Journal of Mathematics and System Science 5 (2015) 60-71
doi: 10.17265/2159-5291/2015.02.002

Atomic Entailment and Atomic Inconsistency and Classical Entailment

T. J. Stepień, L. T. Stepień

The Pedagogical University of Cracow, ul. Podchorazych 2, 30 - 084 Krakow, Poland

Received: December 01, 2014 / Accepted: December 24, 2014 / Published: February 25, 2015.

Abstract: In this paper we put forward a new solution of the well-known problem of relevant logics,i.e., we construct an atomic entailment. Hence, we construct a system of predicate calculus based on the atomic entailment. Next, we establish the definition of atomic inconsistency. The atomic inconsistency establishes an infinite class of inconsistent, but non-trivial systems. In this paper we construct the new definition of the classical entailment, into the bargain.

Key words: Atomic entailment, atomic inconsistency, classical entailment, relevance

1. Introduction

In a number of publications, (see [1] - [7], [9] - [18], [21], [22], [24] - [35], [39] - [45], [53] - [59]), their authors have offered many notions of relevance. Of course, in some publications of these mentioned above, their authors have established the basic properties of the well-known relevant logics. On the other hand, in [14]one can read that although the essence of entailment has been studied from 400 B.C., the problem of establishing such a logic of entailment, which solves the problem of relevance, is still open until now.

Thus, the essential aim is to create such a notion of relevance, which generates a system S of logic, which satisfies the following condition: this system S is generated by this notion of relevance, which is defined by a necessary and sufficient condition.

Therefore in this paper we at first construct a new definition of entailment, i.e. the definition of atomic entailment. Then we construct the definition of the system based on the atomic entailment. Next, we build a system $\overline{S}$ of propositional calculus (see [47], [48]) and a system $\overset{\sqcap}{S}$ of predicate calculus, which are based on the atomic entailment (see [49], [51], [52])[2]. Besides, in this paper, we give also the new definition of the classical entailment.

2. Notational Preliminaries

Let $\rightarrow, \sim, \vee, \wedge, \equiv$ denote the connectives of implication, negation, disjunction, conjunction and equivalence, respectively. We use $\Rightarrow, \neg, \Leftrightarrow, \&, \mathbb{V}, \forall, \exists$ as metalogical symbols. Next $At_0 = \{p, p_1, p_2, \dots, q, q_1, q_2, \dots, s, s_1, s_2, \dots t, \dots\}$ denotes the set of all propositional variables. S_0 is the set of all well-formed formulas, which are built in the usual manner from propositional variables and by means of logical connectives. $P_0(\phi)$ denotes the set of all propositional variables occuring in $\phi (\phi \in S_0)$. R_{S_0} denotes the set of all rules over S_0. Hence, for every $r \in R_{S_0}, \langle \Pi, \phi \rangle \in r$, where $\Pi \subseteq S_0$ and $\phi \in S_0$ and Π is a set of premisses and ϕ is a conclusion. Hence, r_*^0 denotes here the rule of simultaneous substitution for propositional variables.$\langle \{\phi\}, \psi \rangle \in r_*^0 \Leftrightarrow [h^e(\phi) = \psi]$, where h^e is the extension of the mapping $e: At_0 \longrightarrow S_0$ $(e \in \varepsilon_*^0)$

Corresponding author:L. T. Stepień, The Pedagogical University of Cracow, Kraków, Poland. E-mail: sfstepie@cyf-kr.edu.pl , www.ltstepien.up.krakow.pl

[2] In the next paper we will show that the system $\overset{\sqcap}{S}$ can be used for the formalization of The Arithmetic System (see [20]).

to endomorphism $h^e: S_0 \longrightarrow S_0$, where

$$h^e(\phi) = e(\phi), \text{ for } \phi \in At_0$$
$$h^e(\sim\phi) = \sim h^e(\phi)$$
$$h^e(\phi F \psi) = h^e(\phi) F h^e(\psi),$$

for $F \in \{\to, \vee, \wedge, \equiv\}$ and for every $\phi, \psi \in S_0$.

Thus, ε_*^0 is a class of functions $e: At_0 \longrightarrow S_0$(for details, see [36]) (cf. [19]). r_0^0 denotes here the Modus Ponens rule in propositional calculus. $R_{0*} = \{r_0^0, r_*^0\}$ (for details, see [19], [36]). A logical matrix is a pair $\mathfrak{M} = \{U, U'\}$, U is an abstract algebra and U' is a subset of the universe U, i.e. $U' \subseteq U$. Any $\alpha \in U'$ is called a distinguished element of the matrix $\mathfrak{M}$. $E(\mathfrak{M})$ is the set of all formulas valid in the matrix $\mathfrak{M}$. $\mathfrak{M}_2$ denotes the classical two-valued matrix. Hence, Z_2 is the set of all formulas valid in the classical matrix $\mathfrak{M}_2$ (see [19], [36]).

The symbols $x_1, x_2, \ldots$ are individual variables. $a_1, a_2, \ldots$ are individual constants. V is the set of all individual variables. $P_i^n (i, n \in \mathcal{N} = \{1, 2, \ldots\})$ are n-ary predicate letters. The symbols $f_i^n (i, n \in \mathcal{N})$ are n-ary function letters. The symbols $\wedge x_k, \vee x_k$ are quantifiers. $\wedge x_k$ is the universal quantifier and $\vee x_k$ is the existential quantifier. The function letters, applied to the individual variables and individual constants, generate terms. The symbols $t_1, t_2, \ldots$ areterms. T is the set of all terms. The predicate letters, applied to terms, yield simple formulas, i.e. if $P_i^k$is a predicate letter and $t_1, \ldots, t_k$ are terms, then $P_i^k(t_1, \ldots, t_n)$ is a simple formula. Smp is the set of all simple formulas. Next, At_1 is the set of all atomic formulas, where $At_1 = \{P_i^k(x_{j_1}, \ldots, x_{j_k}) : k, i, j_1, \ldots, j_k \in \mathcal{N}\}$. At last S_1 is the set of all well-formed formulas. $FV(\phi)$ denotes the set of all free variables occuring in ϕ, where $\phi \in S_1$. $x_k \in Ff(t_m, \phi)$ expresses that x_k is free for term t_m in ϕ. By x_k/t_m we denote the substitution of the term t_m for the individual variable x_k. $P_1(\phi)$ denotes the set of all predicate letters occuring in $\phi (\phi \in S_1)$. If $FV(\phi) = \{x_1, \ldots, x_k\}$, then $\wedge\phi = \wedge x_1 \ldots \wedge x_k \phi$.

R_{S_1} denotes the set of all rules over S_1. Hence, for every $r \in R_{S_1}$, $\langle \Pi, \phi \rangle \in r$, where $\Pi \subseteq S_1$ and $\phi \in S_1$ and Π is a set of premisses and ϕ is a conclusion. Hence, r_*^1 denotes here the rule of simultaneous substitution for predicate letters. $\langle \{\phi\}, \psi \rangle \in r_*^1 \Leftrightarrow [h^e(\phi) = \psi]$, where h^e is the extension of the mapping $e: Smp \longrightarrow S_1$ $(e \in \varepsilon_*^1)$ to endomorphism $h^e: S_1 \longrightarrow S_1$, where

$$h^e(\phi) = e(\phi), \text{for } \phi \in Smp$$
$$h^e(\sim\phi) = \sim h^e(\phi)$$
$$h^e(\phi F \psi) = h^e(\phi) F h^e(\psi),$$
$$\text{for } F \in \{\to, \vee, \wedge, \equiv\}$$
$$h^e(\wedge x_k \phi) = \wedge x_k h^e(\phi)$$
$$h^e(\vee x_k \phi) = \vee x_k h^e(\phi),$$

for every $\phi, \psi \in S_1$and $k \in \mathcal{N}$ (for details, see [37], [38]).

Next, r_0^1 denotes the Modus Ponens rule in predicate calculus, r_+^1 denotes the generalization rule. $R_{0+} = \{r_0^1, r_+^1\}$, $R_{0*+} = \{r_0^1, r_*^1, r_+^1\}$. We write $X \subset Y$, if $X \subseteq Y$ and $X \neq Y$.

We assume here that for every $\alpha \in S_1$, if $FV(\alpha) = \{x_1, \ldots, x_n\}$, then $\alpha^* = \vee x_1 \ldots \vee x_n \sim \alpha$. Hence, for every $\alpha \in S_1$, if $FV(\alpha) = \emptyset$, then $\alpha^* = \sim \alpha$. Analogically, for every $\alpha \in S_0$, $\alpha^* = \sim \alpha$.

Finally, for any $X \subseteq S_i$ and $R \subseteq R_{S_i}$, $Cn_i(R, X)$ is the smallest subset of S_i containing X and closed under the rules $R \subseteq R_{S_i}$, where $i \in \{0,1\}$. The couple $\langle R, X \rangle$ is called a system, whenever $R \subseteq R_{S_i}$ and $X \subseteq S_i$ and $i \in \{0,1\}$. $Syst \cap A_0$ denotes here the class of all systems $\langle R, X \rangle$, which are based on an atomic entailment, where $R \subseteq R_{S_0}$ and $X \subseteq S_0$. $Syst \cap A_1$ denotes here the class of all systems $\langle R, X \rangle$, which are based on an atomic entailment, where $R \subseteq R_{S_1}$ and $X \subseteq S_1$. $Syst \cap C_1$ denotes here the class of all systems $\langle R, X \rangle$, which are based on a classical entailment, where $R \subseteq R_{S_1}$ and $X \subseteq S_1$. $\phi \left|\frac{A_0}{R,X}\right. \psi$ denotes that ψ results atomically from ϕ on the ground of the system $\langle R, X \rangle$, where $R \subseteq R_{S_0}$ and $X \subseteq S_0$. Next, $\phi \left|\frac{A_1}{R,X}\right. \psi$ denotes that ψ results atomically from ϕ on the ground of the system

$\langle R, X\rangle$, where $R \subseteq R_{S_1}$ and $X \subseteq S_1$. At last, $\phi \left|\frac{C_1}{R,X}\right. \psi$ denotes that ψ results classically from ϕ on the ground of the system $\langle R, X\rangle$, where $R \subseteq R_{S_1}$ and $X \subseteq S_1$. $\overline{S}_1 = \{\phi \in S_1: FV(\phi) = \emptyset\}$, S_1^* denotes the set of all formulas, which are in normal form (see [19] pp. 35 - 42 and 130 - 132, [20] pp. 214 - 222 and [37] pp. 146 - 149).

Definition 2.1. *The function $j: S_1 \longrightarrow S_0$, is defined, as follows:*

$$j\left(P_k^n(t_1, \dots, t_n)\right) = p_k (p_k \in At_0)$$
$$j(\sim\phi) = \ \sim j(\phi)$$
$$j(\phi F \psi) = j(\phi) F j(\psi), \textit{ for } F \in \{\rightarrow, \vee, \wedge, \equiv\}$$
$$j(\wedge x_k \phi) = j(\vee x_n \phi) = j(\phi).$$

By $\langle R, X\rangle \in Cns^A$ we denote here the well-known notion of the absolute consistency (see [36] and [37]). Thus,

Definition 2.2. $\langle R, X\rangle \in Cns^A \Leftrightarrow Cn(R, X) \neq S_i$, *where* $R \subseteq R_{S_i}$, $X \subseteq S_i$ and $i \in \{0,1\}$.

3. Classical Entailment

Definition 3.1. *Let* $Cn_1(R, X) = L \neq \emptyset$ *and* $\phi, \psi \in S_1$. *Then* $\phi \left|\frac{C_1}{R,X}\right. \psi$ *iff the following conditions are satisfied*

(1) $(\forall e \in \varepsilon_*^1)[h^e(\wedge \phi) \in L \Rightarrow h^e(\psi) \in L]$

(2) $(\forall e \in \varepsilon_*^1)[h^e\left((\psi^* \rightarrow \phi^*) \rightarrow \phi^*\right) \in L \Rightarrow h^e(\phi^*) \in L]$.

Definition 3.2. $\langle R, X\rangle \in Syst \cap C_1$ *iff the following condition is satisfied:*

$$(\forall \phi, \psi \in S_1)[\wedge \phi \rightarrow \psi \in Cn_1(R, X) \Leftrightarrow \phi \left|\frac{C_1}{R,X}\right. \psi].$$

4. The classical predicate logic

Let A_2 denote the set of axioms of the classical predicate logic. Hence, $\langle R_{0+}, A_2\rangle$ denotes the classical predicate calculus, where $Cn(R_{0+}, A_2) = L_2$ (see [20] and [37]).

Thus, (see [37] p.57, p.71):

Theorem 4.1. $Cn_1(R_{0*+}, L_2) = L_2$.

Now we notice that on the ground of the classical predicate calculus, the following theorem is valid (cf. [20] pp. 222 - 223):

Theorem 4.2. (on the extensionality of logical expressions). *Let $x_1, \dots, x_n, y_1, \dots, y_l$ be all the free variables, which occur in the expressions α and β, and let C^α be any expression that contains α or an expression obtained from α by the substitution for the variables $x_1, \dots, x_n$ of some other variables different from the bound variables occurring in the expressions α or β, and let C^β differ from C^α only in that in certain places (unnecessarily in all these places) in which in C^α there occurs α or an expression obtained from α by a substitution for the variables $x_1, \dots, x_n$, in the corresponding places in C^β there occurs β or an expression obtained from β by an appropriate substitution, while the variables $y_1, \dots, y_l$ are all the free variables in C^α and C^β. Then the sentence:*

$$\wedge \dots y_1, \dots, y_l \left(\wedge x_1, \dots, x_n (\alpha \equiv \beta) \rightarrow \left(C^\alpha \equiv C^\beta\right)\right)$$

is a theorem in L_2.

5. Atomic Entailment

In [57] one can read that Lewis told that from his very first contact with the logic of "Principia Mathematica", he had been bothered by the paradoxes of material implication. As Whitehead and Russell have it written, a true proposition is implied by arbitrary (true or false) proposition, while a false proposition implies arbitrary (true or false) proposition. Aiming at avoiding these consequences of the material conditional, Lewis wrote his first paper devoted to logic (in this current paper, the Lewis' paper is as [25]). At first, it ought to be noticed here that the results contained in [1] - [7], [9] - [18], [21], [22], [24] - [35], [39] - [45], [53] - [59], and in the other papers, have essentially contributed to the better understanding of the problem of relevance. Thus (see [47], [48], [49], [51], [52]):

Definition 5.1. *Let* $Cn_0(R, X) = L \neq \emptyset$ *and* $\phi, \psi \in S_0$. *Then* $\phi \left|\frac{A_0}{R,X}\right. \psi$ *iff the following conditions* *are satisfied:*

(1) $(\forall e \in \varepsilon_*^0)[h^e(\phi) \in L \Rightarrow h^e(\psi) \in L$ $\& \ P_0(h^e(\phi)) \subseteq P_0(h^e(\psi))]$

(2) $(\forall e \in \varepsilon_*^0)[h^e((\psi^* \to \phi^*) \to \phi^*) \in L \Rightarrow$ $h^e(\phi^*) \in L \ \& \ P_0(h^e(\psi^*)) \subseteq P_0(h^e(\phi^*))]$.

Definition 5.2. $\langle R, X\rangle \in Syst \cap A_0$ *iff the following condition is satisfied:*

$$(\forall \psi, \phi \in S_0)[\phi \to \psi \in Cn_0(R,X) \Leftrightarrow \phi \left|\frac{A_0}{R,X}\right. \psi].$$

Definition 5.3. *Let* $Cn_1(R,X) = L \neq \emptyset$ *and* $\phi, \psi \in S_1$. *Then* $\phi \left|\frac{A_1}{R,X}\right. \psi$ *iff the following conditions are satisfied:*

(1) $(\forall e \in \varepsilon_*^1)[h^e(\wedge \phi) \in L \Rightarrow h^e(\psi) \in L$ $\& \ P_1(h^e(\wedge\phi)) \subseteq P_1(h^e(\psi))]$

(2) $(\forall e \in \varepsilon_*^1)[h^e((\psi^* \to \phi^*) \to \phi^*) \in L \Rightarrow$ $h^e(\phi^*) \in L \ \& \ P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))]$.

Definition 5.4. $\langle R, X\rangle \in Syst \cap A_1$ *iff the following condition is satisfied:*

$$(\forall \psi, \phi \in S_1)[\wedge \phi \to \psi \in Cn_1(R,X) \Leftrightarrow \phi \left|\frac{A_1}{R,X}\right. \psi].$$

6. Atomic Inconsistency

By $Syst \cap AINC$ we denote here the class of all systems $\langle R, X\rangle$, which have the property of atomic inconsistency (see also [8], [23], [60]), where $R \subseteq R_{S_i}$ and $X \subseteq S_i$ and $i \in \{0,1\}$.

Definition 6.1. *Let* $i \in \{0,1\}$ *and* $\alpha \in S_i$. *Then* $S_{i\alpha} = \{\phi \in S_i : P_i(\phi) \subseteq P_i(\alpha)\}$.

Definition 6.2. *Let* $R \subseteq R_{S_i}$ *and* $X \subseteq S_i$ *and* $i \in \{0,1\}$. *Then*

$$\langle R, X\rangle \in Syst \cap AINC \Leftrightarrow$$
$$(\forall \alpha \in S_i)\{[S_{i\alpha} \subseteq Cn(R, X \cup \{\alpha, \sim\alpha\})] \&$$
$$(\forall \beta \in S_i)[P_i(\beta) \nsubseteq P_i(\alpha) \Rightarrow$$
$$\beta \notin Cn(R, X \cup \{\alpha, \sim\alpha\}) \vee$$
$$\sim\beta \notin Cn(R, X \cup \{\alpha, \sim\alpha\})]\}.$$

7. System $\overline{S}$

Let us take the matrix

$\mathfrak{M}_D = \langle \{0,1,2\}, \{1,2\}, f_D^{\to}, f_D^{\equiv}, f_D^{\vee}, f_D^{\wedge}, f_D^{\sim}\rangle$, where:

$f_D^{\to}$	0	1	2
0	1	1	1
1	0	1	0
2	0	1	2

$f_D^{\equiv}$	0	1	2
0	1	0	0
1	0	1	0
2	0	0	2

$f_D^{\vee}$	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

$f_D^{\wedge}$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$f_D^{\sim}$	
0	1
1	0
2	2

In [47] (see [48]) we have defined the system $\overline{S}$ as follows:

Definition 7.1. $\overline{S} = \langle R_{0*}, T_D\rangle$, *where* $T_D = E(\mathfrak{M}_D)$.

Thus, the system $\overline{S}$ is the logic that is obtained from the set of valid formulas in the matrix $\mathfrak{M}_D$, by the rules of substitution and detachment.

It should be noticed here that the matrix $\mathfrak{M}'_D = \langle \{0,1,2\}, \{1,2\}, f_D^{\to}, f_D^{\sim}\rangle$ was investigated by B. Sobocinski (see [46], [47]).

Next, in [47] we have proved the following:

Theorem 7.2. *Let* $\phi, \psi \in S_0$ *and*

$$(\exists e \in \varepsilon_*^0)[h^e(\phi) \in T_D].$$

Then $\phi \to \psi \in Cn_0(R_{0*}, T_D)$ *iff*

$$(\forall e \in \varepsilon_*^0)[h^e(\phi) \in T_D \Rightarrow h^e(\psi) \in T_D$$
$$\& \ P_0(h^e(\phi)) \subseteq P_0(h^e(\psi))].$$

Theorem 7.3. *The system* $\overline{S}$ *is axiomatizable.*

8. System $\overset{\sqcap}{S}$

At first we define the set L_D, putting:

Definition 8.1. $L_D = \{\phi \in S_1 : j(\phi) \in T_D \ \& \ \phi \in L_2\}$.

Next, we define the system $\overset{\sqcap}{S}$, as follows:

Definition 8.2. $\overset{\sqcap}{S} = \langle R_{0+}, L_D\rangle$.

By Theorem 4.1 and by Definition 8.1, one obtains:

Corollary 8.3. $Cn_1(R_{0*+}, L_D) = L_D$.

By Definition 8.1 and by Corollary 8.3, we get

Corollary 8.4. *Let* $\alpha, \beta, \gamma, \phi, \psi, \delta \in S_1$ *and* $Q_i \in \{\wedge x_i, \vee x_i\}$ *and* $i, k, s \in \mathcal{N}$. *Then the following formulas belong to* L_D*:*

(1) $\alpha \to \alpha$
(2) $\alpha \to [(\alpha \to \beta) \to \beta]$
(3) $(\alpha \to \beta) \to [(\beta \to \gamma) \to (\alpha \to \gamma)]$
(4) $[\alpha \to (\beta \to \gamma)] \to [\beta \to (\alpha \to \gamma)]$
(5) $[\alpha \to (\alpha \to \beta)] \to (\alpha \to \beta)$
(6) $\{[(\beta \to \gamma) \to (\alpha \to \gamma)] \to \delta\} \to [(\alpha \to \beta) \to \delta]$
(7) $[\alpha \to (\beta \to \gamma)] \to \{(\delta \to \beta) \to [\alpha \to (\delta \to \gamma)]\}$
(8) $[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
(9) $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
(10) $(\beta \to \gamma) \to \{(\alpha \to \beta) \to [(\gamma \to \delta) \to (\alpha \to \delta)]\}$
(11) $\sim\sim \alpha \to \alpha$
(12) $\alpha \to \sim\sim\alpha$
(13) $(\sim\alpha \to \alpha) \to \alpha$
(14) $(\alpha \to \sim \alpha) \to \sim\alpha$
(15) $(\sim\sim\alpha \to \sim\sim\beta) \to (\alpha \to \beta)$
(16) $(\alpha \to \sim\beta) \to (\sim\sim\alpha \to \sim\beta)$
(17) $(\alpha \to \beta) \to (\alpha \to \sim\sim\beta)$
(18) $(\alpha \to \sim \beta) \to (\beta \to \sim \alpha)$
(19) $(\sim\beta \to \sim\sim \alpha) \to (\sim\beta \to \alpha)$
(20) $(\sim \alpha \to \beta) \to (\sim\beta \to \alpha)$
(21) $\alpha \to \sim(\alpha \to \sim \alpha)$
(22) $(\sim \alpha \to \sim\sim \alpha) \to \alpha$
(23) $(\alpha \to \beta) \to (\sim \beta \to \sim \alpha)$
(24) $(\alpha \to \beta) \to [(\alpha \to \sim \beta) \to \sim\alpha]$
(25) $(\sim\alpha \to \beta) \to [(\sim\beta \to \sim \alpha) \to \sim\sim\beta]$
(26) $(\sim\alpha \to \beta) \to [(\alpha \to \beta) \to \beta]$
(27) $\alpha \to [\beta \to \sim(\alpha \to \sim \beta)]$
(28) $\alpha \equiv \alpha$
(29) $\alpha \equiv \sim\sim\alpha$
(30) $\sim\sim\alpha \equiv \alpha$
(31) $(\alpha \to \beta) \to [(\beta \equiv \gamma) \to (\alpha \to \gamma)]$
(32) $(\alpha \equiv \beta) \to [(\beta \equiv \gamma) \to (\alpha \to \gamma)]$
(33) $(\beta \to \alpha) \to [(\beta \equiv \gamma) \to (\gamma \to \alpha)]$
(34) $(\alpha \equiv \beta) \to (\alpha \to \beta)$
(35) $(\alpha \equiv \beta) \to (\beta \to \alpha)$
(36) $(\alpha \to \beta) \to [(\beta \to \alpha) \to (\alpha \equiv \beta)]$
(37) $(\alpha \equiv \beta) \to [(\alpha \to \gamma) \equiv (\beta \to \gamma)]$
(38) $(\alpha \equiv \beta) \to [(\gamma \equiv \alpha) \equiv (\gamma \equiv \beta)]$
(39) $(\alpha \equiv \beta) \to (\beta \equiv \alpha)$
(40) $\alpha \to (\alpha \vee \beta)$, if $P_1(\alpha) = P_1(\alpha \vee \beta)$
(41) $\alpha \to (\beta \vee \alpha)$, if $P_1(\alpha) = P_1(\beta \vee \alpha)$
(42) $(\alpha \vee \beta) \to [(\alpha \to \beta) \to \beta]$
(43) $(\alpha \to \beta) \to [(\alpha \vee \gamma) \to (\gamma \vee \beta)]$
(44) $(\alpha \to \beta) \to [(\alpha \vee \gamma) \to (\beta \vee \gamma)]$
(45) $(\alpha \to \beta) \to [(\gamma \vee \alpha) \to (\gamma \vee \beta)]$
(46) $[\alpha \vee (\beta \vee \gamma)] \to [(\alpha \vee \beta) \vee \gamma]$
(47) $[\alpha \vee (\beta \vee \gamma)] \to [\alpha \vee (\gamma \vee \beta)]$
(48) $[\alpha \vee (\gamma \vee \beta)] \to [(\alpha \vee \beta) \vee \gamma]$
(49) $[\alpha \vee (\beta \vee \gamma)] \to [\beta \vee (\alpha \vee \gamma)]$
(50) $[\gamma \vee (\alpha \vee \beta)] \to [\beta \vee (\gamma \vee \alpha)]$
(51) $[\beta \vee (\gamma \vee \alpha)] \to [\beta \vee (\alpha \vee \gamma)]$
(52) $[(\beta \vee \alpha) \vee \gamma] \to [\beta \vee (\alpha \vee \gamma)]$
(53) $(\alpha \to \beta) \vee (\beta \to \alpha)$
(54) $(\alpha \to \beta) \to \{(\gamma \to \beta) \to [(\alpha \vee \gamma) \to \beta]\}$
(55) $\sim\alpha \vee \alpha$
(56) $\alpha \vee \sim\alpha$
(57) $(\alpha \vee \beta) \to (\sim\beta \to \alpha)$
(58) $(\alpha \vee \beta) \to (\sim\alpha \to \beta)$
(59) $(\sim\alpha \vee \beta) \to (\alpha \to \beta)$
(60) $\alpha \to (\alpha \wedge \alpha)$
(61) $(\alpha \wedge \beta) \to (\beta \wedge \alpha)$
(62) $\alpha \to [\beta \to (\alpha \wedge \beta)]$
(63) $[(\alpha \wedge \beta) \to (\beta \to \gamma)] \to [(\alpha \wedge \beta) \to \gamma]$
(64) $[\alpha \to (\beta \to \gamma)] \to [(\alpha \wedge \beta) \to \gamma]$
(65) $[(\alpha \wedge \beta) \to \gamma] \to [\alpha \to (\beta \to \gamma)]$
(66) $[(\alpha \to \beta) \wedge \alpha] \to \beta$
(67) $[(\alpha \wedge \gamma) \to \beta] \to [(\alpha \wedge \gamma) \to (\beta \wedge \gamma)]$
(68) $(\alpha \to \beta) \to [(\gamma \wedge \alpha) \to (\gamma \wedge \beta)]$
(69) $(\alpha \to \beta) \to \{(\alpha \to \gamma) \to [\alpha \to (\beta \wedge \gamma)]\}$
(70) $[(\alpha \to \gamma) \wedge (\beta \to \delta)] \to [(\alpha \wedge \beta) \to (\gamma \wedge \delta)]$
(71) $[(\alpha \to \gamma) \wedge (\beta \to \delta)] \to [(\beta \wedge \alpha) \to (\delta \wedge \gamma)]$
(72) $[(\alpha \to \beta) \wedge (\alpha \to \gamma)] \to [\alpha \to (\beta \wedge \gamma)]$
(73) $[(\alpha \wedge \beta) \wedge \gamma] \to \{[(\alpha \wedge \beta) \wedge \gamma] \wedge \beta\}$
(74) $\{[(\alpha \wedge \beta) \wedge \gamma] \wedge \beta\} \to [(\alpha \wedge \gamma) \wedge \beta]$
(75) $\sim(\alpha \wedge \sim\alpha)$
(76) $\sim(\sim\alpha \wedge \alpha)$
(77) $\sim(\alpha \to \beta) \to (\alpha \wedge \sim\beta)$
(78) $[\sim(\alpha \wedge \sim\beta) \wedge \alpha] \to \beta$

(79) $[\alpha \wedge \sim(\alpha \wedge \sim\beta)] \rightarrow \beta$
(80) $(\alpha \wedge \beta) \rightarrow (\sim\sim\alpha \wedge \sim\sim\beta)$
(81) $(\alpha \wedge \beta) \rightarrow \sim(\alpha \rightarrow \sim\beta)$
(82) $(\alpha \wedge \sim\sim\beta) \rightarrow (\alpha \wedge \beta)$
(83) $\sim(\alpha \rightarrow \sim\beta) \rightarrow (\alpha \wedge \beta)$
(84) $(\alpha \rightarrow \sim\sim\beta) \rightarrow \sim(\alpha \wedge \sim\beta)$
(85) $(\alpha \rightarrow \sim\beta) \rightarrow \sim(\alpha \wedge \beta)$
(86) $(\alpha \rightarrow \beta) \equiv (\sim\beta \rightarrow \sim\alpha)$
(87) $(\alpha \equiv \beta) \equiv (\sim\alpha \equiv \sim\beta)$
(88) $(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$
(89) $[\alpha \wedge (\beta \wedge \gamma)] \equiv [(\alpha \wedge \beta) \wedge \gamma]$
(90) $[(\alpha \equiv \beta) \wedge (\gamma \equiv \delta)] \rightarrow [(\alpha \rightarrow \gamma) \equiv (\beta \rightarrow \delta)]$
(91) $(\alpha \equiv \beta) \rightarrow [(\beta \rightarrow \alpha) \wedge (\alpha \rightarrow \beta)]$
(92) $(\alpha \equiv \beta) \rightarrow [(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)]$
(93) $(\alpha \wedge \alpha) \equiv \alpha$
(94) $(\alpha \equiv \beta) \rightarrow [(\alpha \wedge \gamma) \equiv (\beta \wedge \gamma)]$
(95) $(\alpha \equiv \beta) \rightarrow [(\gamma \wedge \alpha) \equiv (\gamma \wedge \beta)]$
(96) $[(\alpha \equiv \beta) \wedge (\gamma \equiv \delta)] \rightarrow [(\alpha \equiv \gamma) \equiv (\beta \equiv \delta)]$
(97) $[(\alpha \rightarrow \gamma) \wedge (\gamma \rightarrow \alpha)] \rightarrow (\alpha \equiv \gamma)$
(98) $[(\alpha \equiv \beta) \wedge (\gamma \equiv \delta)] \rightarrow [(\alpha \wedge \gamma) \equiv (\beta \wedge \delta)]$
(99) $[(\alpha \equiv \beta) \wedge (\beta \equiv \gamma)] \rightarrow [(\alpha \rightarrow \gamma) \wedge (\gamma \rightarrow \alpha)]$
(100) $(\alpha \vee \alpha) \equiv \alpha$
(101) $(\alpha \vee \beta) \equiv (\beta \vee \alpha)$
(102) $(\alpha \equiv \beta) \rightarrow [(\gamma \vee \alpha) \equiv (\gamma \vee \beta)]$
(103) $(\alpha \vee \beta) \equiv (\sim\alpha \rightarrow \beta)$
(104) $(\alpha \rightarrow \beta) \equiv (\sim\alpha \vee \beta)$
(105) $[(\alpha \equiv \beta) \wedge (\gamma \equiv \delta)] \rightarrow [(\alpha \vee \gamma) \equiv (\beta \vee \delta)]$
(106) $\sim(\alpha \wedge \beta) \equiv (\alpha \rightarrow \sim\beta)$
(107) $\sim(\alpha \wedge \beta) \equiv (\beta \rightarrow \sim\alpha)$
(108) $(\alpha \vee \beta) \rightarrow \sim(\sim\alpha \wedge \sim\beta)$
(109) $(\sim\alpha \wedge \sim\beta) \rightarrow \sim(\alpha \vee \beta)$
(110) $\sim(\sim\alpha \vee \sim\beta) \rightarrow (\alpha \wedge \beta)$
(111) $\sim(\alpha \wedge \beta) \rightarrow (\sim\alpha \vee \sim\beta)$
(112) $\sim(\alpha \vee \beta) \rightarrow (\sim\alpha \wedge \sim\beta)$
(113) $\sim(\sim\alpha \wedge \sim\beta) \rightarrow (\alpha \vee \beta)$
(114) $(\alpha \wedge \beta) \rightarrow \sim(\sim\alpha \vee \sim\beta)$
(115) $(\sim\alpha \vee \sim\beta) \rightarrow \sim(\alpha \wedge \beta)$
(116) $(\alpha \vee \beta) \equiv \sim(\sim\alpha \wedge \sim\beta)$
(117) $(\alpha \wedge \beta) \equiv \sim(\sim\alpha \vee \sim\beta)$
(118) $(\alpha \wedge \beta) \equiv \sim(\alpha \rightarrow \sim\beta)$
(119) $(\alpha \rightarrow \beta) \equiv \sim(\alpha \wedge \sim\beta)$

(i) $\bigwedge\phi \rightarrow \phi$
(ii) $\bigwedge x_k\phi \rightarrow \bigvee x_k\,\phi$
(iii) $\bigwedge x_k\phi \equiv \phi$, if $x_k \notin FV(\phi)$
(iv) $\bigvee x_k\phi \equiv \phi$, if $x_k \notin FV(\phi)$
(v) $\bigwedge x_k(\phi \rightarrow \psi) \equiv (\phi \rightarrow \bigwedge x_k\psi)$, if $x_k \notin FV(\phi)$
(vi) $\bigvee x_k(\phi \rightarrow \psi) \equiv (\phi \rightarrow \bigvee x_k\psi)$, if $x_k \notin FV(\phi)$
(vii) $\bigwedge x_k(\phi \rightarrow \psi) \equiv (\bigvee x_k\phi \rightarrow \psi)$, if $x_k \notin FV(\psi)$
(viii) $\bigvee x_k \sim\phi \equiv \sim\bigwedge x_k\phi$
(ix) $\alpha \rightarrow \bigvee x_k\,\alpha$
(x) $(\phi \rightarrow \bigwedge x_k\psi) \rightarrow [(\bigwedge x_k\,\psi \rightarrow \phi) \rightarrow (\phi \rightarrow \psi)]$
(xi) $(\bigvee x_k\phi \rightarrow \psi) \rightarrow [(\psi \rightarrow \bigvee x_k\phi) \rightarrow (\phi \rightarrow \psi)]$
(xii) $\phi x_k/t_s \rightarrow \bigvee x_k\phi$, if $x_k \in Ff(t_s, \phi)$
(xiii) $\bigwedge x_k(\phi \rightarrow \psi) \rightarrow (\bigwedge x_k\phi \rightarrow \bigwedge x_k\psi)$
(xiv) $\bigwedge x_k(\phi \rightarrow \psi) \rightarrow (\bigvee x_k\phi \rightarrow \bigvee x_k\psi)$
(xv) $\bigwedge x_k(\alpha \equiv \beta) \rightarrow (\bigwedge x_k\alpha \equiv \bigwedge x_k\beta)$
(xvi) $\bigwedge x_k(\alpha \equiv \beta) \rightarrow (\bigvee x_k\alpha \equiv \bigvee x_k\beta)$
(xvii) $\sim\bigvee x_k \sim(\phi \rightarrow \psi) \equiv (\bigvee x_k\phi \rightarrow \psi)$, if $x_k \notin FV(\psi)$
(xviii) $\bigwedge x_k(\phi \wedge \psi) \equiv (\bigwedge x_k\phi \wedge \bigwedge x_k\psi)$
(xix) $(\bigwedge x_k\phi \vee \bigwedge x_k\psi) \rightarrow \bigwedge x_k(\phi \vee \psi)$
(xx) $\bigvee x_k(\phi \rightarrow \psi) \equiv (\bigwedge x_k\phi \rightarrow \bigvee x_k\psi)$
(xxi) $\bigvee x_k(\phi \wedge \psi) \rightarrow (\bigvee x_k\phi \wedge \bigvee x_k\psi)$
(xxii) $\bigvee x_k(\phi \vee \psi) \equiv (\bigvee x_k\phi \vee \bigvee x_k\psi)$
(xxiii) $\bigwedge x_k(\phi \vee \psi) \equiv (\phi \vee \bigwedge x_k\psi)$, if $x_k \notin FV(\phi)$
(xxiv) $\bigwedge x_k(\phi \rightarrow \psi) \equiv (\bigvee x_k\phi \rightarrow \psi)$, if $x_k \notin FV(\psi)$
(xxv) $\bigvee x_k(\phi \wedge \psi) \equiv (\phi \wedge \bigvee x_k\psi)$, if $x_k \notin FV(\phi)$
(xxvi) $\bigwedge x_k \bigwedge x_s\,\phi \equiv \bigwedge x_s \bigwedge x_k\,\phi$
(xxvii) $\bigvee x_k \bigvee x_s\,\phi \equiv \bigvee x_s \bigvee x_k\,\phi$
(xxviii) $\bigvee x_k \bigwedge x_s\,\phi \rightarrow \bigwedge x_s \bigvee x_k\phi$
(xxix) $\sim\bigvee x_k\phi \equiv \bigwedge x_k \sim\phi$
(xxx) $\sim\bigvee x_k\sim\phi \equiv \bigwedge x_k\,\phi$
(xxxi) $\sim\bigwedge x_k\sim\phi \equiv \bigvee x_k\phi$
(xxxii) $\{[(\psi^* \rightarrow \phi^*) \rightarrow \phi^*] \rightarrow \phi^*\} \rightarrow (\bigwedge\phi \rightarrow \psi)$
(xxxiii) $\sim Q_i(\bigwedge\phi \wedge \psi) \equiv (\bigwedge\phi \rightarrow \sim Q_i\psi)$
(xxxiv) $\bigwedge x_k(\phi \equiv \psi) \rightarrow [\bigwedge x_k(\phi \rightarrow \psi) \wedge \bigwedge x_k(\psi \rightarrow \phi)]$.

Using Definition 8.1, Corollary 8.3, Corollary 8.4 and using the proof of Theorem 4.2 (see [20], pp.222 -

224), one can obtain

Corollary 8.5 (on the extensionality of logical expressions). *Let $x_1, \ldots, x_n, y_1, \ldots, y_l$ be all the free variables, which occur in the expressions α and β, and let C^α be any expression that contains α or an expression obtained from α by the substitution for the variables $x_1, \ldots, x_n$ of some other variables different from the bound variables occurring in the expressions α or β, and let C^β differ from C^α only in that in certain places (unnecessarily in all these places) in which in C^α there occurs α or an expression obtained from α by a substitution for the variables $x_1, \ldots, x_n$, in the corresponding places in C^β there occurs β or an expression obtained from β by an appropriate substitution, while the variables $y_1, \ldots, y_l$ are all the free variables in C^α and C^β. Then the sentence:*

$$\wedge \ldots y_1, \ldots, y_l (\wedge x_1, \ldots, x_n (\alpha \equiv \beta) \to (C^\alpha \equiv C^\beta))$$

is a theorem in L_D.

Definition 8.6. *Let $\phi \in S_1$ and $\alpha \in Smp$. Next let $v: At_0 \longrightarrow |\mathfrak{M}_2|$ be an arbitrary, but fixed valuation in the matrix $\mathfrak{M}_2$ such that $h^v(j(\phi)) = 1$. Then*

$$e_\phi(\alpha) = \begin{cases} \wedge \phi \wedge \alpha, & \text{if} \quad v(\mathrm{i}(\alpha)) = 0 \\ \wedge \phi \to \alpha, & \text{if} \quad v(\mathrm{i}(\alpha)) = 1. \end{cases}$$

By the definition of the formulas ϕ^*, ψ^*, one can easily obtain right away

Corollary 8.7.

$$(\forall \phi, \psi \in S_1)(\exists e \in \varepsilon_*^1)[h^e(\wedge \phi) \in L_D \mathbin{\mathbb{V}} h^e((\psi^* \to \phi^*) \to \phi^*) \in L_D].$$

Lemma 8.8. *If $\wedge \phi \to \psi \in L_D$, then*

$(\forall e \in \varepsilon_*^1)[h^e(\wedge\phi) \in L_D \Rightarrow h^e(\psi) \in L_D \,\&\, P_1(h^e(\wedge\phi)) \subseteq P_1(h^e(\psi))]$ *and*

$(\forall e \in \varepsilon_*^1)[h^e((\psi^* \to \phi^*) \to \phi^*) \in L_D \Rightarrow h^e(\phi^*) \in L_D \;\&\; P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))]$.

Proof. Let (1) $\wedge \phi \to \psi \in L_D$. Hence, by Corollary 8.3, we obtain that (2) $(\forall e \in \varepsilon_*^1)[h^e(\wedge\phi) \in L_D \Rightarrow h^e(\psi) \in L_D]$. Hence, by the definition of the set L_D and by the definition of the matrix $\mathfrak{M}_D$, it follows that (3) $(\forall e \in \varepsilon_*^1)[h^e(\wedge\phi) \in L_D \Rightarrow h^e(\psi) \in L_D \;\&\; P_1(h^e(\wedge\phi)) \subseteq P_1(h^e(\psi))]$. Let (4) $FV(\phi) = \{x_1, \ldots, x_n\}$ and (5) $FV(\psi) = \{y_1, \ldots, y_m\}$. Hence, by the definition of the formulas ϕ^*, ψ^*, it follows that

(6) $\phi^* = \vee x_1 \ldots \vee x_n \sim\phi$

and

(7) $\psi^* = \vee y_1 \ldots \vee y_m \sim\psi$.

Hence, from (1), by Definition 8.1, Corollary 8.3, Corollary 8.4 and Corollary 8.5, we obtain that (8) $\psi^* \to \phi^* \in L_D$. Hence, by Definition 8.1, Corollary 8.3, Corollary 8.4 and Corollary 8.5, we obtain that

(9) $(\forall e \in \varepsilon_*^1)[h^e((\psi^* \to \phi^*) \to \phi^*) \in L_D \Rightarrow h^e(\phi^*) \in L_D \;\&\; P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))]$,

what together with (3) complete the proof. □

Lemma 8.9. *If $\phi \in S_1^*$ and $(\exists e \in \varepsilon_*^1)[h^e(\phi) \in L_D]$, then $h^{e_\phi}(\phi) \in L_D$.*

Proof. Now we assume that (1) $Q_i \in \{\wedge x_i, \vee x_i\}$ and (2) $\phi \in S_1^*$ and (3) $(\exists e_1 \in \varepsilon_*^1)[h^{e_1}(\phi) \in L_D]$. Hence, by the definition of the set L_D, it follows that (4) $(\exists v: At_0 \longrightarrow |\mathfrak{M}_2|)[h^v(j(\phi)) = 1]$.

Let:

(1.1) $\phi \in Smp$.

Hence, by (4) and Definition 8.6, one can obtain that

(5) $h^{e_\phi}(\phi) = \wedge \phi \to \phi$.

Hence, by Corollary 8.4 (i), in (1.1), it follows that

(6) $h^{e_\phi}(\phi) \in L_D$.

Let

(1.2) $\phi = \sim P_k^n(t_1, \ldots, t_n)$.

Hence, from (4) and Definition 8.6, it follows that

(7) $h^{e_\phi}(\phi) = \sim(\wedge \phi \wedge P_k^n(t_1, \ldots, t_n))$.

Therefore, by Corollary 8.4 (107) and (1.2), it follows that

(8) $h^{e_\phi}(\phi) \equiv (\wedge \phi \to \phi) \in L_D$.

So, using Corollary 8.4 (i), in (1.2), one can obtain that

(9) $h^{e_\phi}(\phi) \in L_D$.

Let

(1.3) $\phi = \phi_1 \vee \phi_2$

and assume inductively that

$$(a_1) h^{e_\phi}(\phi_1) \in L_D$$

or

$$(a_2) h^{e_\phi}(\phi_2) \in L_D.$$

From Definition 8.6 it follows that

(10) $h^{e_\phi}(\phi_1 \vee \phi_2) = h^{e_\phi}(\phi_1) \vee h^{e_\phi}(\phi_2)$.

Next, in (a_1) and (a_2), from (1.3) and by Definition 8.6, it follows that

(11) $P_1(h^{e_\phi}(\phi_1)) = P_1(h^{e_\phi}(\phi_2)) = P_1(\phi)$.

Hence, from (10), by Corollary 8.4 (40) and Corollary 8.4 (41), in (a_1) and (a_2), in (1.3), it follows that

(12) $h^{e_\phi}(\phi) \in L_D$.

Let

(1.4) $\phi = \phi_1 \wedge \phi_2$

and assume inductively that

(13) $h^{e_\phi}(\phi_1), h^{e_\phi}(\phi_2) \in L_D$.

From (1.4) and (13), using Definition 8.6, by Corollary 8.4 (62), in (1.4), one can obtain that

(14) $h^{e_\phi}(\phi) \in L_D$.

Let

(1.5) $\phi = Q_i\phi'$

and assume inductively that

(15) $h^{e_\phi}(\phi') \in L_D$.

Hence, from (1.5), using Definition 8.6, by Corollary 8.4 (ix) and Corollary 8.3, in (1.5), one can obtain that

(1.6) $h^{e_\phi}(\phi) \in L_D$,

which completes the proof. □

Lemma 8.10. *If* $\phi \in S_1$ *and* $(\exists e \in \varepsilon_*^1)[h^e(\phi) \in L_D]$, *then* $h^{e_\phi}(\phi) \in L_D$.

Proof. By Definition 8.6, Corollary 8.4, Corollary 8.5 and Lemma 8.9 and by the well-known Theorem concerning normal form (see [19] pp. 35-42 and 130-132, [20] pp. 214 - 222, and [37] pp.146-149). □

Lemma 8.11. *Let* $\wedge\phi \to \psi \in L_2$ *and*

$$(\forall e \in \varepsilon_*^1)[h^e(\wedge\phi) \in L_D \Rightarrow h^e(\psi) \in L_D \,\&\, P_1(h^e(\phi)) \subseteq P_1(h^e(\psi))]$$ *and*

$$(\forall e \in \varepsilon_*^1)[h^e((\psi^* \to \phi^*) \to \phi^*) \in L_D \Rightarrow h^e(\phi^*) \in L_D \;\&\; P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))].$$ *Then*

$$\wedge\phi \to \psi \in L_D.$$

Proof. By Theorem 7.2, the Definition 8.1, Corollary 8.3, Corollary 8.4, Corollary 8.5, Definition 8.6, Corollary 8.7, Lemma 8.10 and by the definition of the matrix $\mathfrak{M}_D$ and by the definitions of the formulas ϕ^*, ψ^*. □

Lemma 8.12. *Let* $\phi, \psi \in S_1$ *and*

$$(\exists e \in \varepsilon_*^1)[h^e(\phi) \in L_D]$$

and

$$(\forall e \in \varepsilon_*^1)[h^e(\phi) \in L_D \Rightarrow h^e(\psi) \in L_D \,\&\, P_1(h^e(\phi)) \subseteq P_1(h^e(\psi))].$$

Then

$$(\forall e \in \varepsilon_*^1)[h^e(\phi) \in L_2 \;\Rightarrow\; h^e(\psi) \in L_2].$$

Proof. Let (1) $\phi, \psi \in S_1$, (2) $(\exists e \in \varepsilon_*^1)[h^e(\phi) \in L_D]$ and (3) $(\forall e \in \varepsilon_*^1)[h^e(\phi) \in L_D \Rightarrow h^e(\psi) \in L_D \;\&\; P_1(h^e(\phi)) \subseteq P_1(h^e(\psi))]$.

From (1), (2), it follows that (4) $(\exists e_1 \in \varepsilon_*^1)[h^{e_1}(\phi) \in L_2]$. Now suppose that (5) $(\exists e_2 \in \varepsilon_*^1)[h^{e_2}(\phi) \in L_2 \;\&\; h^{e_2}(\psi) \notin L_2]$. Next assume that (6) $h^{e_2}(\phi) = \phi'$ and (7) $h^{e_2}(\psi) = \psi'$. From (5) – (7), it follows that (8) $\phi' \in L_2$ and (9) $\psi' \notin L_2$. From (8) it follows that (10) $(\exists e \in \varepsilon_*^1)[h^e(\phi') \in L_D]$. Hence, by Lemma 8.10, it follows that (11) $h^{e_{\phi'}}(\phi') \in L_D$. Hence, from (3) it follows that (12) $h^{e_{\phi'}}(\psi') \in L_D$. From (8) and (9) and Definition 8.6 and Theorem 4.2, it follows that (14) $h^{e_{\phi'}}(\psi') \notin L_2$. From (12), by the definition of the set L_D, it follows that (15) $h^{e_{\phi'}}(\psi') \in L_2$, what contradicts (14). □

Lemma 8.13. *Let*

$$(\forall e \in \varepsilon_*^1)[h^e((\psi^* \to \phi^*) \to \phi^*) \in L_D \Rightarrow h^e(\phi^*) \in L_D \;\&\; P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))]$$

and

$$(\exists e \in \varepsilon_*^1)[h^e((\psi^* \to \phi^*) \to \phi^*) \in L_D].$$

Then

$$(\forall e \in \varepsilon_*^1)[h^e((\psi^* \to \phi^*) \to \phi^*) \in L_2 \Rightarrow h^e(\phi^*) \in L_2].$$

Proof. The proof of this lemma is analogical to the proof of Lemma 8.12. □

In [50] we have proved the following Lemma:

Lemma 8.14. *Let* $\phi, \psi \in S_1, X \subseteq S_1$ *and* $(\exists v: At_0 \longrightarrow |\mathfrak{M}_2|)[h^v(j(\phi)) = 1]$*and*

$$Cn_1(R_{0+}, L_2 \cup X) = Z_3$$

and

$(\forall e \in \varepsilon_*^1)[h^e(\phi) \in Z_3 \;\Rightarrow\; h^e(\psi) \in Z_3]$.

Then $\bigwedge \phi \to \psi \ \in Z_3$.

In consequence:

Lemma 8.15. *If* $(\exists e \in \varepsilon_*^1)[h^e(\phi) \in L_D]$ *and* $(\forall e \in \varepsilon_*^1)[h^e(\phi) \in L_D \Rightarrow h^e(\psi) \in L_D \,\&\, P_1(h^e(\phi)) \subseteq P_1(h^e(\psi))]$, *then* $\bigwedge \phi \to \psi \ \in L_2$.

Proof. By Corollary 8.4, Lemma 8.12 and Lemma 8.14. □

Lemma 8.16. *Let*

$(\forall e \in \varepsilon_*^1)[h^e\big((\psi^* \to \phi^*) \to \phi^*\big) \in L_D \Rightarrow h^e(\phi^*) \in L_D \ \&\ P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))]$

and

$(\exists e \in \varepsilon_*^1)[h^e\big((\psi^* \to \phi^*) \to \phi^*\big) \in L_D]$.

Then $\psi^* \to \phi^* \in L_2$.

Proof. By Corollary 8.4, Lemma 8.13 and Lemma 8.14. □

Lemma 8.17. Let $\phi^*, \psi^* \in S_1$.

If$(\forall e \in \varepsilon_*^1)[h^e\big((\psi^* \to \phi^*) \to \phi^*\big) \in L_D \Rightarrow h^e(\phi^*) \in L_D \ \&\ P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))]$ *and* $(\forall e \in \varepsilon_*^1)[h^e(\phi) \in L_D \Rightarrow h^e(\psi) \in L_D \,\&\, P_1(h^e(\phi)) \subseteq P_1(h^e(\psi))]$, *then* $\bigwedge \phi \to \psi \ \in L_D$.

Proof. By the definitions of the formulas ϕ^*, ψ^* andCorollary 8.7, Lemma 8.11, Lemma 8.15 and Lemma 8.16. □

Lemma 8.18. Let $\phi^*, \psi^* \in S_0$.

If$(\forall e \in \varepsilon_*^0)[h^e\big((\psi^* \to \phi^*) \to \phi^*\big) \in T_D \Rightarrow h^e(\phi^*) \in T_D \ \&\ P_0(h^e(\psi^*)) \subseteq P_0(h^e(\phi^*))]$ *and* $(\forall e \in \varepsilon_*^0)[h^e(\phi) \in T_D \Rightarrow h^e(\psi) \in T_D \,\&\, P_0(h^e(\phi)) \subseteq P_0(h^e(\psi))]$, *then* $\phi \to \psi \in T_D$.

Proof. Using the similar reasoning as in the proof of Lemma 8.17. □

Lemma 8.19.*If* $\phi \to \psi \in T_D$, *then*

$(\forall e \in \varepsilon_*^0)[h^e(\phi) \in T_D \Rightarrow h^e(\psi) \in T_D \,\&\, P_0\big(h^e(\phi)\big) \subseteq P_0\big(h^e(\psi)\big)]$

and

$(\forall e \in \varepsilon_*^0)[h^e\big((\psi^* \to \phi^*) \to \phi^*\big) \in T_D \Rightarrow h^e(\phi^*) \in T_D \ \&\ P_0(h^e(\psi^*)) \subseteq P_0(h^e(\phi^*))]$.

Proof. The proof of this Lemma is analogical to the proof of Lemma 8.8. □

9. The Main Result

Theorem 9.1. $\langle R_0, T_D\rangle \in Syst \cap A_0$.

Proof. By Lemma 8.18 and Lemma 8.19. □

Theorem 9.2. $\langle R_{0+}, L_D\rangle \in Syst \cap A_1$.

Proof. By Lemma 8.8 and Lemma 8.17. □

Theorem 9.3. $\langle R_{0+}, L_2\rangle \in Syst \cap C_1$.

Proof. By similar reasonings as in the proofs of Lemma 8.8 and Lemma 8.17 (or by Corollary 8.7, the definition of the set L_D and by Lemma 8.14). □

Theorem 9.4. $\langle R_0, T_D\rangle \in Syst \cap AINC$.

Proof. By the definition of the set T_D and by the Definition 6.1 and the Definition 6.2. □

Theorem 9.5. $\langle R_{0+}, L_D\rangle \in Syst \cap AINC$.

Proof. Let

(1) $\alpha \in S_1$

and

(2) $\beta \in S_1$.

Hence, by the definition of the setL_D, it follows that

(3) $\alpha \to (\sim\alpha \to \beta) \in L_D$, where

(4) $P_1(\beta) \subseteq P_1(\alpha)$.

From (1)-(4), it follows that

(5) $S_{1\alpha} \subseteq Cn(R_{0+}, L_D \cup \{\alpha, \sim\alpha\})$.

Let now,

(6) $P_1(\beta) \nsubseteq P_1(\alpha)$.

Next, by the definition of the set L_2, it follows that

(7) $(\alpha \wedge \sim\alpha) \to (\beta \wedge \sim\beta) \in L_2$.

Next, from (6), by the definition of the set T_D, it follows that

(8) $j(\alpha \wedge \sim\alpha) \to j(\beta \wedge \sim\beta) \notin T_D$.

Hence, from (6), (7), by the definition of the set L_D, it follows that

(9) $\beta \notin Cn(R_{0+}, L_D \cup \{\alpha \wedge \sim\alpha\})$

or

(10) $\sim\beta \notin Cn(R_{0+}, L_D \cup \{\alpha \wedge \sim\alpha\})$,

what together with 5), 6) and the Definition 6.2, completes the proof. □

10. Summary

Remark 10.1. *Let* $(\forall e \in \varepsilon_*^0)[h^e(\psi^*) \in L \Rightarrow h^e(\phi^*) \in L \,\&\, P_0\big(h^e(\psi^*)\big) \subseteq P_0\big(h^e(\phi^*)\big)] = \Lambda_0$

and

$(\forall e \in \varepsilon_*^0)[h^e\big((\psi^* \to \phi^*) \to \phi^*\big) \in L \Rightarrow$

$h^e(\phi^*) \in L \ \& \ P_0(h^e(\psi^*)) \subseteq P_0(h^e(\phi^*))] = \Lambda_1$.

By an inspection of Definition 5.1, Definition 5.2, Definition 7.1, Lemma 8.18 and Lemma 8.19, one can easily see that in condition (2) of Definition 5.1 one cannot put Λ_0 instead of Λ_1.

Remark 10.2.

$Let\ (\forall e \in \varepsilon_*^1)[h^e(\psi^*) \in L \Rightarrow h^e(\phi^*) \in L \ \& \ P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))] = \Lambda_0$

and

$$(\forall e \in \varepsilon_*^1)[h^e((\psi^* \to \phi^*) \to \phi^*) \in L \Rightarrow h^e(\phi^*) \in L \ \& P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))] = \Lambda_1.$$

By an inspection of Definition 5.3 and Definition 5.4 and Definition 8.1 and Lemma 8.8 and Lemma 8.17, one can easily see that in condition (2) of Definition 5.3 one cannot put Λ_0 instead of Λ_1.

Remark 10.3.

$$Let(\forall e \in \varepsilon_*^1)[h^e(\psi^*) \in L \Rightarrow h^e(\phi^*) \in L\,] = \Lambda_0$$

$$and\ \ (\forall e \in \varepsilon_*^1)[h^e((\psi^* \to \phi^*) \to \phi^*) \in L \Rightarrow h^e(\phi^*) \in L] = \Lambda_1.$$

By an inspection of Definition 3.1, Definition 3.2 and by Lemma 8.14, Theorem 9.3, one can easily see that in condition (2) of Definition 3.1 one cannot put Λ_0 instead of Λ_1.

Using Definition 2.2 and Definition 6.1 and Definition 6.2, one can obtain the following remark:

Remark 10.4.

$\langle R, X\rangle \in Syst \cap AINC \Rightarrow \langle R, X\rangle \in Cns^A$,

$where\ \ R \subseteq R_{S_i}\ \ and\ \ X \subseteq S_i\ \ and\ \ i \in \{0,1\}$.

References

[1] W. Ackerman. Begrundung einer strengen implication. *The Journal of Symbolic Logic*, 21:113-128, 1956.

[2] A. R. Anderson. Completeness theorems for the system E of entailment and EQ of entailment with quantification. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 6:201-216, 1960.

[3] A. R. Anderson and N. D. Belnap. The Pure Calculus of Entailment. *The Journal of Symbolic Logic*, 27:19-52, 1962.

[4] A. R. Anderson and N. D. Belnap. First degree entailments. *Mathematische Annalen*, 149:302-319, 1963.

[5] A. R. Anderson and N. D. Belnap. *Entailment: The Logic of Relevance and Necessity*, volume I. Princeton University Press, 1975.

[6] A. R. Anderson, N. D. Belnap, and J. M. Dunn. *Entailment: The Logic of Relevance and Necessity*, volume II. Princeton University Press, 1992.

[7] R. Barcan-Marcus. Strict implication, Deducibility and the deduction theorem. *The Journal of Symbolic Logic*, 18:234-236, 1953.

[8] J. Beall. Truth without Detachment, 2012. A Talk delivered at Munich Center for Mathematical Philosophy, 14.03.2012.

[9] K. Bimbo. Relevance Logics and Relation Algebras. In *Philosophy of Logic, ed. D. Jacquette*, vol. 5, pages 723-789. Elsevier/North-Holland, Amsterdam, 2007. Included in series Handbook of the Philosophy of Science (eds. D. M.Gabbay, P. Thagard and J. Woods).

[10] K. Bimbo, J. M. Dunn, and R. D. Maddux. Relevance Logics and Relation Algebras. *The Review of Symbolic Logic*, 2:102-131, 2009.

[11] L. Borkowski.Comments on Conditional Sentences and on Material and Strict Implication, in*The Book in Honour of Kazimierz Ajdukiewicz*., 102-104, Warszawa, 1964.

[12] R. T. Brady. Entailment, classicality and the paradoxes, delivered to the Australasian Association of Philosophy Conference, A.N.U., Canberra, 1989.

[13] Ross T. Brady. Relevant implication and the case for a weaker logic. *Journal of Philosophical Logic*, 25:151-183, 1996.

[14] Jingde Cheng. The fundamental role of entailment in knowledge representation and reasoning. *Journal of Computing and Information*, 2:853-873, 1996.

[15] Jingde Cheng. Temporal relevant logic as the logic basis for reasoning about dynamics of concurrent systems. In *Proceedings of the 1998 IEEE-SMC Annual International Conference on Systems, Man, and Cybernetics, San Diego, USA, 11-14 October 1998, 1998*.

[16] Z. Czerwinski. On paradox of implication. *Studia Logica*, VII:264-271, 1958.

[17] A. E. Duncan-Jones. Is strict implication the same as entailment ? *Analysis*, II:70-78, 1935.

[18] J. M. Dunn. A modification of Parry's analytic

implication. *Notre Dame Journal of Formal Logic*, 13:195-205, 1972.

[19] Yu. L. Ershov and E. A. Palyutin. *Mathematical Logic*. Mir Publishers, MOSCOW, 1984.

[20] A. Grzegorczyk. *An Outline of Mathematical Logic. Fundamental Results and Notions Explained with All Details*, D. Reidel Publishing Company, Dordrecht-Holland/Boston-USA, PWN, Warszawa, 1974.

[21] I. Hacking. what is strict implication ? *The Journal of Symbolic Logic*, 28:51-71, 1963.

[22] R. Hirsch and Szabolc Mikulas. Positive Fragments of Relevance Logic and Algebras of Binary Relations. *The Review of Symbolic Logic*, 4:81-105, 2011.

[23] Anthony Hunter and S. Konieczny. Approaches to measuring inconsistent information. In *Inconsistency Tolerance*, 191-236, Springer, 2005.

[24] S. A. Kripke. The problem of entailment. *The Journal of Symbolic Logic*, 24:324, 1959.

[25] C. I. Lewis. Implication and the algebra of logic. *Mind*, *New Series*, 21:522-531, 1912.

[26] C. Lewy. Entailment. *Aristotelian Society Supplementary Volume*, 32:123-142, 1958.

[27] S. Mccall. A new variety of implication. *The Journal of Symbolic Logic*, 29:151-152, 1964.

[28] R. K. Meyer. Entailment and relevant implication. *Logique et Analyse*, 11:472-479, 1968.

[29] R. K. Meyer. Two questions from Anderson and Belnap. *Reports on Mathematical Logic*, 7:71-86, 1976.

[30] R. K. Meyer and J. M. Dunn. Entailment and material implication. *Notices of The American Mathematical Society*, 15:1021-1022, 1966.

[31] R. K. Meyer and R. Routley. Classical relevant logics (I). *Studia Logica*, XXXII:51-68, 1973.

[32] J. M. MÉndez and G. Robles. Relevance Logics, Paradoxes of Consistency and the K Rule II. A Non-constructive Negation. *Logics and Logical Philosophy*, 15:175-191, 2006.

[33] J. M. MÉndez, Francisco Salto, and G. Robles. Anderson and Belnap's Minimal Positive Logic with Minimal Negation. *Reports on Mathematical Logic*, 36:117-130, 2002.

[34] J. Norman and R. Sylvan, Editors. *Directions in Relevant Logic*. Kluwer Academic Publishers, 1989.

[35] R. Z. Parks and J. R. Chidgey. Necessity and ticket entailment. *Notre Dame Journal of Formal Logic*, 13:224-226, 1972.

[36] W. A. Pogorzelski. *The Classical Propositional Calculus*. PWN, Warszawa, 1975.

[37] W. A. Pogorzelski. *The Classical Calculus of Quantifiers*. PWN, Warszawa, 1981.

[38] W. A. Pogorzelski and T. Prucnal. The substitution rule for predicate letters in the first-order predicate calculus. *Reports on Mathematical Logic*, 5:77-90,1975.

[39] G. Pottinger. On analysing relevance constructively. *Studia Logica*, 38:171-185, 1979.

[40] G. Priest. Sense, Entailment and Modus Ponens. *Journal of Philosophical Logic*, 9:415-435, 1980.

[41] A. N. Prior. Facts, propositions, and entailment. *Mind*, 57:62-68, 1947.

[42] R. Routley. The American Plan Completed: alternative classic-style semantics, without stars, for relevant and paraconsistent logics. *Studia Logica*, XLIII1/2:131-158, 1984.

[43] R. Routley and R. K. Meyer. Towards a general semantical theory of implication and conditionals. I. Systems with normal conjunctions and disjunctions and aberrant and normal negations. *Reports on Mathematical Logic*, 4:67-90,1975.

[44] R. Routley and R. K. Meyer. Towards a general semantical theory of implication and conditionals. II. Systems with normal conjunctions and disjunctions and aberrant and normal negations. *Reports on Mathematical Logic*, 9:47-62,1977.

[45] T. J. Smiley. Entailment and deducibility. *Proceedings of The Aristotelian Society*, 59:233-254, 1959.

[46] B. Sobocinski. Axiomatization of partial system of three-valued calculus of propositions. *The Journal of Computing System*, 1:22-55, 1952.

[47] T. Stepien. System $\bar{S}$. *Reports on Mathematical Logic*, 15:59-65, 1983.

[48] T. Stepien. System $\bar{S}$. *Zentralblatt für Mathematik*, 471, 1983.

[49] T. Stepien. Logic based on atomic entailment. *Bulletin of the Section of Logic*, 14:65-71, 1985.

[50] T. Stepien. Derivability. *Reports on Mathematical Logic*, 33:79-99, 1999.

[51] T. Stepien. Logic Based On Atomic Entailment And Paraconsistency. *11th international Congress Of Logic,Methodology and Philosophy Of Science* (August 1999, Krakow, Poland).

[52] T. J. Stepien and L. T. Stepien. Atomic Entailment and Classical Entailment, *The Bulletin of Symbolic Logic*, 17:317-318, 2011.

[53] N. Tennant. Perfect validity, entailment and paraconsistency. *Studia Logica*, XLIII:181-200, 1984.

[54] A. Urquhart. Semantics for relevant logics. *The Journal of Symbolic Logic*, 37:159-169, 1972.

[55] A. Urquhart. The Undecidability of Entailment and Relevant Implication, *The Journal of Symbolic Logic*, 49:1059-1073, 1984.

[56] A. Urquhart. Failure of Interpolation in Relevant Logics. *Journal of Philosophical Logic*, 22:449-479, 1993.

[57] A. Urquhart. Anderson and Belnap's Invitation to Sin. *Journal of Philosophical Logic*, 39:453-472, 2010.

[58] Andreas Weiermann and Martin Bunder. Ackermannian lower bounds for a combinatorial problem related to the decidability of certain relevant logics, 2005. *http://cage.ugent.be/~weierman/relevancelogic.pdf*.

[59] P. Weingartner. Two simple ideas of relevance. In *Proceedings of the Xxxth Conference on the History of Logic, Dedicated To Roman Suszko*, Cracow, 19-21 October 1984, 1984.

[60] G. Xiao and Y. Ma, Inconsistency measurement based on variables in minimal unsatisfable subsets. 2012. *A Talk delivered at European Conference on Artificial Intelligence, ECAI'12, France (2012)*.