Atomic Entailment and Atomic Inconsistency and Classical Entailment

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Abstract: In this paper we put forward a new solution of the well-known problem of relevant logics, i.e., we construct an atomic entailment. Hence, we construct a system of predicate calculus based on the atomic entailment. Next, we establish the definition of atomic inconsistency. The atomic inconsistency establishes an infinite class of inconsistent, but non-trivial systems. In this paper we construct the new definition of the classical entailment, into the bargain.

Key words: Atomic entailment, atomic inconsistency, classical entailment, relevance

1. Introduction

In a number of publications, (see [1] - [7], [9] - [18], [21], [22], [24] - [35], [39] - [45], [53] - [59]), their authors have offered many notions of relevance. Of course, in some publications of these mentioned above, their authors have established the basic properties of the well-known relevant logics. On the other hand, in [14] one can read that although the essence of entailment has been studied from 400 B.C., the problem of establishing such a logic of entailment, which solves the problem of relevance, is still open until now.

Thus, the essential aim is to create such a notion of relevance, which generates a system S of logic, which satisfies the following condition: this system S is generated by this notion of relevance, which is defined by a necessary and sufficient condition.

Therefore in this paper we at first construct a new definition of entailment, i.e. the definition of atomic entailment. Then we construct the definition of the system based on the atomic entailment. Next, we build a system S of propositional calculus (see [47], [48]) and a system S of predicate calculus, which are based on the atomic entailment (see [49], [51], [52]). Besides, in this paper, we give also the new definition of the classical entailment.

2. Notational Preliminaries

Let →, ¬, V, ∧, ≡ denote the connectives of implication, negation, disjunction, conjunction and equivalence, respectively. We use ⇒, ¬, ⇔, &, V, ∀ as metalogical symbols. Next, Σ0 = {p0, p1, p2, ..., q0, q1, q2, ..., s0, s1, s2, ..., t0} denotes the set of all propositional variables. S0 is the set of all well-formed formulas, which are built in the usual manner from propositional variables and by means of logical connectives. P0(φ) denotes the set of all propositional variables occurring in φ(φ ∈ S0). Next, A0 = {p0, p1, p2, ..., q0, q1, q2, ..., s0, s1, s2, ..., t0} denotes the set of all propositional variables. S0 is the set of all well-formed formulas, which are built in the usual manner from propositional variables and by means of logical connectives. P0(φ) denotes the set of all propositional variables occurring in φ(φ ∈ S0). R denotes the set of all rules over S0. Hence, for every r ∈ R, (Π, φ) ∈ r, where Π ⊆ S0 and φ ∈ S0 and Π is a set of premisses and φ is a conclusion. Hence, r denotes here the rule of simultaneous substitution for propositional variables. ⟨(φ), ψ⟩ ∈ r0 ⇐⇒ [he(φ) = ψ], where he is the extension of the mapping e: A0 → S0 (e ∈ ε0).
to endomorphism \( h^e : S_0 \rightarrow S_0 \), where
\[
h^e(\phi) = e(\phi), \quad \text{for } \phi \in At_0
\]
\[
h^e(\neg \phi) = \neg h^e(\phi)
\]
\[
h^e(\phi \lor \psi) = h^e(\phi) h^e(\psi),
\]
for \( F \in \{ \to, \lor, \land, \equiv \} \) and for every \( \phi, \psi \in S_0 \).

Thus, \( e^0 \) is a class of functions \( e : At_0 \rightarrow S_0 \) (for details, see [36]) (cf. [19]). \( r^0 \) denotes here the Modus Ponens rule in propositional calculus. \( R_{0^+} = \{ r^0 \} \) (for details, see [19], [36]). A logical matrix is a pair \( M = \{ U, U' \}, \ U \) is an abstract algebra and \( U' \) is a subset of the universe \( U \), i.e. \( U' \subseteq U \). Any \( a \in U' \) is called a distinguished element of the matrix \( M. E([M]) \) is the set of all formulas valid in the matrix \( M \). \( M_2 \) denotes the classical two-valued matrix. Hence, \( Z_2 \) is the set of all formulas valid in the classical matrix \( M_2 \) (see [19], [36]).

The symbols \( x_1, x_2, ... \) are individual variables. \( a_1, a_2, ... \) are individual constants. \( V \) is the set of all individual variables. \( P_1^n(i, n \in N = \{ 1, 2, ... \}) \) are \( n \)-ary predicate letters. The symbols \( f_i^n(i, n \in N) \) are \( n \)-ary function letters. The symbols \( \land x_k, \lor x_k \) are quantifiers. \( \land x_k \) is the universal quantifier and \( \lor x_k \) is the existential quantifier. The function letters, applied to the individual variables and individual constants, generate terms. The symbols \( t_1, t_2, ... \) are terms. \( T \) is the set of all terms. The predicate letters, applied to terms, yield simple formulas, i.e. if \( P_1^n \) is a predicate letter and \( t_1, ..., t_k \) are terms, then \( P_1^n(t_1, ..., t_n) \) is a simple formula. \( Smp \) is the set of all simple formulas. \( At_1 \) is the set of all atomic formulas, where \( At_1 = \{ P_1^k(x_1, ..., x_k) : k, i, j_1, ..., j_k \in N \} \). At last \( S_1 \) is the set of all well-formed formulas. \( FV(\phi) \) denotes the set of all free variables occurring in \( \phi \), where \( \phi \in S_1 \). \( x_k \in Ff(t_m, \phi) \) expresses that \( x_k \) is free for term \( t_m \) in \( \phi \). By \( x_k/t_m \) we denote the substitution of the term \( t_m \) for the individual variable \( x_k \). \( P_1(\phi) \) denotes the set of all predicate letters occurring in \( \phi (\phi \in S_1) \). If \( FV(\phi) = \{ x_1, ..., x_k \} \), then \( \land \phi = \land x_1 \land x_k \phi \).

\( R_{S_1} \) denotes the set of all rules over \( S_1 \). Hence, for every \( r \in R_{S_1} \), \( (\Pi, \phi) \in r \), where \( \Pi \subseteq S_1 \) and \( \phi \in S_1 \) and \( \Pi \) is a set of premisses and \( \phi \) is a conclusion. Hence, \( r^1 \) denotes here the rule of simultaneous substitution for predicate letters. \( \{(\phi), \psi \in r^1 \Leftrightarrow [h^e(\phi) = \psi] \) , where \( h^e \) is the extension of the mapping \( e : Smp \rightarrow S_1 (e \in e^1) \) to endomorphism \( h^e : S_1 \rightarrow S_1 \), where
\[
h^e(\phi) = e(\phi), \quad \text{for } \phi \in S_0
\]
\[
h^e(\neg \phi) = \neg h^e(\phi)
\]
\[
h^e(\phi \lor \psi) = h^e(\phi) h^e(\psi),
\]
for \( F \in \{ \to, \lor, \land, \equiv \} \) and for every \( \phi, \psi \in S_1 \).
\[ (R, X), \text{ where } R \subseteq R_{S_1} \text{ and } X \subseteq S_1. \text{ At last, } \phi \bigg|_{R, X} \psi \]
denotes that \( \psi \) results classically from \( \phi \) on the
ground of the system \( (R, X) \), where \( R \subseteq R_{S_1} \) and \( X \subseteq S_1. \ S_1^* \) denotes the
set of all formulas, which are in normal form (see [19]
pp. 35 - 42 and 130 - 132, [20] pp. 214 - 222 and [37]
p. 116 - 149).

**Definition 2.1.** The function \( j: S_1 \rightarrow S_0 \), is defined,
as follows:
\[
j(P_k^a(t_1, ..., t_n)) = p_k(p_k \in A_{t_0})
j(\sim \phi) = \sim j(\phi)
j(\phi \psi) = j(\phi) J j(\psi), \text{ for } F \in \{\sim, \lor, \land, \equiv\}
j(\bigwedge x_n \phi) = j(\bigvee x_n \phi) = j(\phi).
\]

By \( (R, X) \in Cns^A \) we denote here the well-known
notion of the absolute consistency (see [36] and [37]).

**Definition 2.2.** \( (R, X) \in Cns^A \Rightarrow Cn(R, X) \neq S_1, \)
where \( R \subseteq R_{S_1} \), \( X \subseteq S_i \) and \( i \in \{0, 1\} \).

### 3. Classical Entailment

**Definition 3.1.** Let \( Cn_1(R, X) = L \neq \emptyset \) and
\( \phi, \psi \in S_1. \text{ Then } \phi \bigg|_{R, X} \psi \text{ iff the following conditions}
are satisfied}
(1) \( (\forall e \in e^i)[h^e(\land \phi) \in L \Rightarrow h^e(\psi) \in L] \)
(2) \( (\forall e \in e^i)[h^e(\sim \phi) \in L \Rightarrow h^e(\sim \psi) \in L] \).

**Definition 3.2.** \( (R, X) \in Syst \cap C_i \) iff the following
condition is satisfied:
\[
(\forall \phi, \psi \in S_1)[\land \phi \rightarrow \psi \in Cn_1(R, X) \Leftrightarrow \phi \bigg|_{R, X} \psi].
\]

### 4. The classical predicate logic

Let \( A_2 \) denote the set of axioms of the classical
predicate logic. Hence, \( (R_{0+}, A_2) \) denotes the
classical predicate calculus, where \( Cn(R_{0+}, A_2) = L_2 \)
(see [20] and [37]).

Thus, (see [37] p.57, p.71):

**Theorem 4.1.** \( Cn_1(R_{0+}, L_2) = L_2 \).

Now we notice that on the ground of the classical
predicate calculus, the following theorem is valid (cf.

**Theorem 4.2.** (on the extensionality of logical
expressions). \( \{x_1, ..., x_n, y_1, ..., y_l \} \) all the free
variables, which occur in the expressions \( \alpha \) and \( \beta \),
and let \( C^\alpha \) be any expression that contains \( \alpha \) or an
expression obtained from \( \alpha \) by the substitution for
the variables \( x_1, ..., x_n \) of some other variables
different from the bound variables occurring in the
expressions \( \alpha \) or \( \beta \), and let \( C^\beta \) differ from \( C^\alpha \)
only in that in certain places (unnecessarily in all these
places) in which in \( C^\alpha \) there occurs \( \alpha \) or an
expression obtained from \( \alpha \) by a substitution for
the variables \( x_1, ..., x_n \), in the corresponding places in
\( C^\beta \) there occurs \( \beta \) or an expression obtained from \( \beta 
by an appropriate substitution, while the variables
\( y_1, ..., y_l \) are all the free variables in \( C^\alpha \) and \( C^\beta \).

Then the sentence:
\[
\bigwedge \{x_1, ..., x_n (\alpha \equiv \beta) \rightarrow (C^\alpha \equiv C^\beta)\}
\]
is a theorem in \( L_2 \).

### 5. Atomic Entailment

In [57] one can read that Lewis told that from his
very first contact with the logic of “Principia
Mathematica”, he had been bothered by the paradoxes
of material implication. As Whitehead and Russell
have it written, a true proposition is implied by
arbitrary (true or false) proposition, while a false
proposition implies arbitrary (true or false)
proposition. Aiming at avoiding these consequences
of the material conditional, Lewis wrote his first paper
devoted to logic (in this current paper, the Lewis’
paper is as [25]). At first, it ought to be noticed here
that the results contained in [1] - [7], [9] - [18], [21],
[22], [24] - [35], [39] - [45], [53] - [59], and in the
other papers, have essentially contributed to the better
understanding of the problem of relevance. Thus (see
[47], [48], [51], [52]):

**Definition 5.1.** Let \( Cn_0(R, X) = L \neq \emptyset \) and
\( \phi, \psi \in S_0. \text{ Then } \phi \bigg|_{R, X} \psi \text{ iff the following conditions
are satisfied:}
(1) \( (\forall \psi \in S_0)[\land \phi \rightarrow \psi \in Cn_0(R, X) \Leftrightarrow \phi \bigg|_{R, X} \psi].
\]

\( \bigwedge \{x_1, ..., x_n (\alpha \equiv \beta) \rightarrow (C^\alpha \equiv C^\beta)\} \)

is a theorem in \( L_2 \).
are satisfied:
(1) \((\forall e \in e^0)[h^e(\phi) \in L \Rightarrow h^e(\psi) \in L \land P_0(h^e(\phi)) \subseteq P_0(h^e(\psi))]\)
(2) \((\forall e \in e^0)[h^e((\phi^* \rightarrow \phi^*) \rightarrow \phi^*) \in L \Rightarrow h^e(\phi^*) \in L \land P_1(h^e(\phi^*)) \subseteq P_1(h^e(\phi^*))]\).

**Definition 5.2.** \((R, X) \in \text{Syst} \cap A_0\) iff the following condition is satisfied:
\((\forall \phi, \psi \in S_0)[\phi \rightarrow \psi \in Cn_0(R, X) \Leftrightarrow \phi \left|_{R, X} \psi\right].

**Definition 5.3.** Let \(Cn_1(R, X) = L \neq \emptyset\) and \(\phi, \psi \in S_1\). Then \(\phi \left|_{R, X} \psi\right.\) iff the following conditions are satisfied:
(1) \((\forall e \in e^1)[h^e(\Lambda \phi) \in L \Rightarrow h^e(\psi) \in L \land P_1(h^e(\Lambda \phi)) \subseteq P_1(h^e(\psi))]\)
(2) \((\forall e \in e^1)[h^e((\phi^* \rightarrow \phi^*) \rightarrow \phi^*) \in L \Rightarrow h^e(\phi^*) \in L \land P_1(h^e(\phi^*)) \subseteq P_1(h^e(\phi^*))]\).

**Definition 5.4.** \((R, X) \in \text{Syst} \cap A_1\) iff the following condition is satisfied:
\((\forall \phi, \psi \in S_1)[\Lambda \phi \rightarrow \psi \in Cn_1(R, X) \Leftrightarrow \phi \left|_{R, X} \psi\right].

6. Atomic Inconsistency

By \(\text{Syst} \cap AINC\) we denote here the class of all systems \((R, X)\), which have the property of atomic inconsistency (see also [8], [23], [60]), where \(R \subseteq R_S\) and \(X \subseteq S_1\) and \(i \in \{0, 1\}.

**Definition 6.1.** Let \(i \in \{0, 1\}\) and \(\alpha \in S_1\). Then \(S_{ia} = \{\phi \in S_1 : P_i(\phi) \subseteq P_i(\alpha)\}.

**Definition 6.2.** Let \(R \subseteq R_S\) and \(X \subseteq S_1\) and \(i \in \{0, 1\}. Then \((R, X) \in \text{Syst} \cap AINC \Leftrightarrow \)
\((\forall \alpha \in S_i)[S_{ia} \subseteq Cn(R, X \cup \{\alpha, \sim \alpha\})] \land \)
\((\forall \beta \in S_i)[P_i(\beta) \not\subseteq P_i(\alpha) \Rightarrow \beta \in Cn(R, X \cup \{\alpha, \sim \alpha\})] \land \)
\(\sim \beta \not\in Cn(R, X \cup \{\alpha, \sim \alpha\})].

7. System \(\overline{\text{S}}\)

Let us take the matrix \(\mathcal{M}_D = \{(0,1,2), (1,2), \overline{f}_D^0, \overline{f}_D^1, \overline{f}_D^2, f_D^1\}, where:
\[
\begin{array}{ccc|ccc}
\overline{f}_D^0 & 0 & 1 & 2 \\
0 & 1 & 1 & 2 \\
1 & 0 & 1 & 0 \\
2 & 0 & 1 & 2 \\
\end{array}
\begin{array}{ccc|ccc}
\overline{f}_D^1 & 0 & 1 & 2 \\
0 & 1 & 1 & 2 \\
1 & 0 & 1 & 0 \\
2 & 0 & 1 & 2 \\
\end{array}
\begin{array}{ccc|ccc}
\overline{f}_D^2 & 0 & 1 & 2 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
2 & 0 & 1 & 2 \\
\end{array}
\]

In [47] (see [48]) we have defined the system \(\overline{\text{S}}\) as follows:

**Definition 7.1.** \(\overline{\text{S}} = (R_0, T_D), where T_D = E(\mathcal{M}_D).

Thus, the system \(\overline{\text{S}}\) is the logic that is obtained from the set of valid formulas in the matrix \(\mathcal{M}_D\), by the rules of substitution and detachment.

It should be noticed here that the matrix \(\mathcal{M}_D = \{(0,1,2), (1,2), f_D^0, f_D^1, f_D^2\}\) was investigated by B. Sobociński (see [46], [47]).

Next, in [47] we have proved the following:

**Theorem 7.2.** Let \(\phi, \psi \in S_0\) and
\((\exists e \in e^0)[h^e(\phi) \in T_D].

Then \(\phi \rightarrow \psi \in Cn_0(R_0, T_D) \Leftrightarrow \)
\((\forall e \in e^0)[h^e(\phi) \in T_D \Rightarrow h^e(\psi) \in T_D \land \)
\(P_0(h^e(\phi)) \subseteq P_0(h^e(\psi))].

**Theorem 7.3.** The system \(\overline{\text{S}}\) is axiomatizable.

8. System \(\text{\overline{S}}\)

At first we define the set \(L_D\), putting:

**Definition 8.1.** \(L_D = \{\phi \in S_1: j(\phi) \in T_D \land \phi \in L_2\}.

Next, we define the system \(\text{\overline{S}}\), as follows:

**Definition 8.2.** \(\text{\overline{S}} = (R_0+, L_D)\).

By Theorem 4.1 and by Definition 8.1, one obtains:

**Corollary 8.3.** \(Cn_1(R_0+, L_D) = L_D\).

By Definition 8.1 and by Corollary 8.3, we get
Corollary 8.4. Let $\alpha, \beta, \gamma, \phi, \psi, \delta \in S_1$ and $Q_i \in \{\land x_i, \lor x_i\}$ and $i, k, s \in \mathbb{N}$. Then the following formulas belong to $L_D$:

1. $\alpha \to \alpha$
2. $\alpha \to [(\alpha \to \beta) \to \beta]$
3. $(\alpha \to \beta) \to [(\beta \to \gamma) \to (\alpha \to \gamma)]$
4. $[\alpha \to (\beta \to \gamma)] \to [\beta \to (\alpha \to \gamma)]$
5. $[\alpha \to (\alpha \to \beta)] \to (\alpha \to \beta)$
6. $[(\beta \to \gamma) \to (\alpha \to \gamma)] \to [(\alpha \to \beta) \to \delta]$}
7. $[\alpha \to (\beta \to \gamma)] \to \{[\delta \to \beta] \to [\alpha \to (\delta \to \gamma)]\}$
8. $[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
9. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
10. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\gamma \to \delta) \to (\alpha \to \delta)]$

$\sim \alpha \to \alpha$

9. $(\sim \alpha \to \alpha) \to \sim \alpha$
10. $(\sim \alpha \to \alpha) \to \sim \beta \to \beta$
11. $(\sim \alpha \to \sim \alpha) \to \sim \alpha$
12. $(\sim \alpha \to \sim \alpha) \to \sim \alpha$
13. $(\sim \alpha \to \sim \alpha) \to \sim \alpha$
14. $(\sim \alpha \to \sim \alpha) \to \sim \alpha$
15. $(\sim \alpha \to \sim \alpha) \to \sim \beta \to \beta$
16. $(\sim \alpha \to \sim \alpha) \to \sim \beta \to \beta$
17. $(\sim \alpha \to \sim \alpha) \to \sim \alpha$
18. $(\sim \alpha \to \sim \alpha) \to \sim \alpha$
19. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
20. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
21. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
22. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
23. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
24. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
25. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
26. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
27. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
28. $(\beta \to \gamma) \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
29. $(\alpha \equiv \alpha)$
30. $(\alpha \equiv \alpha)$
31. $(\alpha \equiv \alpha)$
32. $(\alpha \equiv \alpha)$
33. $(\alpha \equiv \alpha)$
34. $(\alpha \equiv \alpha)$
35. $(\alpha \equiv \alpha)$
36. $(\alpha \equiv \alpha)$
37. $(\alpha \equiv \alpha)$
38. $(\alpha \equiv \beta) \to [(\gamma \equiv \alpha) \equiv (\gamma \equiv \beta)]$
(79) \([\alpha \land \sim(\alpha \land \sim\beta)] \rightarrow \beta\)

(80) \((\alpha \land \beta) \rightarrow (\sim\alpha \land \sim\beta)\)

(81) \((\alpha \land \beta) \rightarrow \sim(\alpha \rightarrow \sim\beta)\)

(82) \((\alpha \land \sim\beta) \rightarrow (\alpha \land \beta)\)

(83) \(\sim(\alpha \rightarrow \sim\beta) \rightarrow (\alpha \land \beta)\)

(84) \((\alpha \rightarrow \sim\beta) \rightarrow (\alpha \land \sim\beta)\)

(85) \((\alpha \rightarrow \sim\beta) \rightarrow \sim(\alpha \land \beta)\)

(86) \((\alpha \rightarrow \beta) \equiv (\sim\beta \rightarrow \sim\alpha)\)

(87) \((\alpha \equiv \beta) \equiv (\sim\alpha \equiv \sim\beta)\)

(88) \((\alpha \land \beta) \equiv (\beta \land \alpha)\)

(89) \([\alpha \land (\beta \land \gamma)] \equiv [(\alpha \land \beta) \land \gamma]\)

(90) \([(\alpha \equiv \beta) \land (\gamma \equiv \delta)] \rightarrow [(\alpha \rightarrow \gamma) \equiv (\beta \rightarrow \delta)]\)

(91) \((\alpha \equiv \beta) \rightarrow [(\beta \rightarrow \alpha) \land (\alpha \rightarrow \beta)]\)

(92) \((\alpha \equiv \beta) \rightarrow [(\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)]\)

(93) \((\alpha \land \alpha) \equiv \alpha\)

(94) \((\alpha \equiv \beta) \rightarrow [(\alpha \land \gamma) \equiv (\beta \land \gamma)]\)

(95) \((\alpha \equiv \beta) \rightarrow [(\gamma \land \alpha) \equiv (\gamma \land \beta)]\)

(96) \([(\alpha \equiv \beta) \land (\gamma \equiv \delta)] \rightarrow [(\alpha \equiv \gamma) \equiv (\beta \equiv \delta)]\)

(97) \([(\alpha \rightarrow \gamma) \land (\gamma \rightarrow \alpha)] \rightarrow (\alpha \equiv \gamma)\)

(98) \([(\alpha \equiv \beta) \land (\gamma \equiv \delta)] \rightarrow [(\alpha \land \gamma) \equiv (\beta \land \delta)]\)

(99) \([(\alpha \equiv \beta) \land (\beta \equiv \gamma)] \rightarrow [(\alpha \rightarrow \gamma) \land (\gamma \rightarrow \alpha)]\)

(100) \((\alpha \lor \alpha) \equiv \alpha\)

(101) \((\alpha \lor \beta) \equiv (\beta \lor \alpha)\)

(102) \((\alpha \equiv \beta) \rightarrow [(\gamma \lor \alpha) \equiv (\gamma \lor \beta)]\)

(103) \((\alpha \lor \beta) \equiv (\sim \alpha \rightarrow \beta)\)

(104) \((\alpha \rightarrow \beta) \equiv (\sim \alpha \lor \beta)\)

(105) \([(\alpha \equiv \beta) \land (\gamma \equiv \delta)] \rightarrow [(\alpha \lor \gamma) \equiv (\beta \lor \delta)]\)

(106) \(\sim(\alpha \land \beta) \equiv (\alpha \rightarrow \sim\beta)\)

(107) \(\sim(\alpha \land \beta) \equiv (\beta \rightarrow \sim\alpha)\)

(108) \((\alpha \lor \beta) \rightarrow (\sim \alpha \land \sim \beta)\)

(109) \((\sim \alpha \land \sim \beta) \rightarrow (\alpha \lor \beta)\)

(110) \(\sim(\alpha \lor \sim \beta) \rightarrow (\alpha \land \beta)\)

(111) \(\sim(\alpha \land \beta) \rightarrow (\sim \alpha \lor \sim \beta)\)

(112) \(\sim(\alpha \lor \beta) \rightarrow (\sim \alpha \lor \sim \beta)\)

(113) \(\sim(\sim \alpha \land \sim \beta) \rightarrow (\alpha \lor \beta)\)

(114) \((\alpha \land \beta) \rightarrow (\sim \alpha \lor \sim \beta)\)

(115) \((\sim \alpha \lor \sim \beta) \rightarrow (\alpha \land \beta)\)

(116) \((\alpha \lor \beta) \equiv (\sim \alpha \land \sim \beta)\)

(117) \((\alpha \land \beta) \equiv (\sim \alpha \lor \sim \beta)\)

(118) \((\alpha \land \beta) \equiv (\sim \alpha \lor \sim \beta)\)

(119) \((\alpha \rightarrow \beta) \equiv (\sim \alpha \lor \sim \beta)\)

(120) \((\alpha \lor \beta) \equiv (\sim \alpha \land \sim \beta)\)

(i) \(\alpha \rightarrow \beta\)

(ii) \(\alpha \lor \beta\)

(iii) \(\alpha \land \beta\)

(iv) \(\sim \alpha \land \sim \beta\)

(v) \(\sim \alpha \lor \sim \beta\)

(vi) \(\sim \alpha \lor \sim \beta\)

(vii) \(\sim \alpha \land \sim \beta\)

(viii) \(\sim \alpha \lor \sim \beta\)

(ix) \(\sim \alpha \land \sim \beta\)

(x) \(\sim \alpha \lor \sim \beta\)

(xi) \(\sim \alpha \land \sim \beta\)

(xii) \(\sim \alpha \lor \sim \beta\)

(xiii) \(\sim \alpha \land \sim \beta\)

(xiv) \(\sim \alpha \lor \sim \beta\)

(xv) \(\sim \alpha \land \sim \beta\)

(xvi) \(\sim \alpha \lor \sim \beta\)

(xvii) \(\sim \alpha \land \sim \beta\)

(xviii) \(\sim \alpha \lor \sim \beta\)

(xix) \(\sim \alpha \land \sim \beta\)

(xx) \(\sim \alpha \lor \sim \beta\)

(xxi) \(\sim \alpha \land \sim \beta\)

(xxii) \(\sim \alpha \lor \sim \beta\)

(xxiii) \(\sim \alpha \land \sim \beta\)

(xxiv) \(\sim \alpha \lor \sim \beta\)

(xxv) \(\sim \alpha \land \sim \beta\)

(xxvi) \(\sim \alpha \lor \sim \beta\)

(xxvii) \(\sim \alpha \land \sim \beta\)

(xxviii) \(\sim \alpha \lor \sim \beta\)

(xxix) \(\sim \alpha \land \sim \beta\)

(xxxxv) \(\sim \alpha \lor \sim \beta\)

Using Definition 8.1, Corollary 8.3, Corollary 8.4 and using the proof of Theorem 4.2 (sec [20], pp.222 -
Proof. the variables expression obtained from $\Sigma$ by an appropriate substitution, while the variables $\Lambda D$ Definition 8.6. Then the sentence:

\[ (3) \rightarrow \psi \]

8.3, we obtain that (2) \[ \psi^{*} \rightarrow \phi^{*} \in L_{D} \]. Hence, by Definition 8.1, Corollary 8.3, Corollary 8.4 and Corollary 8.5, we obtain that (8) \[ \psi^{*} \rightarrow \phi^{*} \in L_{D} \]. Hence, by the definition of the set $L_{D}$, it follows that (4) \[ (3) \rightarrow \psi \].

Hence, from (1), by Definition 8.1, Corollary 8.3, Corollary 8.4 and Corollary 8.5, we obtain that (8) \[ \psi^{*} \rightarrow \phi^{*} \in L_{D} \]. Hence, by Definition 8.1, Corollary 8.3, Corollary 8.4 and Corollary 8.5, we obtain that (9) \[ \forall e \in \varepsilon \coprod \exists e \in \varepsilon \coprod \phi^{*} \rightarrow \phi^{*} \in L_{D} \Rightarrow h^{e}(\phi^{*}) \in L_{D} & P_{1}(h^{e}(\phi^{*})) \subseteq P_{1}(h^{e}(\phi^{*}))\] what together with (3) complete the proof. □

Lemma 8.9. If $\phi \in \Pi^{1}_{2}$ and \[ (\exists e \in \varepsilon \coprod [h^{e}(\phi) \in L_{D}]) \], then $h^{e}(\phi) \in L_{D}$.

Proof. Now we assume that (1) \[ Q_{1} \in \{ \Lambda x_{i}, \Lambda x_{i} \} \] and (2) $\phi \in \Pi^{1}_{2}$ and (3) \[ (\exists e \in \varepsilon \coprod [h^{e}(\phi) \in L_{D}]) \]. Hence, by the definition of the set $L_{D}$, it follows that (4) \[ (3) \rightarrow \psi \].

Let:

(1.1) $\phi \in \Pi^{1}_{2}$.

Hence, by (4) and Definition 8.6, one can obtain that

(5) $h^{e}(\phi) = \Lambda \phi \rightarrow \phi$.

Hence, by Corollary 8.4 (i), in (1.1), it follows that

(6) $h^{e}(\phi) \in L_{D}$.

Let

(1.2) $\phi = \Lambda \phi \rightarrow \phi$.

Hence, from (1.2) and Definition 8.6, it follows that

(7) $h^{e}(\phi) = \Lambda \phi \rightarrow \phi$.

Therefore, by Corollary 8.4 (107) and (1.2), it follows that

(8) $h^{e}(\phi) \in L_{D}$.

So, using Corollary 8.4 (i), in (1.2), one can obtain that

(9) $h^{e}(\phi) \in L_{D}$.

Let

(1.3) $\phi = \phi_{1} \vee \phi_{2}$

and assume inductively that

$$(a_{1})h^{e}(\phi_{1}) \in L_{D}$$

or

$$(a_{2})h^{e}(\phi_{2}) \in L_{D}.$$
From Definition 8.6 it follows that
\[ h^\phi_\phi(\phi_1 \lor \phi_2) = h^\phi_\phi(\phi_1) \lor h^\phi_\phi(\phi_2). \]

Next, in (a.1) and (a.2), from (1.3) and by Definition 8.6, it follows that
\[ P_1(h^\phi_\phi(\phi_1)) = P_1(h^\phi_\phi(\phi_2)) = P_1(\phi). \]

Hence, from (10), by Corollary 8.4 (40) and Corollary 8.4 (41), in (a.1) and (a.2), in (1.3), it follows that
\[ h^\phi_\phi(\phi) = P_1(\phi). \]

Let
\[ (1.4) \phi = \phi_1 \land \phi_2 \]
and assume inductively that
\[ (13) h^\phi_\phi(\phi_1), h^\phi_\phi(\phi_2) \in L_D. \]

From (1.4) and (13), using Definition 8.6, by Corollary 8.4 (62), in (1.4), one can obtain that
\[ (14) h^\phi_\phi(\phi) \in L_D. \]

Let
\[ (1.5) \phi = Q_1 \phi' \]
and assume inductively that
\[ (15) h^\phi_\phi(\phi') \in L_D. \]

Hence, from (1.5), using Definition 8.6, by Corollary 8.4 (ix) and Corollary 8.3, in (1.5), one can obtain that
\[ (16) h^\phi_\phi(\phi) \in L_D, \]
which completes the proof. □

**Lemma 8.10.** If \( \phi \in S_1 \) and
\[ (\exists e \in e^1_1)[h^e(\phi) \in L_D], \text{ then } h^e_\phi(\phi) \in L_D. \]


**Lemma 8.11.** Let \( \land \phi \rightarrow \psi \in L_2 \) and
\[ (\forall e \in e^1_1)[h^e(\land \phi) \in L_D \Rightarrow h^e(\psi) \in L_D \& P_1(h^e(\psi)) \in P_1(h^e(\psi))] \]
\[ (\forall e \in e^1_1)[h^e((\land \phi) \rightarrow \psi) \in L_D \Rightarrow h^e(\phi^*) \in L_D \& P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))]. \]

Then
\[ \land \phi \rightarrow \psi \in L_2. \]

**Proof.** By Theorem 7.2, the Definition 8.1, Corollary 8.3, Corollary 8.4, Corollary 8.5, Definition 8.6, Corollary 8.7, Lemma 8.10 and by the definition of the matrix \( M_2 \) and by the definitions of the formulas \( \phi^*, \psi^*. \) □

**Lemma 8.12.** Let \( \phi, \psi \in S_1 \) and
\[ (\exists e \in e^1_1)[h^e(\phi) \in L_D] \]
and
\[ (\forall e \in e^1_1)[h^e(\phi) \in L_D \Rightarrow h^e(\psi) \in L_0 \& P_1(h^e(\phi)) \subseteq P_1(h^e(\psi))]. \]

Then
\[ (\forall e \in e^1_1)[h^e(\phi) \in L_2 \Rightarrow h^e(\psi) \in L_2]. \]

**Proof.** Let (1) \( \phi, \psi \in S_1 \), (2) \( (\exists e \in e^1_1)[h^e(\phi) \in L_D] \) and (3) \( (\forall e \in e^1_1)[h^e(\phi) \in L_D \Rightarrow h^e(\psi) \in L_D \& P_1(h^e(\phi)) \subseteq P_1(h^e(\psi))]. \)

From (1), (2), it follows that (4) \( (\exists e_1 \in e^1_1)[h^{e_1}(\phi) \in L_D]. \) Now suppose that (5) \( (\exists e_2 \in e^1_1)[h^{e_2}(\phi) \in L_2 \& h^{e_2}(\psi) \in L_2]. \)

Next assume that (6) \( h^{e_2}(\phi) = \phi' \) and (7) \( h^{e_2}(\psi) = \psi'. \)

From (5) – (7), it follows that (8) \( \phi' \in L_2 \) and (9) \( \psi' \in L_2. \)

From (8) it follows that (10)
\[ (\exists e \in e^1_1)[h^e(\phi') \in L_2]. \]

Hence, by Lemma 8.10, it follows that (11) \( h^e(\phi') \in L_2. \)

Hence, from (3) it follows that (12) \( h^e(\psi') \in L_2. \)

From (8) and (9) and Definition 8.6 and Theorem 4.2, it follows that (14) \( h^e(\psi') \in L_2. \)

From (12), by the definition of the set \( L_2, \) it follows that (15) \( h^e(\psi') \in L_2, \) what contradicts (14). □

**Lemma 8.13.** Let
\[ (\forall e \in e^1_1)[h^e((\psi^* \rightarrow \phi') \rightarrow \phi^*) \in L_D \Rightarrow h^e(\phi') \in L_0 \& P_1(h^e(\psi^*)) \subseteq P_1(h^e(\phi^*))]. \]

Then
\[ (\forall e \in e^1_1)[h^e((\psi^* \rightarrow \phi') \rightarrow \phi^*) \in L_2 \Rightarrow h^e(\phi') \in L_2]. \]

**Proof.** The proof of this lemma is analogous to the proof of Lemma 8.12. □

In [50] we have proved the following Lemma:

**Lemma 8.14.** Let \( \phi, \psi \in S_1, X \subseteq S_1 \) and
\[ (\exists v: A_0 \rightarrow (M_{D_1} | h^v(j(\phi)) = 1 \& Cn_1(R_{D_4}, L_2 \cup X) = Z_3) \]
and
\[ (\forall e \in e^1_1)[h^e(\phi) \in Z_3 \Rightarrow h^e(\psi) \in Z_3]. \]
Then $\Lambda \phi \rightarrow \psi \in Z_3$.

In consequence:

**Lemma 8.15.** If $\exists e \in e^i_D[h^e(\phi) \in L_D] \text{ and } (\forall e \in e^i_D)[h^e(\phi) \in L_D \Rightarrow h^e(\psi) \in L_D \& P_1(h^e(\phi)) \subseteq P_1(h^e(\psi))]$, then $\Lambda \phi \rightarrow \psi \in L_2$.


**Theorem 9.1.** $\langle R_0, L_D \rangle \in \text{Syst} \cap A_1$.

**Proof.** By Lemma 8.18 and Lemma 8.19.

**Theorem 9.2.** $\langle R_0+ , L_D \rangle \in \text{Syst} \cap A_1$.

**Proof.** By Lemma 8.8 and Lemma 8.17.

**Theorem 9.3.** $\langle R_0+ , L_2 \rangle \in \text{Syst} \cap C_1$.

**Proof.** By similar reasoning as in the proofs of Lemma 8.8 and Lemma 8.17 (or by Corollary 8.7, the definition of the set $L_D$ and by Lemma 8.14).

**Theorem 9.4.** $\langle R_0, T_D \rangle \in \text{Syst} \cap AINC$.

**Proof.** By the definition of the set $T_D$ and the Definition 6.1 and the Definition 6.2.

**Theorem 9.5.** $\langle R_0+, L_D \rangle \in \text{Syst} \cap AINC$.

**Proof.** Let

(1) $\alpha \in S_1$

and

(2) $\beta \in S_1$.

Hence, by the definition of the set $L_D$, it follows that

(3) $\alpha \rightarrow (\sim \alpha \rightarrow \beta) \in L_D$, where

(4) $P_1(\beta) \subseteq P_1(\alpha)$.

From (1)-(4), it follows that

(5) $S_{1\alpha} \subseteq \text{Cn}(R_0+, L_D \cup \{\alpha, \sim \alpha\})$.

Let now,

(6) $P_1(\beta) \not\subseteq P_1(\alpha)$.

Next, by the definition of the set $L_2$, it follows that

(7) $(\alpha \land \sim \alpha) \rightarrow (\beta \land \sim \beta) \not\in L_2$.

Next, from (6), by the definition of the set $T_D$, it follows that

(8) $j(\alpha \land \sim \alpha) \rightarrow j(\beta \land \sim \beta) \not\in T_D$.

Hence, from (6), (7), by the definition of the set $L_D$, it follows that

(9) $\beta \not\in \text{Cn}(R_0+, L_D \cup \{\alpha \land \sim \alpha\})$

or

(10) $\sim \beta \not\in \text{Cn}(R_0+, L_D \cup \{\alpha \land \sim \alpha\})$.

what together with (5), (6) and the Definition 6.2, completes the proof.

10. Summary

**Remark 10.1.** Let $(\forall e \in e^i_D)[h^e(\psi^*) \in L \Rightarrow h^e(\phi^*) \in L \& P_0(h^e(\psi^*)) \subseteq P_0(h^e(\phi^*))] = \Lambda_0$

and

$(\forall e \in e^i_D)[h^e((\psi^* \rightarrow \phi^*) \rightarrow \phi^*) \in L \Rightarrow$
Remark 10.3.

Let \( \forall e \in \varepsilon \cdot \left[ h^e(\psi^*) \in L \Rightarrow h^e(\phi^*) \in L \right] \land P_0\left(h^e(\psi^*)\right) \subseteq P_0\left(h^e(\phi^*)\right) \) = \( \Lambda_0 \).

By an inspection of Definition 5.1, Definition 5.2, Definition 7.1, Lemma 8.18 and Lemma 8.19, one can easily see that in condition (2) of Definition 5.1 one cannot put \( \Lambda_0 \) instead of \( \Lambda_1 \).

**Remark 10.2.**

Let \( \left( \forall e \in \varepsilon \right) \left( h^e(\psi^*) \in L \Rightarrow h^e(\phi^*) \in L \right) \land P_0\left(h^e(\psi^*)\right) \subseteq P_0\left(h^e(\phi^*)\right) \) = \( \Lambda_0 \).

By an inspection of Definition 5.3 and Definition 5.4 and Lemma 8.1 and Lemma 8.17, one can easily see that in condition (2) of Definition 5.3 one cannot put \( \Lambda_0 \) instead of \( \Lambda_1 \).

**Remark 10.3.**

Let \( \left( \forall e \in \varepsilon \right) \left( h^e(\psi^*) \in L \Rightarrow h^e(\phi^*) \in L \right) \land \left( \forall e \in \varepsilon \right) \left[ h^e(\left( \psi^* \rightarrow \phi^* \right) \rightarrow \phi^* \right] \in L \Rightarrow h^e(\phi^*) \in L \) = \( \Lambda_0 \).

By an inspection of Definition 3.1, Definition 3.2 and by Lemma 8.14, Theorem 9.3, one can easily see that in condition (2) of Definition 3.1 one cannot put \( \Lambda_0 \) instead of \( \Lambda_1 \).

Using Definition 2.2 and Definition 6.1 and Definition 6.2, one can obtain the following remark:

**Remark 10.4.**

\[ \langle R, X \rangle \in \text{Syst} \cap \text{AINC} \Rightarrow \langle R, X \rangle \in \text{Cns}^A, \]

where \( R \subseteq R_S \) and \( X \subseteq S_i \) and \( i \in \{0, 1\} \).

References


[60] G. Xiao and Y. Ma, Inconsistency measurement based on variables in minimal unsatisfiable subsets. 2012. *A Talk delivered at European Conference on Artificial Intelligence, ECAI’12, France (2012).*