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Stabilization of Hyperbolic Chaos by the Pyragas Method

Sergey Belyakin, Arsen Dzanoev, Sergey Kuznetsov

Physics Faculty, Moscow State University, Moscow 119992, Russia

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Abstract: For a long time it was a common opinion that hyperbolic attractors are artificial mathematical constructions . However, in the recent papers there were proposed physically realizable systems that possess, in their phase space, the set with features that are very similar to hyperbolic type of attractors. As is known, invariant sets are called hyperbolic attractors of the dynamical system if they are closed, topologically transitive subsets, and every their trajectory possesses uniform hyperbolicity. Very familiar types of the hyperbolic attractors are Smale-Williams' solenoid and Plykin's attractor. Further, it is well known that chaotic systems are very sensitive to the external perturbations. This property is used for controlling nonlinear systems and chaos suppression. Thus, an important question arises: Is it possible to suppress chaos in systems with hyperbolic attractors because these attractors are structurally stable subsets? In the present contribution we study the possibility of stabilization of chaotic oscillations in systems with the Smale-Williams hyperbolic attractors by means of the Pyragas method with a delay . It is shown that by means of external perturbation the dynamical system could be controllable: the hyperbolic attractor degenerates into a periodic one.

Key words: Dynamical system, hyperbolic attractors, Pyragas method

1. Introduction

Hyperbolicity is a fundamental feature of chaotic systems. It is as follows: a tangent space Σ of such systems is a combination of three subspaces; stable E^s, unstable E^u and neutral E^0. Close trajectories which correspond to E^s converge exponentially to each other when $t \to +\infty$, and those which correspond to E^u- when $t \to -\infty$. In the subspace E^0 the vectors contract and expand more slowly than the exponential velocity. When the degree of contraction and expansion in the subspaces E^s and E^u changes from point to point along the trajectory, such systems are called non- uniformly hyperbolic. Dynamic systems with uniform hyperbolicity of all the trajectories are called Anosov systems.

If the system under consideration is dissipative, i.e. some contraction of the phase volume takes place, then in this case chaos is provided by the availability of the hyperbolic attractor. Each trajectory belonging to such an attractor possesses the property of hyperbolicity. The hyperbolic attractor is structurally stable and possesses SRB - measure (i.e. the measure of Sinai – Ruelle – Bowen) and corresponding systems are K – systems, for which the conditions of the central limit theorem are obeyed [1].

2. Strange and Hyperbolic Attractors

Strange (chaotic) attractors which are seen in the models of real physical systems possess some degree of hyperbolicity, however, this hyperbolicity has a form different from that of uniform hyperbolicity. Such attractors are, in fact, complicatedly organized sets, but they belong to a quasi - stochastic type (i. e. they are quasi - attractors) [2].

Hyperbolic sets were consttructed rather long ago [1] and, for a long time it was thought that the systems

Corresponding author: Sergey Belyakin, Physics Faculty, Moscow State University, E-mail: best@physics.msu.ru.

with such a structure of phase space were abstract mathematic constructions. However, recently were suggested physical models having attracting subsets possessing the properties of hyperbolicity [3, 4].

As it is known, for the resent 20 years one of the mostly demanded fields of inquiry in the theory of complex and chaotic systems has been the solution of the problem of controlling chaotic dynamics and chaos suppression [2] by minor external influences. The investigation of recent years showed that the problem can be solved for a particular class of dynamic systems, however, the question about the systems with hyperbolic attractors remains open. It is connected with the fact that such attractors are rough and their structure can not change qualitatively with minor external influences.

In the present contribution consideration is being given to an autonomous physical system which is characterized by the presence of the attractor of a hyperbolic type. We study the possibility of controlling and stabilizing the dynamics of the systems of this type by the Pyragas method [5].

The set Λ is called a hyperbolic attractor of a dynamic system if Λ - is a closed topologic transitive hyperbolic set and there exists such a vicinity $U \supset \Lambda$ that $\Lambda = \bigcap_{t\geq 0} f^n U$. Smale - Williams' solenoid and Plykin's attractor are well- known hyperbolic attractors. Smale - Williams' solenoid is obtained by the transformation of the toroidal domain $T = S^1 \times D^2$ into itself where S^1 - a unitary circle and D^2 - a unitary disc in R^2 .Then $f: T \to T$, $f(x, y, \phi) = (\frac{1}{k}x + \frac{1}{2}\cos\phi, \frac{1}{k}y + \frac{1}{2}\sin\phi, 2\phi)$, where $k>2$ determines the compression "by thickness", sets the solenoid as a subset $T \subset R^3$.

3. The Use of the Pyragas Method for the Formation of Regular Dynamics in Autonomous Hyperbolic Attractors

Let us take into consideration the system of the type:

$$\begin{aligned} \dot{x} &= \omega y + (1 - a_2 + \tfrac{1}{2}a_1 - \tfrac{1}{50}a_1^2)x + \varepsilon x_1 y_1, \\ \dot{y} &= -\omega x + (1 - a_2 + \tfrac{1}{2}a_1 - \tfrac{1}{50}a_1^2)y, \\ \dot{x}_1 &= \omega y_1 + (a_1 - 1)x_1 + \varepsilon x, \\ \dot{y}_1 &= -\omega x_1 + (a_1 - 1)y_1, \\ a_1 &= x^2 + y^2,\ a_2 = x_1^2 + y_1^2. \end{aligned} \tag{1}$$

Here x, y - dynamic variables, ε - coefficient of connection and ω - inherent frequency oscillations. These dynamic equations were suggested as a modified system of two interacting subsystems of the "predator - prey" type [3,4].

In the present contribution is shown that by means of delay of the type:

$$D_y(t) = K[y(t - \tau) - y(t)] \tag{2}$$

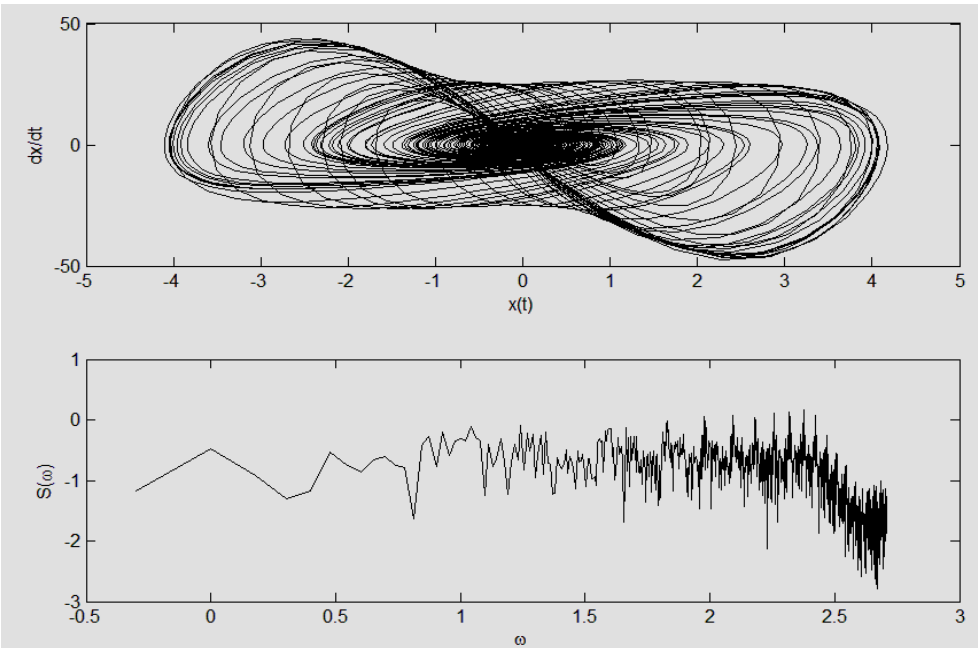
which is introduced into the second equation of the system (1), it is possible to bring the system to stable conditions. The problem is solved by means of the following analytical approach.

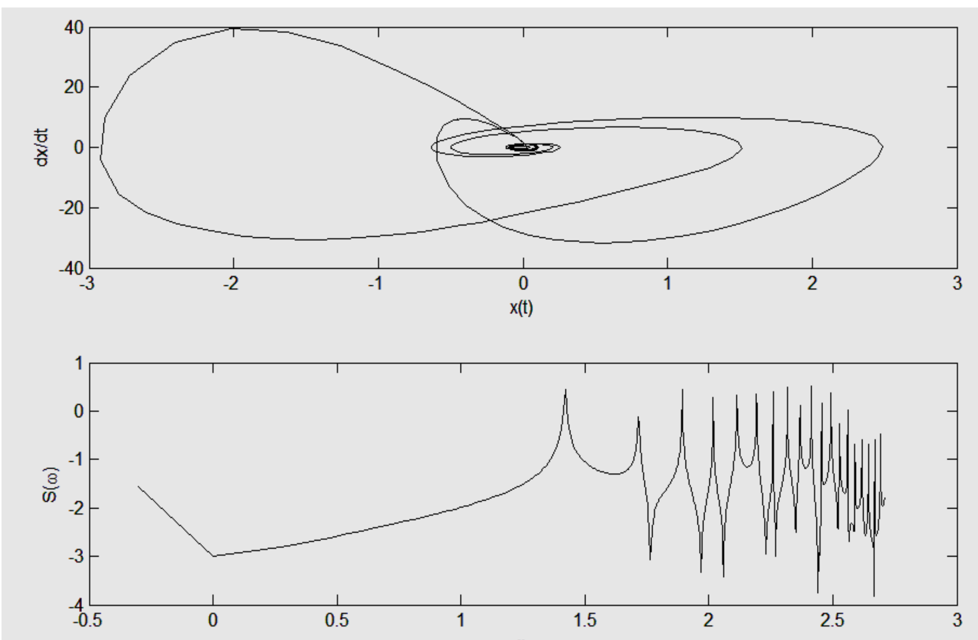
Let there be a smooth family of non - linear controlled systems of ordinary differential equations $\dot{x} = F(x, \mu, u)$, $x \in M \subset R^m$, $\mu \in L \subset R^k$, $u \in U \subset R^n$, $F \in C^\infty$ depending on the vector of controlling parameters u. Suppose that it is necessary to stabilize unstable limiting cycle $x^*(t, \mu^*)$ of the period T , which is the solution of the family when $u=0$ and $\mu = \mu^*$. Let the system have a regular attractor when the parameters are of the same value $u=0$ a $\mu = \mu^*$. Then the stabilization of the cycle $x^*(t, \mu^*)$ is carried out by means of the feedback with the delay being in the form of $u(t) = \mathrm{K}(x(t) - x(t - T))$, where K - is the matrix of coefficients. Therewith the initial conditional $x(0)$ is chosen in a sufficiently small vicinity of the cycle. Then the solution $x(t)$ of the system $\dot{x} = F(x(t),\ \mu^*,\ \mathrm{K}(x(t) - x(t - T))$ with the feedback with $\mu = \mu^*$ can converge to the sought - for unstable cycle.

4. Stabilization of Hyperbolic Dynamics

In our case instead of the matrix K the amplitude of delay K is used (2). In order to study the possibility of suppression of chaos in the system (1) familiar criteria of dynamics were used: Lyapunov exponents and spectral density obtained by fast Fourier transformation.

On the figure on the left a phase portrait of the system (1) and corresponding spectral density are shown when $\varepsilon = 0.3, \omega = 2\pi$. It is easy to see that with the given values of the parameters it has particularly pronounced chaotic properties. Let us introduce then the disturbance of the type (2) and consider the system with $K = -4.5, \tau = 0.4, \varepsilon = 0.3, \omega = 2\pi$. In this case the dynamics of the system undergoes qualitative changes: the hyperbolic attractor degenerates into a limiting cycle (fig.1 on the right), and the spectrum transforms from a continuous one corresponding to chaotic oscillations into an equidistant one with frequencies which correspond to the basic frequency and its harmonics.





The areas of synchronization (K/τ) for the system (2) are marked with dark bands.

5. Autonomous System with the Smale – Williams' Attractor

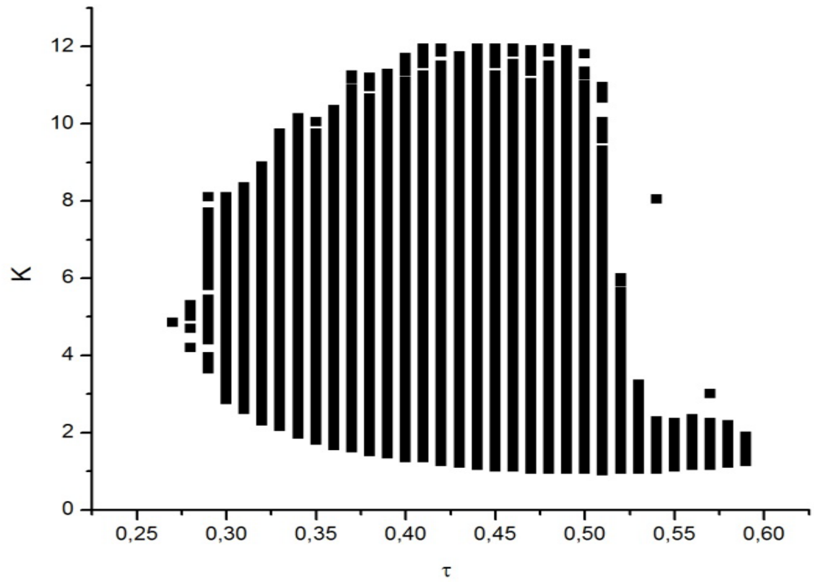
In accordance with the paper [4] for constructing an autonomous system with the Smale - Williams' attractor in the Poicare representation we will take as the starting point the modified system of "predator - prey" type:

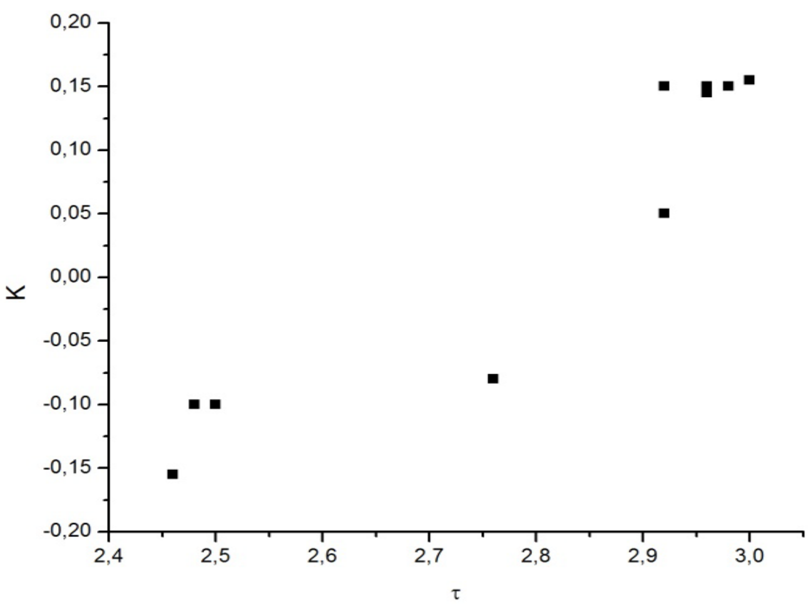
$$\dot{a}_1 = 2\mu_1(a_1, a_2)a_1,\ \dot{a}_2 = 2\mu_2(a_1)a_2 \qquad (3)$$

where

$$\mu_1 = 1 - a_2 + \tfrac{1}{2}a_1 - \tfrac{1}{50}(1-a_1)^2,\ \mu_2 = a_1 - 1,$$
$$a_1 = x^2 + y^2,\ a_2 = x_1^2 + y_1^2$$

We will interpret non - negative variables $a_{1,2}$ as squares of the modules of complex variables z_1 and z_2 so that the values $a_{1,2} = \left|z_{1,2}\right|^2$ will satisfy exactly the equations (3). In addition we will introduce connection between the subsystems 1and 2 by adding for the first variable - a term containing z_2^2, and into the second equation - a term proportional to z_1:





$$\dot{z}_1 = \mu_1(|z_1|^2, |z_2|^2)z_1 + \tfrac{1}{2}\varepsilon_1 z_2^2,$$
$$\dot{z}_2 = \mu_2(|z_1|^2)z_2 + \varepsilon_2 z_1. \qquad (4)$$

As the numerical solution to the equations (3) shows, in the steady - state conditions the system demonstrates periodical auto - oscillations and the representative point visits the vicinity of the origin of coordinates on the plane of the variables z_1, z_2 time after time. After each passage of this sort there happens first - some excitation of the first subsystem, then the excitation of the second one, further the attenuation of the first subsystem and, finally, - some slower attenuation of the second one. In the modified version of the model with complex variables (4) the activation of the second subsystem takes place in the presence of the effect due to the added term in the second equation that is why the argument of the variable z_2 inherits the value of the argument of the variable z_1 corresponding to the first subsystem. Then, at the stage of attenuation the second subsystem provides a seeding signal for the first one during the next passage of the orbit in the vicinity of the origin of coordinates. Since the corresponding term contains the square of the complex variable, the transmission of the excitation is accompanied by the doubling of the argument of the complex number. Further the process reoccurs. At every new cycle the angular variable corresponding to the argument of complex amplitudes is multiplied by the factor 2, which corresponds to the expanding representation of the circle - the Bernulli representation.

It is possible to determine the Poincare representation, the unitary application of which corresponds to the cycle of exchange of excitation between subsystems. In real variables this representation is three - dimensional, moreover there is an angular variable the transformation of which is described by the Bernulli representation whereas on the other directions the contraction of the phase volume takes place. That is why the attractor in the phase space of the Poincare representation is the Smale - Williams' solenoid.

Let us go over from complex variables to real, supposing that $z_1 = x + iy$ and $z_2 = x_1 + iy_1$. Then, instead of (3,4) we obtain the system of equations

$$\frac{dx}{dt} = \mu_1(x, y, x_1, y_1)x - \tfrac{1}{2}\varepsilon_1(x_1^2 - y_1^2),$$
$$\frac{dy}{dt} = \mu_1(x, y, x_1, y_1)y - \varepsilon_1 x_1 y_1,$$
$$\frac{dx_1}{dt} = \mu_2(x, y)x_1 - \varepsilon_2 x, \qquad (5)$$
$$\frac{dy_1}{dt} = \mu_2(x, y)y_1 - \varepsilon_2 y,$$

where the coefficients appearing in the equations are determined by the expressions

$$\mu_1 = 1 - a_2 + \tfrac{1}{2}a_1 - \tfrac{1}{50}(1 - a_1)^2,\ \mu_2 = a_1 - 1,$$
$$a_1 = x^2 + y^2,\ a_2 = x_1^2 + y_1^2 \qquad (6)$$

The parameters of the model we will set equal to $\varepsilon_1 = 0.01,\ \varepsilon_2 = 0.1$.

The design of a device has been developed as an analogue machine implementing the integration of differential equations (3, 4). For it the simulation of dynamics in the programming medium Multisim has been carried out, and conformity with the equations including the realization of the representation for the angular variable has been demonstrated. At present the approbation of the laboratory model based on the design is being carried out. The model is supposed to be tested within the framework of the program of the experiments.

For the system (5, 6) which we are based on an investigation in the framework of numerical simulation should be carried out. It includes the following items.

(1) Building up realizations, phase portraits and diagrams with angular variables.

(2) Calculation of statistic characteristics: Fourier spectrum, correlation functions, functions of distribution.

(3) Calculation of Lyapunov exponents (the entire spectrum) and the analysis of their dependence on parameters.

6. The Use of the Pyragas Method for Creating Regular Dynamics in Systems with Autonomous Hyperbolic Attractors

Let us consider an autonomous system of the type:

$$\begin{aligned}
\dot{x} &= (1 - a_2 + \tfrac{1}{2}a_1 - \tfrac{1}{50}a_1^2)x - \tfrac{1}{2}\varepsilon_1(x_1^2 - y_1^2),\\
\dot{y} &= (1 - a_2 + \tfrac{1}{2}a_1 - \tfrac{1}{50}a_1^2)y - \varepsilon_1 x_1 y_1,\\
\dot{x}_1 &= (a_1 - 1)x_1 - \varepsilon_2 x,\\
\dot{y}_1 &= (a_1 - 1)y_1 - \varepsilon_2 y,\\
a_1 &= x^2 + y^2, a_2 = x_1^2 + y_1^2.
\end{aligned} \tag{7}$$

Here x, y - dynamic variables, $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.1$ - coefficients of connection. Those dynamic equations were suggested as a modified system of two interacting subsystems of "predator - prey" type [3, 4].

In the present contribution is shown that by means of the delay of the type

$$D_y(t) = K[y(t-\tau) - y(t)] \tag{8}$$

the following can be implemented:

(1) The choice of the method of control. As such it is possible to use an external signal or the introduction of additional delayed feedback. (Both methods can be realized primarily during the schematic simulation in the medium Multisim, and then in a real experiment). It might also be interesting to think about the realization of a more complicated scheme of control of the type suggested in the work [6] for the stabilization of unstable periodic orbits belonging to the attractor.

(2) Carrying out the analysis of the modification of dynamics when including control, for example, with the use of the Lyapunov exponents. Obviously, it will be useful to construct charts on the plane of parameters where by means of color and the tones of brightness the value of the Lyapunov exponent is coded. In parallel, the analysis and systematization of obtained results are carried out. The answer to the question about the degree to which hyperbolic chaos is controllable should be obtained. One more question of interest is that of scenarios of the origins and destruction of hyperbolic chaos in the given concrete system.

The phase portraits and Fourier spectrums presented demonstrate the behavior of the system (8). *K=0* corresponds to the chaos Fig. 1, Fig. 2 corresponds to *K=-0.7* and *τ=0.5* the stable state, Fig. 3 corresponds

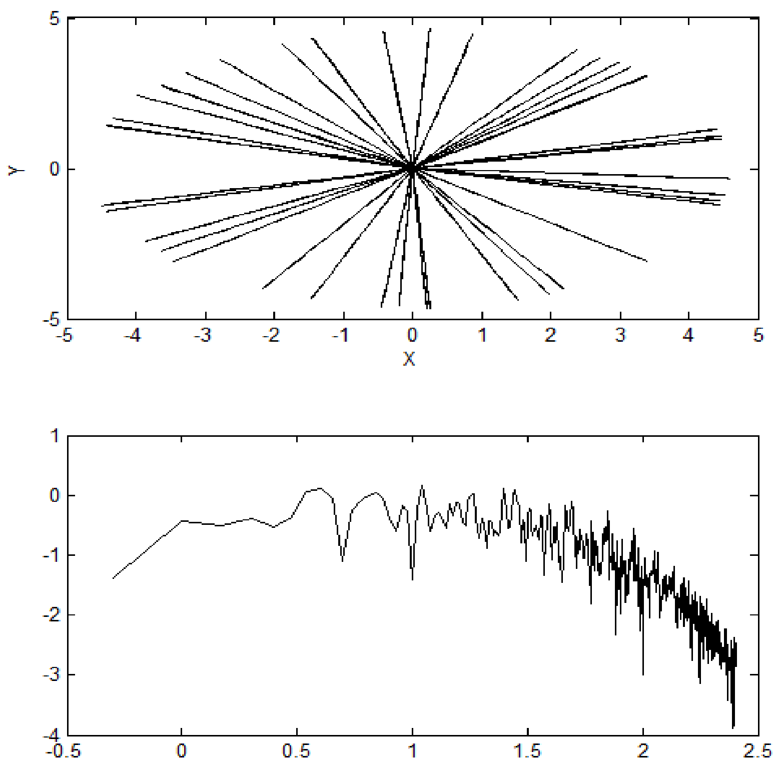


Fig. 1 Example Hyperbolic oscillator.

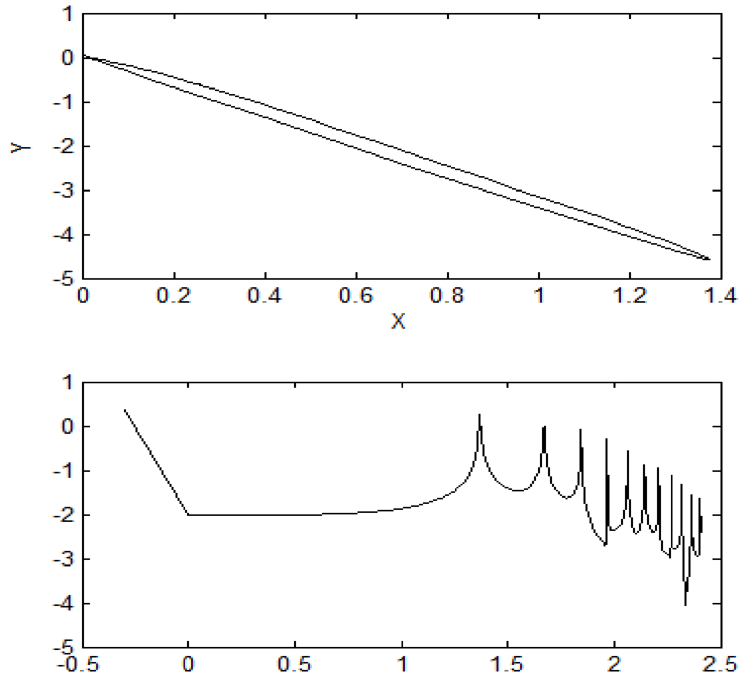


Fig. 2 Example Hyperbolic oscillator.

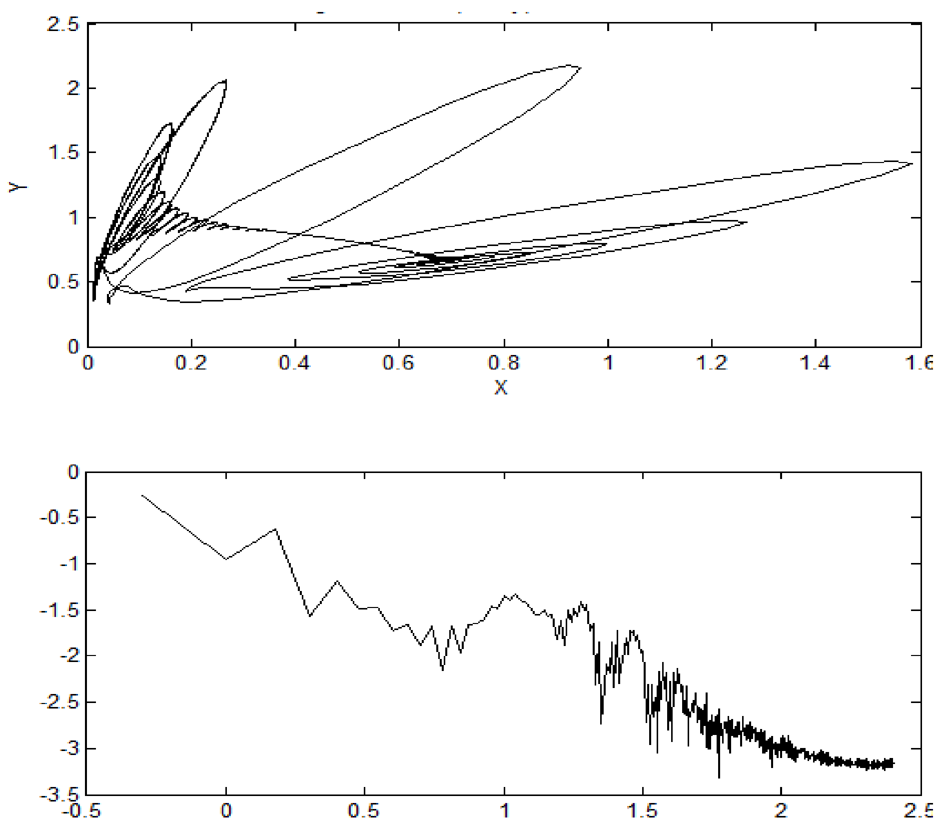
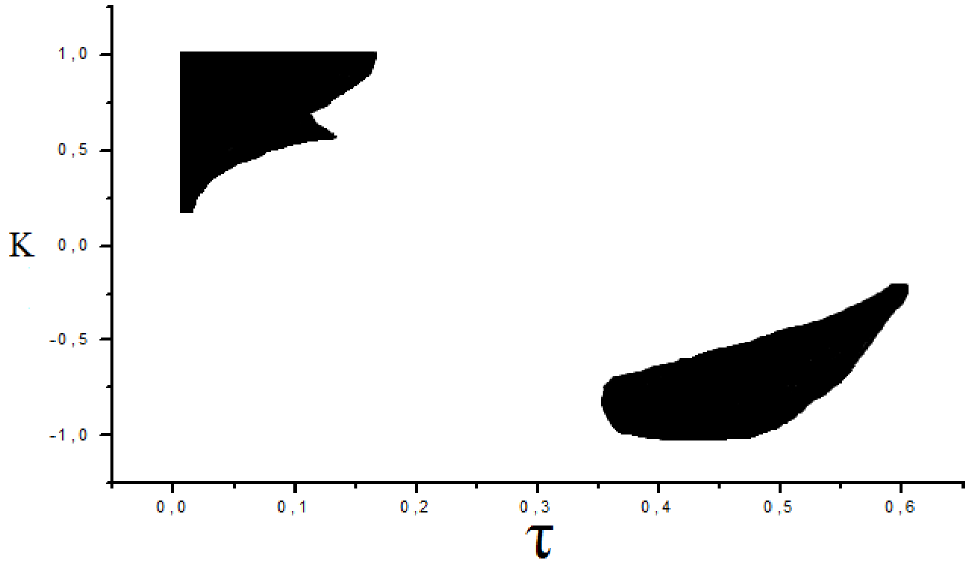


Fig. 3 Example Hyperbolic oscillator.



to *K=1.7* and *τ=0.7* to the hyperbolic chaotic state. The portrait of the system (7), when introducing disturbance of the type (8) is also presented. The blank part with the given values of the parameters depicts chaotic features. In dark areas the dynamics of the system undergoes qualitative changes, and the hyperbolic attractor degenerates into the limiting cycle, and the spectrum from the continuous corresponding to chaotic oscillations changes into an equidistant one with the frequencies corresponding to the basic frequency and its harmonics.

7. Conclusion

Thus, the disturbance of the type (8) stabilizes the dynamics of the system with a hyperbolic type of attractor. This result is possible to be interpreted in the way that either the inserted disturbances are not minor for the attractor of the given type, or the attractor in the investigated area of parameters does not belong to a strictly hyperbolic type.

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