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An Alternative for Time Series Models

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Two methods for construction of new stochastic processes with discrete time are presented. One of the methods employs as the defining tool "triangular (more specifically 'pseudoaffine') transformations" which are extended from the Euclidean R^n to infinite dimension space. They transform any well-known discrete time stochastic process into the constructed one. The other more flexible method is the "method of parameter dependence", extended to infinite dimension. Properties of the obtained stochastic processes (by either method) indicate the possibility to apply them for financial analysis, as an alternative for the classical time series models. The advantage of the presented models over the existing ones first of all relies on expected better accuracy. This follows from the fact that the typically held assumption on Markovianity in the existing models can easily be relaxed. The defined processes may incorporate a quite long memory including, among others, the k-Markovian cases for $k \geq 2$. Regardless the non-Markovianity of the models they still are tractable in an analytical or numerical way. The stochastic processes defined in this paper provide more flexible and more general tools than the existing time series models for modeling financial problems. Among others, they make it possible to incorporate the influence of environmental (explanatory) random variables on the underlying stochastic models' behavior. These additional features turn out to be describable by the method of parameter dependence. Some suggestions for an associated preliminary statistical analysis are included.

Keywords: stochastic dependence, stochastic processes, alternative for time series financial models, parameter dependence method of construction, k-Markovianity

Introduction

In this work, a pattern for construction of new stochastic models is proposed. The models are a modification of the classical time series frameworks for financial analysis (Tsay, 2005). As such they are considered as a possible alternative to these known ones. They can be obtained by two different methods. One of the methods employs triangular transformations (J. K. Filus, L. Z. Filus, & Arnold, 2010), as the defining tool and may therefore be more useful in a further statistical analysis and possible simulation studies. This method is described in section two and three. The other, described in section four, relies on application of the "parameter dependence method" (J. K. Filus & L. Z. Filus, 2012, 2013), which is more flexible than the first method in the sense that it produces more models. The models obtained by either of the two methods are stochastic processes whose terms have financial meanings especially the meaning of log returns for a single asset.

All the stochastic processes obtained by the triangular transformations method may also be obtained by the parameter dependence (not conversely), but the possibility of a nice statistical and simulation analysis as

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provided by the transformations is sometimes lost. This was the reason why both methods were introduced. Any of the two is very general. The patterns employed allow the author to define wide classes of conditional probability distributions of any term X_t, given realizations $X_1, \ldots, X_{T-1}$ of all past terms $X_1, \ldots, X_{t-1}$ of the defined stochastic processes. Notice that such conditional distributions are very seldom explicitly given in efficient forms in the literature. The classical exception lies within the pattern of the multivariate normal case. The obtained conditional distributions are then used for further construction of joint probability distributions of all the random vectors $(X_1, \ldots, X_t)$, $t = 2, 3, \ldots$ if an initial distribution of X_1 is given.

Perhaps the most amazing fact that follows is the easy possibility of defining non-Markovian (as well as the Markovian) stochastic processes incorporating long pasts and still analytically tractable. Additionally, the method of parameter dependence allows the author to include into the model, typically occurring in practice, "state random variables" that describe a "stochastic environment" in which the processes evolve over time.

The generality of these new models (from a financial perspective) inclined the author rather to concentrate on the formulation of fundamental ideas as beginning to possibly new theories. Therefore, in order to avoid unnecessary dissipation, the number of examples was purposefully limited. Statistical analysis problems of the new stochastic models are only mentioned. Also references are limited, somewhat, especially because the results presented are possibly at first in a financial setting. However, somewhat similar, from a pure mathematical point of view, but generally different results were published in the work by J. K. Filus and L. Z. Filus (2008).

Defining Transformations

Then the author considers a sequence of log returns R_t of a single asset, $t = 0, 1, \ldots, T$ (Tsay, 2005), as given by the following sequence $T = 1, 2, \ldots$ of transformations:

$$\begin{gathered} R_0 = V_0 \\ R_1 = V_1(R_0)X_1 + B_1(R_0) \\ R_2 = V_2(R_0, R_1)X_2 + B_2(R_0, R_1) \\ \vdots \\ R_T = V_T(R_0, R_1, \ldots, R_{T-1})X_T + B_T(R_0, R_1, \ldots, R_{T-1}) \\ T = 1, 2, \ldots \end{gathered} \tag{1}$$

where the random variables $X_1, \ldots, X_T$ are assumed to be independent and identically distributed.

This is then a general white noise pattern which is a source of randomness for the considered log returns $R_1, \ldots, R_T, \ldots$.

V_0 represents a nonnegative constant initial value, while the functions $V_1, \ldots, V_T$ are arbitrary positive and piecewise continuous with respect to each argument. If the variance of each random variable X_t is 1, then $V_1, \ldots, V_T$ will have the "conditional volatilities" interpretation conditioned on realizations of past returns R_0, $R_1, \ldots$ prior to a given R_t. Also conditioned on the same realizations of the past returns are the conditional expectations

$$E[R_t \mid R_0, R_1, \ldots, R_{t-1}] = B_t(R_0, R_1, \ldots, R_{t-1})$$

where $B_1, \ldots, B_T$ are arbitrary, piecewise continuous with respect to each argument, functions of realizations of past returns.

Example

The functions $V_t(\ )$ and $B_t(\ )$ ($t = 1, \dots, T$) may be arbitrary continuous. However, in practical applications one could choose, for example, the following simple functions:

$$V_t(r_0, r_1, \dots, r_{t-1}) = 1 + a_0 r_0^2 + a_1 r_1^2 + \dots + a_{t-1} r_{t-1}^2$$

$$B_t(r_0, r_1, \dots, r_{t-1}) = b_0 r_0 + b_1 r_1 + \dots + b_{t-1} r_{t-1}$$

where the coefficients $a_0, a_1, \dots, a_{t-1}$ are real nonnegative and $b_0, b_1, \dots, b_{t-1}$ are arbitrary real. These coefficients are to be statistically estimated. Also, if appropriate, one can choose as model:

$$V_t(r_0, r_1, \dots, r_{t-1}) = \exp[a_0 r_0^2 + a_1 r_1^2 + \dots + a_{t-1} r_{t-1}^2]$$

and

$$B_t(r_0, r_1, \dots, r_{t-1}) = \exp[b_0 r_0 + b_1 r_1 + \dots + b_{t-1} r_{t-1}]$$

with arbitrary real coefficients $a_0, a_1, \dots, a_{t-1}$ and $b_0, b_1, \dots, b_{t-1}$. Other examples of such functions can easily be given.

Returning to the main subject, it is noticed that the sequence of the random vector transformations $(X_1, \dots, X_T) \rightarrow (R_1, \dots, R_T)$ ($T = 1, 2, \dots$) defined by equation (1), is the pseudoaffine version of sequence of triangular transformations $R^T \rightarrow R^T$ (Filus et al., 2010).

Here it is proposed to apply them as a general financial model for values of log returns. This model can be seen as a slightly different version of time series and is proposed to be named "triangular model". It is realized that all the transformations (1) are easily invertible and their inverses are given as follows:

$$
\begin{aligned}
R_0 &= V_0\\
X_1 &= [R_1 - B_1(R_0)] / V_1(R_0)\\
X_2 &= [R_2 - B_2(R_0, R_1)] / V_2(R_0, R_1)\\
&\vdots\\
X_T &= [R_T - B_T(R_0, R_1, \dots, R_{T-1})] / V_T(R_0, R_1, \dots, R_{T-1})\\
T &= 1, 2, \dots
\end{aligned}
\tag{1*}
$$

For realizations $x_1, \dots, x_T$ and $r_0, r_1, \dots, r_T$ of the underlying random variables, denoted by the corresponding capital letters, the jacobians, $J_T(r_1, \dots, r_T) = \partial(x_1, \dots, x_T) / \partial(r_1, \dots, r_T)$, have the simple form of the inverse of the volatilities' products

$$J_T(r_0, r_1, \dots, r_{T-1}) = [V_1(r_0)\ V_2(r_0, r_1)\ \dots\ V_T(r_0, r_1, \dots, r_{T-1})]^{-1} \tag{2}$$

for each $T = 1, 2, \dots$

One can see that if the sequence of probability densities (pdf) of the random vectors $(X_1, \dots, X_T)$ is known (which is mostly the case), then from equations (1^*) and (2) one immediately can derive the corresponding sequence of joint pdfs of the random vectors of the returns $(R_1, \dots, R_T)$, $T = 1, 2, \dots$.

In such a way, one defines a wide class of stochastic processes $\{R_T\}_T = 1, 2, \dots$ (The Kolmogorov consistency theorem easily applies to this case). The author considers these processes as "modified time series" processes for log returns $R_1, R_2, \dots$. Clearly, the model given by equation (1) is heteroscedastic as the underlying conditional volatilities, $V_t(r_0, r_1, \dots, r_{t-1})$, $t = 1, 2, \dots$ (conditioned on elementary events $R_0 = V_0 = r_0$, $R_1 = r_1, \dots, R_{t-1} = r_{t-1}$), are, in general, distinct.

It follows from equation (1) that the introduced model is, in general, not Markovian but still analytically tractable.

Actually, when using model (1), one can incorporate in each conditional pdf $g_T(r_T \mid r_1, \ldots, r_{T-1})$, at present time T, all the past information on the returns and underlying calculations is still performable. However, this computational advantage is overshadowed by limitations of a statistical nature. As T grows, the number of parameters to be estimated also grows without bounds, so some restrictions on the past must be provided. For that one can apply the notion of k-Markovianity that limits the past to the last k observations ($k = 1, 2, \ldots$). The case $k = 1$ means the ordinary Markovianity. The general k-Markovian version of model (1) can be defined as the following sequence of transformations:

$$
\begin{aligned}
R_0 &= V_0 \\
R_1 &= V_1(R_0)X_1 + B_1(R_0) \\
R_2 &= V_2(R_0, R_1)X_2 + B_2(R_0, R_1) \\
&\vdots \\
R_j &= V_j(R_0, R_1, \ldots, R_{j-1})X_j + B_j(R_0, R_1, \ldots, R_{j-1}) \text{ if } j - 1 \le k \\
&\vdots \\
R_t &= V_t(R_{t-k}, \ldots, R_{t-1})X_t + B_t(R_{t-k}, \ldots, R_{t-1}) \text{ if } t - 1 \ge k \\
&\vdots \\
R_T &= V_T(R_{T-k}, \ldots, R_{T-1})X_T + B_T(R_{T-k}, \ldots, R_{T-1}) \\
& k = 1, 2, \ldots, T = 1, 2, \ldots, k < T
\end{aligned} \tag{3}
$$

The k-Markovian conditional pdfs of $R_t \mid R_0, R_1, \ldots, R_{t-1}$ as derived from equation (3) are given by:

$$g_t(r_t \mid r_1, \ldots, r_{t-1}) \text{ if } t - 1 \le k \text{ and } g_t(r_t \mid r_{t-k}, \ldots, r_{t-1}) \text{ if } t - 1 \ge k.$$

Thus, in this setting, the (conditional) distribution of the present asset log return R_T only depends on the last k moments (months, years) in the past. The earlier times are considered irrelevant and are neglected. Nevertheless, even in the case $k = 2$ (bi-Markovian), the amount of information incorporated in the stochastic model is significantly bigger than that in the Markovian case, so one may expect more accurate predictions.

Examples

The following examples are based on equations (1) and (3).

Example 1

It is assumed that, for each T, the random variables $X_1, \ldots, X_T$ are independent, each having the standard normal $N(0, 1)$ pdf. Using standard calculations based on the knowledge of equations (1^*) and (2), one first obtains the (unconditional) normal pdf $g_1(r_1) = N[B_1(R_0), V_1(R_0)]$ for R_1 and then for each $t = 2, 3, \ldots, T$, one obtains the conditional pdf:

$$g_t(r_t \mid r_1, \ldots, r_{t-1}) =$$
$$[V_t(r_0, r_1, \ldots, r_{t-1})\sqrt{2\pi}]^{-1}\exp[-(1/2)\{(r_t - B_T(r_0, r_1, \ldots, r_{t-1})) / V_t(r_0, r_1, \ldots, r_{t-1})\}^2] \tag{4}$$

It is realized that the latter conditional pdf is normal with respect to the single variable r_t. The joint probability density $g_T(r_1, \ldots, r_T)$ for each random vector $(R_1, \ldots, R_T)$, $T = 2, 3, \ldots$ is given by the common formula:

$$g_T(r_1, \ldots, r_T) = g_1(r_1) \Pi_{t=1}^{T} g_t(r_t \mid r_1, \ldots, r_{t-1}), \tag{5}$$

where $g_t(r_t \mid r_1, \ldots, r_{t-1})$ is given by equation (4). The so obtained T-dimensional pdf is the FF-normal (former name "pseudonormal") (Kotz, Balakrishnan, & Johnson, 2000).

Example 2

Considering the following "pseudolinear" part of the pseudoaffine transformation (1) which one obtains by setting in equation (1) all the "pseudotranslation" coefficients $B_t(R_0, R_1, \ldots, R_{t-1})$ to zero, one then has the pseudolinear transformations:

$$\begin{gathered} R_0 = V_0 \\ R_1 = V_1(R_0)X_1 \\ R_2 = V_2(R_0, R_1)X_2 \\ \begin{matrix} . & . & . \\ . & . & . \\ . & . & . \end{matrix} \\ R_T = V_T(R_0, R_1, \ldots, R_{T-1})X_T \\ T = 1, 2, \ldots \end{gathered} \tag{6}$$

It is investigated how the transformations (6) act on set of independent Pareto distributed random variables $X_t(t = 1, 2, \ldots, T;\ T = 1, 2, \ldots)$ so, in this case, the expected values of X_t' s are positive. Recall that the Pareto density is given by

$$f_t(x_t) = 1 / \beta (1 + x_t / \beta\gamma)^{1+\gamma} \tag{7}$$

where β and γ are positive real parameters. Using equation (6), for every $t = 1, \ldots, T$, expresses x_t as $x_t = r_t / V_{t-1}(r_0, r_1, \ldots, r_{t-1})$ (assuming $V_{t-1}(r_0, r_1, \ldots, r_{t-1}) \neq 0$).

Also it is realized that the jacobian of inverse to equation (6) equals to the inverse product:

$$J_T(r_0, r_1, \ldots, r_{T-1}) = [V_1(r_0)\, V_2(r_0, r_1) \ldots V_T(r_0, r_1, \ldots, r_{T-1})]^{-1}$$

As the next step, one obtains (for each $t = 1, 2, \ldots, T$) the conditional pdfs $g_t(r_t \mid r_0, r_1, \ldots, r_{t-1})$ of each *rv* R_t, given the past realizations $r_0, r_1, \ldots, r_{t-1}$ of the *rvs* $R_0, R_1, \ldots, R_{t-1}$, as follows:

$$\begin{aligned} g_t(r_t \mid r_0, r_1, \ldots, r_{t-1}) &= f(x_t) \mid \partial x_t / \partial r_t \mid = f(r_t / V_{t-1}(r_1, r_2, \ldots, r_{t-1})) \mid V_{t-1}(r_1, r_2, \ldots, r_{t-1}) \mid^{-1} \\ &= 1 / \{ \beta \mid V_{t-1}(r_1, r_2, \ldots, r_{t-1})\mid [1 + r_t / \beta \mid V_{t-1}(r_1, r_2, \ldots, r_{t-1}) \mid \gamma]^{1+\gamma} \} \end{aligned} \tag{8}$$

So, the effect of each $t^{\text{-th}}$ line in transformation (6) on the rv X_t is to change its Pareto density (7) for the (conditional) Pareto density (8) of R_t. The two Pareto densities (7) and (8) only differ by the scale parameters, namely: β in (7) was transformed into the product $\beta \mid V_{t-1}(r_1, r_2, \ldots, r_{t-1}) \mid$ in equation (8).

Given the conditional densities (8), one obtains the joint density of each random vector $(R_0, R_1, \ldots, R_T)$, $T = 1, 2, \ldots$ using formula (5). In such a way, the "Pareto stochastic process" $\{R_T\}_T = 1, 2, \ldots$ is well defined.

Example 3

In the same way as for the independently Pareto distributed random variables $X_1, \ldots, X_T$ $(T = 1, 2, \ldots)$, one can apply transformation (6) to any sequence of independent identically, and exponentially distributed random variables that will be denoted by the same symbols X_t's. If, for any $t = 1, 2, \ldots, g_t(x_t)$ is the exponential density of X_t given by the expression $(1/\theta) \exp[-x_t / \theta]$ then it can easily be verified that the corresponding conditional density of $R_t \mid R_0, R_1, \ldots, R_{t-1}$ will be given as follows:

$$h_t(r_t \mid r_0, r_1, \ldots, r_{t-1}) = (1 / \theta \mid V_{t-1}(r_0, r_1, \ldots, r_{t-1}) \mid)\exp[- r_t / \theta \mid V_{t-1}(r_0, r_1, \ldots, r_{t-1}) \mid]$$

It is then clear that as in Example 2, the parameter θ is multiplied by the "coefficient" $\mid V_{t-1}(r_0, r_1, \ldots, r_{t-1}) \mid$.

The same actually will happen with the parameter σ in Example 1, if one would assume that all the random variables X_t in (1) are normal $N(0, \sigma)$. Also in this case, the parameter σ will be turned to the conditional volatility of R_t:

$$\sigma \mid V_{t-1}(r_0, r_1, \ldots, r_{t-1}) \mid$$

This regularity for the parameter transformations will be applied in the next section.

Parameters Dependence Models

In all three examples in the previous section, transformation (1) or (6) was used in order to obtain the conditional densities, say, $\phi_t(r_t \mid r_0, r_1, \ldots, r_{t-1})$ describing the stochastic dependence of the return R_t on the past.

It is realized that in this derivation, the underlying operations only result in changing the value of a parameter of the given density of X_t, into other value that depends on the past return values $r_0, r_1, \ldots, r_{t-1}$. This observation opens the way for the method of conditioning (on values $r_0, r_1, \ldots, r_{t-1}$), which is significantly more efficient than the method of triangular transformations (1) or (6). This method, called the "parameter dependence", is presented in J. K. Filus and L. Z. Filus (2012, 2013). In the considered framework, one can describe this method as follows.

It is supposed that there is given a sequence of independent random variables (now, instead of X_t, denoted by $R_{f\,t}$ $t = 1, 2, \ldots$) all having the same arbitrary probability density $f_t(r_t\,;\,\alpha)$, $\alpha \in A$. In this situation, any past in this artificial "no memory process" has no influence on the current density $f_t(r_t\,;\,\alpha)$ of $R_{f\,t}$. The density depends on a constant (original) scalar or vector parameter α. Instead of applying transformation (1) or (6) to the random vectors $(R_{f1}, \ldots, R_{fT})$, one can "directly transform" each density $f_t(r_t\,;\,\alpha)$ into a conditional density $\phi_t(r_t \mid r_0, r_1, \ldots, r_{t-1})$ of $R_t \mid r_0, r_1, \ldots, r_{t-1}$ just by setting the parameter α of $f_t(r_t\,;\,\alpha)$ to "become" a function of the values $r_0, r_1, \ldots, r_{t-1}$. In such a way, one defines the sequence of conditional pdfs by the formula:

$$\phi_t(r_t \mid r_0, r_1, \ldots, r_{t-1}) = f_t(r_t\,;\,\alpha_t(r_0, r_1, \ldots, r_{t-1})),\ t = 1, 2, \ldots \tag{9}$$

which, for an arbitrary function $\alpha_t(r_0, r_1, \ldots, r_{t-1})$, defines a legitimate density with respect to r_t if all the values $\alpha_t(r_0, r_1, \ldots, r_{t-1})$ still belong to the set A of the parameters α of $f_t(r_t\,;\,\alpha)$.

Each sequence of the so obtained conditional densities $\{\phi_t(r_t \mid r_0, r_1, \ldots, r_{t-1})\}_t = 1, 2, \ldots$ defines a corresponding stochastic process $\{R_t\}_t = 1, 2, \ldots$

The parameter dependence method allows for relatively free choice for the functions $\alpha_t(r_0, r_1, \ldots, r_{t-1})$ and therefore the class of the so obtained stochastic processes is much wider than that obtained by the triangular transformation from the same sequence of independent random variables X_t or R_{ft}. On the other hand, the factor that, in applications, often may limit the range of choices of the functions $\alpha_t(r_0, r_1, \ldots, r_{t-1})$ is reality.

Every "educative guess" for such a function must be statistically verified. So, first of all, the chosen function itself usually has its own parameters (parametric approach) that must be estimated by any statistical method such as, the maximum likelihood method. Then the properly arranged parametric hypothesis should be verified. Finally, the choice of the best fitting to data function $\alpha_t(r_0, r_1, \ldots, r_{t-1})$ should be based on statistical methods as to be the best one from several candidates (the choices made in the beginning). This then should be declared as the final model.

It is common that the general stochastic model for log returns of a given single asset from a portfolio is a joint probability distribution (Tsay, 2005).

$$P(R_1 < r_1, \ldots, R_T < r_T \mid Y_1, \ldots, Y_k) = G_T(r_1, \ldots, r_T; Y_1, \ldots, Y_k)$$
$$= G_1(r_1; Y_1, \ldots, Y_k)\ \Pi_{t=2}^{T}\ G_t(r_t \mid r_1, \ldots, r_{t-1}\ ; Y_1, \ldots, Y_k) \qquad (10)$$

where $G_1(r_1; Y_1, \ldots, Y_k)$ is the cdf of the random variable R_1 and, for $t = 2, 3, \ldots, T$, $G_t(r_t \mid r_1, \ldots, r_{t-1}; Y_1, \ldots, Y_k)$ is the conditional distribution function of R_t, given realizations $r_1, \ldots, r_{t-1}$ of the random variables $R_1, \ldots, R_{t-1}$.

However, the above joint and conditional distributions also depend on the state random variables $Y_1, \ldots, Y_k$ that summarize the "environment" in which asset return is determined (Tsay, 2005, p. 13).

One can apply the parameter dependence method to define the conditional distribution functions

$$P(R_1 < r_1, \ldots, R_T < r_T \mid y_1, \ldots, y_k)$$

where:

$y_1, \ldots, y_k$ are (measured) realizations of the states $Y_1, \ldots, Y_k$.

For that it is enough to set parameter $\alpha_t(r_0, r_1, \ldots, r_{t-1})$ (which already determines the conditional distribution $G_t(r_t \mid r_1, \ldots, r_{t-1}))$ to be additionally dependent on the values $y_1, \ldots, y_k$. Thus, for a given t, the conditional distribution of $R_t \mid r_1, \ldots, r_{t-1}; y_1, \ldots, y_k$ will be determined by a parameter(s) α_t of R_t's distribution as follows:

$$G_t(r_t \mid r_1, \ldots, r_{t-1}\ ; y_1, \ldots, y_k) = G_t(r_t\ ; \alpha_t\ (r_1, \ldots, r_{t-1}\ ; y_1, \ldots, y_k)) \qquad (11)$$

If the values (realizations) $y_1, \ldots, y_k$ are measured then the joint distribution (10) is already determined. If not, one needs to have joint probability density $f(y_1, \ldots, y_k)$ of the random vector $(Y_1, \ldots, Y_k)$. It seems that often one may assume stochastic independence of the components $Y_1, \ldots, Y_k$ of this vector.

Finally, as typically, it may be needed to multiply the resulting G_T's distribution (10) conditioned on $y_1, \ldots, y_k$ by the density $f(y_1, \ldots, y_k)$. As an example of the parameter function $\alpha_t\ (r_1, \ldots, r_{t-1}; y_1, \ldots, y_k)$, one may consider the following:

$$\alpha_t\ (r_1, \ldots, r_{t-1}; y_1, \ldots, y_k) = \alpha\ (1 + a_1 r_1^{\,2} + \ldots + a_{t-1} r_{t-1}^{\,2})\exp[b_1 y_1 + \ldots + b_k y_k]$$

where α is the constant original parameter of the density $f_t(r_t\ ; \alpha)$ of the random variable $R_{f\,t}$.

$\{R_{f\,t}\}_t = 1, 2, \ldots$ is the original stochastic process with the independent terms. Furthermore, $a_1, \ldots, a_{t-1}$ and $b_1, \ldots, b_k$ are real coefficients. Obviously, when all the coefficients $b_1, \ldots, b_k$ are small enough then the impact of the states $y_1, \ldots, y_k$ on the parameter (so on the conditional distribution) is insignificant.

According to the knowledge, the above application of the parameter dependence method to incorporate the random states $Y_1, \ldots, Y_k$ impact on the returns' distributions is not yet present in literature.

Final Remark

The core achievement when employing either of the two methods, is opening the way for easy constructions of the conditional probability distributions of $X_t \mid X_1, \ldots, X_{t-1}$, given in compact analytical forms ready for the calculations. Underlying calculations can be analytical or, if necessary, relatively simple numerical. Having the conditional distribution functions (11) in analytical forms allows for extending many classical regression models**,** usually being in the form of conditional expectation, say,

$$E[R_t \mid r_1, \ldots, r_{t-1}\ ; y_1, \ldots, y_k]$$

by replacing them with the full probability distribution (11).

It is noticed that the latter regression is the expected value of (11), so it is only part of the wider model considered here. In what is called "enforced regression" (J. K. Filus & L. Z. Filus, 2014), the numerical characteristics like conditional expectations or covariance coefficients can be replaced by richer functional characteristics such as the conditional distributions or joint probability distributions respectively. This idea is, apparently, different from that (nonparametric) considered by Koenker and Bassett (1978), and followers, for a wider discussion of this subject (J. K. Filus & L. Z. Filus, 2014).

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