogorzelski, *Classical Propositional Calculus*, Pwn, Warszawa 1975, (In Polish).

[7] W. A. Pogorzelski, T. Prucnal, "The Subsitution Rule For Predicate Letters In The First-Order Predicate Calculus", *Reports On Mathematical Logic*, No. 5, (1975).

[8] W. A. Pogorzelski, *Classical Calculus Of Quantifiers*, Pwn, Warszawa 1981. (In Polish)

[9] W. A. Pogorzelski, P. Wojtylak, *Elements Of The Theory Of Completeness In Propositional Logic*, Silesian University, Katowice 1982.

[10] W. Rautenberg, *A Concise Introduction To Mathematical*

Logic, Springer 2006.

[11] E. Skvortsova, "On The Scope Of The Structural Completeness In The Functional Calculi With Identity", *Referativnyj Zhurnal*, No. 12 (2005), 54. (In Russian)

[12] S. M. Srivastava, *A Course On Mathematical Logic*. Springer 2008.

[13] T. Stepien, "Derivability", *Reports On Mathematical Logic*, No. 33 (1999), 79.

[14] T. J. Stepien, "On The Scope Of The Structural Completeness In The Functional Calculi With Identity" *International Conference Kolmogorov And Contemporary Mathematics, (Moscow, June 16-21, 2003) In Commemoration Of The Centenial Of Andrei Nikolaevich Kolmogorov (25.Iv.1903 - 20.X.1987). Abstracts*, Russian Academy Of Sciences (Ras), Moscow State University (Msu), 695 (2003).

[15] R. Wójcicki, *Lectures On Propositional Calculi*, Ossolineum 1984.