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# Substitution Marginal Rate and Its Usage in the Marginal Preference Calculation

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**Abstract:** The slope of indifference curve is known as a marginal rate of substitution (MRS). MRS defining ratio always describes slope of indifferent curve. i.e. MRS matches the module of indifferent curves slope. Utility function $u(x_1, x_2)$ is used to calculate marginal rate of substitution (MRS), because MRS gives the slope of appropriate indifference curve, it can be interpreted as a norm, in which costumer is ready to substitute good 1 by small amount of good 2. The word "marginal" in economic means "differential". Here we have partial differentiation, because in time of calculation of good 1`s marginal utility the amount of good 2 remains the same. We can calculate MRS in two ways using differential and function. In the first case consider change ( $dx_1, dx_2$ ) during which utility is unchanged. For the second method let the curve of indifference present by $x_2(x_1)$ function. The function shows how many of $x_2$ is needed for each unit of $x_1$ to stay on this concrete curve of indifference. We obtain two equations for the term of MRS and budget constraint and two $x_1$ and $x_2$ variables. To define the optimal choice of $x_1$ and $x_2$ as a function of the price and income, we need to solve those two equations. The problem of maximization can be solved by using differential.

**Key words:** Marginal rate of substitution, problem of maximization, indifference curve

## 1. Introduction

Economic is developing with construction of models of social phenomena that are the simplifications of reality. Solving the given task economists are using two principles: (i) the principles of optimization, according to which consumers are trying to make the best decision for themselves, (ii) and principle of equilibrium, according to which the price adjustment regulations are continued until the demand and supply are equal.

The economic model of consumer behavior is simple [1]: consumers always choose the best available product. Let consider three axiom of user preference:

Axiom of completeness- Assume that it is possible to compare any two baskets.

Reflexivity- Any basket is as good as the basket identical to it.

Transitivity – if we want to have theory, where the consumer always chooses "the best", the preferences should satisfy the axiom of transitivity. If preferences were not transitive it would be impossible to choose "the best" from the array of baskets. From the experience, consumer choice full theory can be formulated based on the three axioms mentioned above and based on the technical assumption that it is convenient to display preferences graphically using constructions known as the indifference curves.

Often it is convenient to mark the slope angle on each point of indifference curve. The slope of indifference curve is known as Marginal Rate of Substitution (*MRS*). This name is derived from the fact, that this is a value, with what consumer wishes to substitute one merchandise with another.

Suppose we reduce consumer's merchandise-1 by $\Delta x_1$ and after that increased by $\Delta x_2$, this allows consumer to return on its indifference curve, and bring him/her to the same satisfaction as before. The ratio

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$\Delta x_2 / \Delta x_1$ is called the norm, with what consumer is ready to substitute merchandise-1 by merchandise-2. Let $\Delta x_1$ be a very small slow varying marginal change. Then the ratio $\Delta x_2 / \Delta x_1$ gives marginal rate of the substitution of merchandise-1 with merchendise-2. With decreasing v the ratio $\Delta x_2 / \Delta x_1$ reaches the angle of indifference curve, as shown in Fig.1. The numerator and denominator are assumed to be small numbers in this ratio and describe marginal change of initial consumer basket. Thus, the ratio defined by MRS always describes the slope of the indifference curve. To be more precise

$$MRS = \lim \left|\frac{\Delta x_2}{\Delta x_1}\right| = \left|\frac{dx_2}{dx_1}\right|$$

I.e. MRS is the module of the slope of the indifference curve.

First, let's be clear on what "marginal utility" means. Take the consumer that consumes some basket of merchandise ($x_1, x_2$). We ask a question: by how much the utility of this consumer changes, if we provide him/her more merchandise than merchandise-1? The value of this change is called marginal utility of merchandise-1, and is called as $MU_1$ given by following ratio:

$$MU_1 = \frac{\Delta U}{\Delta x_1} = \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1}$$

Which gives us the change of utility value ($\Delta U$) that corresponds to a small change of merchandise-1($\Delta x_1$). Note that, in this calculation the value of merchandise-2 remains unchanged.

The utility function $u(x_1, x_2)$ can be used to calculate MRS that provides the slope of the indifference curve for given basket. Thus it can be interpreted as a norm, by which consumer is ready to replace merchandise-1 by small amount of merchandise-2.

Such interpretation makes it possible to easily calculate the MRS. Consider the change in the consumption of the both merchandise, while the benefits remain unchanged, i.e. change of consumption moves along the indifference curve, resulting in:

$$MU_1 \Delta x_1 + MU_2 \Delta x_2 = \Delta U = 0$$

for the slope of the indifference curve we get:

$$MRS = \frac{\Delta x_2}{\Delta x_1} = -\frac{MU_1}{MU_2} \qquad (1)$$

The MRS has negative sign, because increasing consumption of merchandise-1 reduces consumption ofmerchandise-2, but because economists are often using absolute value of MRS, we are going to assume it as positive.

The word "marginal" often means "differentiate" so, marginal utility of the merchandise-1 can be written in the following form:

$$MU_1 = \lim_{\Delta x_1 \to 0} \frac{u(x_1 + \Delta x_1 \cdot x_2) - u(x_1, x_2)}{\Delta x_1} = \frac{\partial u(x_1, x_2)}{\partial x_1}$$

Here we used partial differentiating because while calculating marginal utility of merchandise-1 the amount of merchandise-2 remains unchanged.

Now we can calculate MRS again, that can be done in two ways – using differential or using functions. In first case consider ($dx_1, dx_2$) change, while utility remains unchanged, thus following equation must be fulfilled:

$$dU = \frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial u(x_1, x_2)}{\partial x_2} dx_2 = 0$$

The first term in the equation is related to the growth of utility corresponding to a small change $dx_1$ and the second term - to the growth of utility corresponding to change $dx_2$. The sum of these two terms is equal to zero. From here for the ratio $dx_2 / dx_1$, it follows:

$$\frac{dx_2}{dx_1} = -\frac{\partial u(x_1, x_2) / \partial x_1}{\partial u(x_1, x_2) / \partial x_2},$$

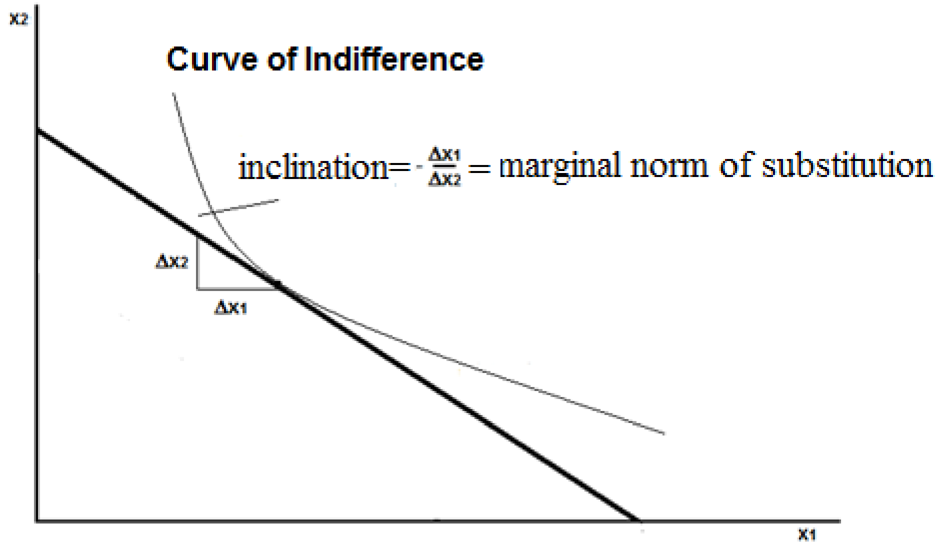


**Fig.1 Marginal rate of substitution (MRS). Marginal norm of substitution gives us the slope of the indifference curve.**

it is an analogue to the equation (1).

For the second method let's present the indifference curve with the function $x_2(x_1)$. This function how many $x_2$ points are needed for any amount of $x_1$ for staying on the specific curve of indifference. Thus the $x_2(x_1)$ function should satisfy specific equation:

$$u(x_1, x_2(x_1)) \equiv k .$$

Where - $k$ is a utility size (number) of given utility curve. If we differentiate both sides of this equation with respect of $x_1$, we will obtain:

$$\frac{\partial u(x_1,x_2)}{\partial x_1} + \frac{\partial u(x_1,x_2)}{\partial x_2}\frac{\partial x_2(x_1)}{\partial x_1} = 0$$

From the latest we can obtain the ratio $\partial x_2(x_1)/\partial x_1$:

$$\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\partial u(x_1,x_2)/\partial x_1}{\partial u(x_1,x_2)/\partial x_2}$$

What we already have above.

It is very important to be able to solve problem of the preferences maximization and get the algebraic examples of the demand function. It should be noted, that optimal choice ( $x_1, x_2$ ) should satisfy the following term:

$$MRS(x_1,x_2) = -\frac{p_1}{p_{2,}} \tag{2}$$

Where, $p_1$ and $p_2$ are the prices of the merchandises $x_1$ and $x_2$ correspondingly. After required operations we obtain:

$$\frac{\partial u(x_1,x_2)/\partial x_1}{\partial u(x_1,x_2)/\partial x_2} = \frac{p_1}{p_2} \tag{3}$$

It is also known, that the optimal choice should also satisfy budget constraints:

$$p_1x_1 + p_2x_2 = m \tag{4}$$

In the result we obtained two equations – for the MRS and for the budget constraint – and two unknowns $x_1$ and $x_2$ . These two equations must be solved in order to figure out the optimum choice of $x_1$ and $x_2$ as a function of price and income. There are many ways to solve this system of equation with two unknowns. The one method, which can always be used but is not the easiest, is to express one unknown from the budget constraint and plug it in MRS equation.

The budget constraint can be written in the following way:

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2} x_1 \qquad (5)$$

If we plug value for $x_2$ in the equation (3), we obtain:

$$\frac{\partial u(x_1, m/p_2 - (p_1/p_2)x_1)/\partial x_1}{\partial u(x_1, m/p_2 - (p_1/p_2)x_1)/\partial x_2} = \frac{p_1}{p_2}$$

In this equation we have one unknown $x_1$, so we can express it with known ( $p_1, p_2, m$ ) variables, and then we can do the same for $x_2$ variable.

It also possible to solve the utility maximization problem [2] by using differentials, for that first we should present the utility problem as a limited maximization problem

$$\max_{x_1 \to x_2} u(x_1, x_2)$$

so that

$$p_1 x_1 + p_2 x_2 = m$$

This problem requires the variables $x_1$ and $x_2$ to be selected in such manner that they, first satisfy budget constraint and second produce different values $x_1$ and $x_2$ so that they satisfy constraint leading to highest meaning of function $u(x_1, x_2)$ . For this we should solve the budget constraint for one variable and then plug it in the utility function For instance, if we know the value of $x_1$, then $x_2$ that satisfies the budget constraint, can be expressed by the following linear function:

$$x_2(x_1) = \frac{m}{p_1} - \frac{p_1}{p_2} x_1 \qquad (6)$$

In order to obtain unlimited maximization, we should plug the calculated value in utility function instead of $x_2$

$$\max_{x_1} u(x_1, \frac{m}{p_2} - \left(\frac{p_1}{p_2}\right) x_1)$$

It is an unlimited maximization only for $x_1$, because we use $x_2(x_1)$ function and for any value of $x_1$, the $x_2$ always satisfy the budget constraint. This problem can be solved by differentiating with respect to $x_1$ variable and then equalizing it to zero. We obtain:

$$\frac{\partial u(x_1, x_2(x_1))}{\partial x_1} + \frac{\partial u(x_1, x_2(x_1))}{\partial x_2} \frac{\partial x_2}{\partial x_1} = 0 \quad (7)$$

The first term in this equation shows how the increasing of the $x_1$ value increases utility .The second term in the equation consists of two parts: first one is a utility growth value $\partial u / \partial x_2$ that is increasing with $x_2$, the second one is $\partial x_2 / \partial x_1$ value that causes $x_2$ to increase with increasing of $x_1$ in order to satisfy the budget constraint equality. In order to calculate the latest deferential we can differentiate the equation (6)

$$\frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$$

If we plug it in the equation (7) we obtain:

$$\frac{\partial u(\dot{x}_1, \dot{x}_2)/\partial x_1}{\partial u(\dot{x}_1, \dot{x}_2)/\partial x_2} = \frac{p_1}{p_2}$$

Which means, that marginal substitution rate ( $\dot{x}_1, \dot{x}_2$ ) between $x_1$ and $x_2$ equals to the ratio of prices of the optimal choice. This is the same condition we had above: The slope of the indifference curve should be equal to the slope of the budget constraint line. Of course, optimal choice should satisfy the budget constraint $p_1 \dot{x}_1 + p_2 \dot{x}_2 = m$ , which again leaves us with the equation with two unknowns.

The other way of solving utility maximization problem is to use the Lagrangian multiplication factors starting with introducing the Lagrangian function.

$$L = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - m).$$

The variable $\lambda$ is called Lagrangian multiplication factor, because it is multiplied by the budget constraint. According to Lagrangian theorem optimal choice ( $\dot{x}_1, \dot{x}_2$ ) should satisfy three conditions:

$$\frac{\partial L}{\partial x_1} = \frac{\partial u(\dot{x}_1, \dot{x}_2)}{\partial x_1} - \lambda p_1 = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial u(\dot{x}_1, \dot{x}_2)}{\partial x_2} - \lambda p_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = p_1 \dot{x}_1 + p_2 \dot{x}_2 - m = 0$$

Each of these three equations contains the Lagrangian differentials with respect to $x_1$, $x_2$ and $\lambda$ variables correspondingly, and are equal to zero. We have three equations with three unknowns and it is possible to express $x_1$ and $x_2$ variables with $p_1$ , $p_2$ and $m$ variables. If we divide first equation by the second one, we obtain:

$$\frac{\partial u(\dot{x}_1, \dot{x}_2) / \partial x_1}{\partial u(\dot{x}_1, \dot{x}_2) / \partial x_2} = \frac{p_1}{p_2}$$

This concludes, that MRS equals to the ratio of the prices that was already mentioned above. And the budget constraint results to the second equation, so we are back to the two equations with two unknowns.

In conclusion, the optimal basket is characterized with the constrain that slope of the indifference curve (MRS) is equal to the slope of the budget line. It should also be noted, that if the prices of two merchandises are the same for all consumers, then all consumers have the same marginal rate of substitution, and, therefore, all consumers are ready to substitute two merchandises with the equal amounts.

## References

[1] Hall R. Varian, Intermediate Microeconomics. W. W. Norton & Company. 1993.

[2] Eugene F. Fama, Merton H. Miller. The Theory of Finance, Dryden Press, 1971.